Chapter 3

Common solutions of split equilibrium problem and fixed point problem for a nonexpansive semigroup

3.1 Introduction

Throughout the rest part of thesis unless otherwise stated, let $H_1$ and $H_2$ be real Hilbert spaces and let $C$ and $Q$ be nonempty, closed and convex subsets of $H_1$ and $H_2$, respectively.

In 2010, Censor, Gibali and Reich [38] introduced and studied the following split variational inequality problem (in short, $S\text{P VIP}$): Find $x^* \in C$ such that

$$\langle f(x^*), x - x^* \rangle \geq 0, \quad \forall x \in C,$$

(3.1.1)

and such that

$$y^* = Ax^* \in Q \text{ solves } \langle g(y^*), y - y^* \rangle \geq 0, \quad \forall y \in Q,$$

(3.1.2)

where $f : H_1 \to H_1$ and $g : H_2 \to H_2$ are nonlinear mappings and $A : H_1 \to H_2$ is a bounded linear operator.

To solve the $S\text{P VIP}(3.1.1)-(3.1.2)$ without a product space formulation, Censor et al. [38] proposed the following iterative algorithm: Let $\lambda > 0$, choose a point $x_0 \in H_1$. Given the current iterate $x_n$, compute

$$x_{n+1} = U(x_n + \delta A^*(T - I)Ax_n),$$

(3.1.3)
for $n = 1, 2, 3, \ldots$, where $\delta \in (0, \frac{1}{L})$ with $L$ being the spectral radius of the operator $A^*A$, $A^*$ is the adjoint operator of $A$, $T := P_Q(I - \lambda g)$, $U := P_C((I - \lambda f)$. This iterative algorithm can be viewed as an extension of Dolidze’s algorithm for variational inequalities [74], which suggests the involvement of averaged operators.

The special cases of $S_P\text{VIP}(3.1.1)-(3.1.2)$ are split zero problem and split feasibility problem which have already been studied and used in practice as a model in intensity-modulated radiation therapy treatment planning, see [36,37]. This formalism is also at the core of modeling of many inverse problems arising for phase retrieval and other real-world problems; for instance, in sensor networks in computerized tomography and data compression; see, e.g. [25,26,51]. Recently, Moudafi [126] introduced an iterative method, an extension of iterative methods given by Censor et al. [38], Censor and Segal [40], Moudafi [125] for a split monotone variational inclusions.

In this chapter, we consider the following split equilibrium problem (in short, $S_P\text{EP}$) due to Moudafi [126]:

Let $F_1 : C \times C \to \mathbb{R}$ and $F_2 : Q \times Q \to \mathbb{R}$ be nonlinear bifunctions and $A : H_1 \to H_2$ be a bounded linear operator, then the split equilibrium problem ($S_P\text{EP}$) is to find $x^* \in C$ such that

$$F_1(x^*, x) \geq 0, \quad \forall x \in C,$$

(3.1.4)

and such that

$$y^* = Ax^* \in Q \text{ solves } F_2(y^*, y) \geq 0, \quad \forall y \in Q. \quad \text{(3.1.5)}$$

When looked separately, (3.1.4) is the classical equilibrium problem (EP) and we denoted its solution set by $\text{Sol}(EP(3.1.4))$. The $S_P\text{EP}(3.1.4)-(3.1.5)$ constitutes a pair of equilibrium problems which have to be solved so that the image $y^* = Ax^*$ under a given bounded linear operator $A$, of the solution $x^*$ of the EP(3.1.4) in $H_1$ is the solution of another EP(3.1.5) in another space $H_2$, we denote the solution set of EP(3.1.5) by $\text{Sol}(EP(3.1.5))$.

The solution set of $S_P\text{EP}(3.1.4)-(3.1.5)$ is denoted by $\text{Sol}(S_P\text{EP}(3.1.4)-(3.1.5)) = \{ p \in \text{Sol}(EP(3.1.4)) : Ap \in \text{Sol}(EP(3.1.5))\}$. 52
Example 3.1.1. Let $H_1 = H_2 = \mathbb{R}$ with the inner product defined by $\langle x, y \rangle = xy$, $\forall x, y \in \mathbb{R}$. Let $C = [0, 2]$ and $Q = (-\infty, 0]$: let $F_1 : C \times C \rightarrow \mathbb{R}$ and $F_2 : Q \times Q \rightarrow \mathbb{R}$ be defined by $F_1(x, y) = x^2 - y$, $\forall x, y \in C$ and $F_2(u, v) = (u + 6)(v - u)$, $\forall u, v \in Q$ and let, for each $x \in \mathbb{R}$, we define $A(x) = -3x$. It is easy to observe that $\text{Sol}(\text{EP}(3.1.4)) = [\sqrt{2}, 2]$, $A(2) = -6$ and $\text{Sol}(\text{EP}(3.1.5)) = \{-6\}$. Hence $\text{Sol}(\text{SP}\text{EP}((3.1.4)-(3.1.5))) = \{2\} \neq \emptyset$.

In 2006, Marino and Xu [116] introduced the following implicit and explicit iterative methods based on viscosity approximation method, for fixed point problem (FPP) for a nonexpansive self mapping $T$ on $H_1$:

$$x_t = t\gamma f(x_t) + (I - tB)Tx_t,$$

(3.1.6)

and

$$x_{n+1} = \alpha_n\gamma f(x_n) + (I - \alpha_nB)Tx_n,$$

(3.1.7)

where $f$ is a contraction mapping on $H_1$ with constant $\alpha > 0$, $B$ is a strongly positive self-adjoint bounded linear operator on $H_1$ with constant $\gamma > 0$ and $\gamma \in (0, \frac{\gamma}{\alpha})$. They proved that the net $(x_t)$ and the sequence $\{x_n\}$ generated by (3.1.6) and (3.1.7) respectively, converge strongly to the unique solution of the variational inequality

$$\langle (B - \gamma f)z, x - z \rangle \geq 0, \quad \forall x \in \text{Fix}(T),$$

which is the optimality condition for the minimization problem

$$\min_{x \in \text{Fix}(T)} \frac{1}{2} \langle Bx, x \rangle - h(x),$$

where $h$ is the potential function for $\gamma f$.

Recently, Cianciaruso et al. [47] introduced the following implicit and explicit iterative methods for approximating a common solution of EP(3.1.4) and FPP for a nonexpansive semigroup $S = \{T(s) : 0 \leq s < \infty\}$ defined in Definition 1.2.10:
\[
\begin{aligned}
F_1(u_t, y) + \frac{1}{r_t} \langle y - u_t, u_t - x_t \rangle, \quad \forall y \in C, \\
x_t = t \gamma f(x_t) + (I - tB) \frac{1}{s_t} \int_0^{s_t} T(s)u_t ds,
\end{aligned}
\]

(3.1.8)

where \((s_t)\) and \((r_t)\) are the continuous nets in \((0, 1)\);

and

\[
\begin{aligned}
F_1(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle, \quad \forall y \in C, \\
x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n B) \frac{1}{s_n} \int_0^{s_n} T(s)u_n ds,
\end{aligned}
\]

where \(\{\alpha_n\}\), \(\{s_n\}\) and \(\{r_n\}\) are the sequences in \((0, 1)\), respectively.

Based on these methods, they established the strong convergence theorems for approximating a common solution of EP\((3.1.4)\) and FPP for a nonexpansive semigroup \(S\), which generalize the work of Chen and Song [43], Plubtieng and Punpaeng [146,147], Shimizu and Takahashi [152]. Further related work, see Kamraksa and Wangkeeree [87].

Motivated by the work of Censor et al. [38], Cianciaruso et al. [47] Moudafi [126], Plubtieng and Punpaeng [147], and by the ongoing research in this direction, we suggest and analyze implicit and explicit iterative methods for approximating a common solution of \(S_p\)EP\((3.1.4)-(3.1.5)\) and FPP for a nonexpansive semigroup \(S\) in real Hilbert spaces.

Further, based on these methods, we prove the strong convergence theorems for approximating a common solution of \(S_p\)EP\((3.1.4)-(3.1.5)\) and FPP for a nonexpansive semigroup \(S\). Some consequences from these theorems are also derived. Furthermore, we justify our main results through a numerical example. The results and methods presented here extend and generalize the corresponding results and methods given in [47,108,147].

### 3.2 Preliminaries

Assume that \(F_1 : C \times C \to \mathbb{R}\) satisfies Assumption 2.2.1. For \(r > 0\) and for all \(x \in H_1\), define a mapping \(T_r : H_1 \to C\) as follows:

\[
T_r(x) = \{ z \in C : F_1(z, y) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0, \quad \forall y \in C \}. 
\]
It follows from Lemma 2.2.2 that $T^{F_1}_r$ is single-valued and firmly nonexpansive; $T^{F_1}_r(x) \neq \emptyset$ for every $x \in H_1$; $\text{Sol}(\text{EP}(3.1.4))$ is closed and convex, and $\text{Fix}(T^{F_1}_r) = \text{Sol}(\text{EP}(3.1.4))$.

Further, assume that $F_2 : Q \times Q \to \mathbb{R}$ satisfying Assumption 2.2.1. For $s > 0$ and for all $w \in H_2$, define a mapping $T^{F_2}_s : H_2 \to Q$ as follows:

$$T^{F_2}_s(w) = \left\{ d \in Q : F_2(d, e) + \frac{1}{s} \langle e - d, d - w \rangle \geq 0, \forall e \in Q \right\}.$$ 

Then, we easily observe that $T^{F_2}_s$ is single-valued and firmly nonexpansive; $T^{F_2}_s(w) \neq \emptyset$ for every $w \in H_2$; $\text{Sol}(\text{EP}(3.1.5))$ is closed and convex, and $\text{Fix}(T^{F_2}_s) = \text{Sol}(\text{EP}(3.1.5))$ where $\text{Sol}(\text{EP}(3.1.5))$ is the solution set of the following equilibrium problem:

Find $y^* \in Q$ such that $F_2(y^*, y) \geq 0$, $\forall y \in Q$.

**Lemma 3.2.1.** [48] Let $F_1 : C \times C \to \mathbb{R}$ be a bifunction satisfying Assumption 2.2.1 hold and let $T^{F_1}_r$ be defined as in Lemma 2.2.2 for $r > 0$. Let $x, y \in H_1$ and $r_1, r_2 > 0$. Then

$$\|T^{F_1}_{r_2}y - T^{F_1}_{r_1}x\| \leq \|y - x\| + \left| \frac{r_2 - r_1}{r_2} \right| \|T^{F_1}_{r_2}y - y\|.$$ 

We denote the identity operator on $H_1$ as well as $H_2$ by the same symbol $I$. Further, assume that $\text{Sol}(\text{SpEP}(3.1.4)-(3.1.5)) \neq \emptyset$.

### 3.3 Implicit iterative method

We prove a strong convergence theorem based on the proposed implicit iterative method for computing an approximate common solution of $\text{SpEP}(3.1.4)-(3.1.5)$ and FPP for a nonexpansive semigroup $S$ in real Hilbert spaces.

**Theorem 3.3.1.** Let $H_1$ and $H_2$ be two real Hilbert spaces and $C \subseteq H_1$ and $Q \subseteq H_2$ be nonempty, closed and convex sets. Let $A : H_1 \to H_2$ be a bounded linear operator. Assume that $F_1 : C \times C \to \mathbb{R}$ and $F_2 : Q \times Q \to \mathbb{R}$ are bifunctions satisfying Assumption 2.2.1 and $F_2$ is upper semicontinuous in first argument. Let $S = \{T(s) : 0 \leq s < \infty\}$ be a nonexpansive semigroup on $C$ such that $\Delta := \text{Fix}(S) \cap \text{Sol}(\text{SpEP}(3.1.4)-(3.1.5)) \neq \emptyset$, 55
where Fix$(S)$ is defined by (1.2.10). Let $f : H_1 \to H_1$ be a contraction mapping with constant $\alpha \in (0, 1)$ and $B$ be a strongly positive self-adjoint bounded linear operator on $H_1$ with constant $\gamma > 0$, such that $0 < \gamma < \frac{\gamma}{\alpha} < \gamma + \frac{1}{\alpha}$. Assume that $(r_t)$ and $(s_t)$ are the continuous nets of positive real numbers such that $\liminf_{t \to 0} r_t = r > 0$ and $\lim s_t = +\infty$. Let the nets $(u_t)$ and $(x_t)$ be implicitly generated by

$$u_t = T_{r_t}^{F_1}(x_t + \delta A^*(T_{r_t}^{F_2} - I)Ax_t),$$  \hfill (3.3.1)$$

$$x_t = t\gamma f(x_t) + (I - tB)\frac{1}{s_t} \int_0^{s_t} T(s)u_t ds,$$  \hfill (3.3.2)

where $\delta \in (0, 1/L)$, $L$ is the spectral radius of the operator $A^*A$ and $A^*$ is the adjoint of $A$. Then $x_t$ and $u_t$ converge strongly to $z \in \Delta$, where $z = P_\Delta(I - B + \gamma f)z$, which is the unique solution of the variational inequality

$$\langle (\gamma f - B)z, x^* - z \rangle \leq 0, \quad \forall x^* \in \Delta.$$  \hfill (3.3.3)

**Proof.** We first show that $x_t$ is well defined. For $t \in (0, 1)$ such that $t < \|B\|^{-1}$, define a mapping $S_t : H_1 \to H_1$ by

$$S_t x = t\gamma f(x) + (I - tB)\frac{1}{s_t} \int_0^{s_t} T(s)(T_{r_t}^{F_1}(x + \delta A^*(T_{r_t}^{F_2} - I)Ax)) ds, \quad \forall x \in H_1.$$  

We claim that $S_t$ is contractive with constant $(1 - t(\gamma - \alpha))$. Indeed since $T_{r_t}^{F_1}$ and $T_{r_t}^{F_2}$ both are firmly nonexpansive, they are averaged. For $\delta \in (0, \frac{1}{L})$, the mapping $(I + \delta A^*(T_{r_t}^{F_2} - I)A)$ is averaged, see [126]. It follows from Proposition 1.2.1 (ii) that the mapping $T_{r_t}^{F_1}(I + \delta A^*(T_{r_t}^{F_2} - I)A)$ is averaged and hence is nonexpansive. Further, for any $x, y \in H_1$, it follows from Lemma 1.2.5 that

$$\|S_t x - S_t y\| \leq \left\|t\gamma f(x) + (1 - tB)\frac{1}{s_t} \int_0^{s_t} T(s)T_{r_t}^{F_1}(x + \delta A^*(T_{r_t}^{F_2} - I)Ax) ds \
- t\gamma f(y) + (1 - tB)\frac{1}{s_t} \int_0^{s_t} T(s)T_{r_t}^{F_1}(y + \delta A^*(T_{r_t}^{F_2} - I)Ay) ds\right\| \leq t\gamma \|f(x) - f(y)\| + (1 - t\gamma) \left\|\frac{1}{s_t} \int_0^{s_t} T(s)|T_{r_t}^{F_1}(x + \delta A^*(T_{r_t}^{F_2} - I)Ax)\right\|$$

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\[
-T_{r_t}^{F_1}(y + \delta A^*(T_{r_t}^{F_2} - I)Ay)ds
\leq t\gamma\alpha\|x - y\|
+(1 - t\gamma)\|T_{r_t}^{F_1}(I + \delta A^*(T_{r_t}^{F_2} - I)A)x - T_{r_t}^{F_1}(I + \delta A^*(T_{r_t}^{F_2} - I)A)y\|
\leq t\gamma\alpha\|x - y\| + (1 - t\gamma)\|x - y\|
= [1 - t(\hat{\gamma} - \gamma\alpha)]\|x - y\|.
\]

Since \(0 < 1 - t(\hat{\gamma} - \gamma\alpha) < 1\), it follows that \(S_t\) is a contraction mapping. Therefore by Banach contraction principle, \(S_t\) has a unique fixed point \(x_t\), i.e., \(x_t\) is the unique solution of the fixed point equation (3.3.2).

Next, we show that \((x_t)\) is bounded. Let \(p \in \Delta\), we have \(p = T_{r_t}^{F_1}p, Ap = T_{r_t}^{F_2}Ap\) and \(p = T(s)p\).

We estimate

\[
\|u_t - p\|^2 = \|T_{r_t}^{F_1}(x_t + \delta A^*(T_{r_t}^{F_2} - I)Ax_t) - T_{r_t}^{F_1}p\|^2
\leq \|x_t + \delta A^*(T_{r_t}^{F_2} - I)Ax_t - p\|^2
\leq \|x_t - p\|^2 + \delta^2\|A^*(T_{r_t}^{F_2} - I)Ax_t\|^2
+2\delta\langle x_t - p, A^*(T_{r_t}^{F_2} - I)Ax_t\rangle. \tag{3.3.4}
\]

Thus, we have

\[
\|u_t - p\|^2 \leq \|x_t - p\|^2 + \delta^2\langle (T_{r_t}^{F_2} - I)Ax_t, AA^*(T_{r_t}^{F_2} - I)Ax_t\rangle
+2\delta\langle x_t - p, A^*(T_{r_t}^{F_2} - I)Ax_t\rangle. \tag{3.3.5}
\]

Now, we have

\[
\delta^2\langle (T_{r_t}^{F_2} - I)Ax_t, AA^*(T_{r_t}^{F_2} - I)Ax_t\rangle \leq L\delta^2\langle (T_{r_t}^{F_2} - I)Ax_t, (T_{r_t}^{F_2} - I)Ax_t\rangle
= L\delta^2\|T_{r_t}^{F_2} - I)Ax_t\|^2. \tag{3.3.6}
\]
Denoting \( \Lambda := 2\delta \langle x_t - p, A^*(T_{r_t}^{F_2} - I)Ax_t \rangle \) and using (1.2.3), we have

\[
\Lambda = 2\delta \langle A(x_t - p), (T_{r_t}^{F_2} - I)Ax_t \rangle \\
= 2\delta \langle A(x_t - p) + (T_{r_t}^{F_2} - I)Ax_t - (T_{r_t}^{F_2} - I)Ax_t, (T_{r_t}^{F_2} - I)Ax_t \rangle \\
= 2\delta \left\{ \langle T_{r_t}^{F_2}Ax_t - Ap, (T_{r_t}^{F_2} - I)Ax_t \rangle - \| (T_{r_t}^{F_2} - I)Ax_t \|^2 \right\} \\
\leq 2\delta \left\{ \frac{1}{2}\| (T_{r_t}^{F_2} - I)Ax_t \|^2 - \| (T_{r_t}^{F_2} - I)Ax_t \|^2 \right\} \\
\leq -\delta \| (T_{r_t}^{F_2} - I)Ax_t \|^2.
\]

(3.3.7)

Using (3.3.5), (3.3.6) and (3.3.7), we obtain

\[
\| u_t - p \|^2 \leq \| x_t - p \|^2 + \delta(L\delta - 1)\| (T_{r_t}^{F_2} - I)Ax_t \|^2.
\]

(3.3.8)

Since \( \delta \in (0, \frac{1}{L}) \), we obtain

\[
\| u_t - p \|^2 \leq \| x_t - p \|^2.
\]

(3.3.9)

Now setting \( z_t := \frac{1}{s_t} \int_0^{s_t} T(s)u_tds \), we obtain

\[
\| z_t - p \| = \left\| \frac{1}{s_t} \int_0^{s_t} T(s)u_tds - p \right\| \\
\leq \frac{1}{s_t} \int_0^{s_t} \| T(s)u_t - T(s)p \|ds \\
\leq \| u_t - p \| \leq \| x_t - p \|.
\]

(3.3.10)

Further, we estimate

\[
\| x_t - p \| = \| t\gamma f(x_t) + (1-tB) \frac{1}{s_t} \int_0^{s_t} T(s)u_tds - p \| \\
\leq t\| \gamma f(x_t) - Bp \| + (1-t\tilde{\gamma}) \frac{1}{s_t} \int_0^{s_t} \| T(s)u_t - T(S)p \|ds \\
\leq t \| \gamma f(x_t) - f(p) \| + \| \gamma f(p) - Bp \| + (1-t\tilde{\gamma})\| u_t - p \| \\
\leq t\gamma \| x_t - p \| + t\| \gamma f(p) - Bp \| + (1-t\tilde{\gamma})\| x_t - p \| \\
\leq [1 - t(\tilde{\gamma} - \gamma\alpha)] \| x_t - p \| + t\| \gamma f(p) - Bp \|
\]

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\[
\leq \frac{1}{\gamma - \gamma^2} \| \gamma f(p) - Bp \|. \quad (3.3.11)
\]

Hence the net \((x_i)\) is bounded and consequently, we deduce that the nets \((u_t)\), \((z_t)\) and \((f(x_i))\) are bounded.

Next, we have

\[
\| x_t - z_t \| = \| t(\gamma f(x_t) - Bz_t) + (1 - tB)(z_t - z_t) \| \\
\leq t\| \gamma f(x_t) - Bz_t \| \\
\rightarrow 0 \text{ as } t \rightarrow 0.
\]  

(3.3.12)

Next, we show that \(\| x_t - u_t \| \rightarrow 0 \text{ as } t \rightarrow 0\). It follows from (3.3.8) and Lemma 1.2.7 (i) that

\[
\| x_t - p \|^2 \\
\leq (1 - t\gamma^2)^2 \left[ \frac{1}{s} \int_0^s T(s)u_t ds - p \right]^2 \\
+ 2t(\gamma f(x_t) - Bp + \gamma f(p) - \gamma f(p), x_t - p) \\
\leq (1 + t^2\gamma^2 - 2t\gamma)\| u_t - p \|^2 + 2t\gamma\| x_t - p \|^2 + 2t(\gamma f(p) - Bp, x_t - p) \\
\leq (1 + t^2\gamma^2)\| u_t - p \|^2 + 2t\gamma\| x_t - p \|^2 + 2t(\gamma f(p) - Bp, x_t - p) \\
\leq \| u_t - p \|^2 + 2t\gamma\| x_t - p \|^2 + t\gamma^2\| x_t - p \|^2 \\
+ 2t\| \gamma f(p) - Bp \| \| x_t - p \| \\
\leq \| x_t - p \|^2 + \delta(\gamma - 1)\| (T_{r_t}^{F_2} - I)Ax_t \|^2 + 2t\gamma\| x_t - p \|^2 + t\gamma^2\| x_t - p \|^2 \\
+ 2t\| \gamma f(p) - Bp \| \| x_t - p \|. 
\]  

(3.3.13)

Since \((x_t)\) is bounded, we may assume that \(\psi := \sup_{0 < t < 1} \| x_t - p \|\). Therefore, preceding inequality reduces to

\[
\delta(1 - \gamma)(T_{r_t}^{F_2} - I)Ax_t \leq 2t\gamma\| x_t - p \|^2 + \gamma^2\| x_t - p \|^2 + 2\| \gamma f(p) - Bp \| \| x_t - p \|. 
\]

Since \(\delta(1 - \gamma) > 0\), preceding inequality implies that
\[
\lim_{t \to 0} \|(T_{r_i}^{F_2} - I)Ax_t\| = 0. \tag{3.3.14}
\]

Next, we have

\[
\|u_t - p\|^2 = \|T_{r_i}^{F_1}(x_t + \delta A^*(T_{r_i}^{F_2} - I)Ax_t) - p\|^2
= \|T_{r_i}^{F_1}(x_t + \delta A^*(T_{r_i}^{F_2} - I)Ax_t) - T_{r_i}^{F_1}p\|^2
\leq \langle u_t - p, x_t + \delta A^*(T_{r_i}^{F_2} - I)Ax_t - p \rangle
= \frac{1}{2} \left\{ \|u_t - p\|^2 + \|x_t + \delta A^*(T_{r_i}^{F_2} - I)Ax_t - p\|^2
- \|\langle u_t - p \rangle - \{x_t + \delta A^*(T_{r_i}^{F_2} - I)Ax_t - p\|^2 \right\}
= \frac{1}{2} \left\{ \|u_t - p\|^2 + \|x_t - p\|^2 - \|u_t - x_t - \delta A^*(T_{r_i}^{F_2} - I)Ax_t\|^2 \right\}
= \frac{1}{2} \left\{ \|u_t - p\|^2 + \|x_t - p\|^2 - \|u_t - x_t\|^2
+ \delta^2 \|A^*(T_{r_i}^{F_2} - I)Ax_t\|^2 - 2\delta \langle u_t - x_t, A^*(T_{r_i}^{F_2} - I)Ax_t \rangle \right\}
\leq \frac{1}{2} \left\{ \|u_t - p\|^2 + \|x_t - p\|^2 - \|u_t - x_t\|^2
- \delta^2 \|A^*(T_{r_i}^{F_2} - I)Ax_t\|^2 + 2\delta \|A(u_t - x_t)\|(T_{r_i}^{F_2} - I)Ax_t \right\}.
\]

Hence, we have

\[
\|u_t - p\|^2 \leq \|x_t - p\|^2 - \|u_t - x_t\|^2 - \delta^2 \|A^*(T_{r_i}^{F_2} - I)Ax_t\|^2
+ 2\delta \|A(u_t - x_t)\|(T_{r_i}^{F_2} - I)Ax_t
\leq \|x_t - p\|^2 - \|u_t - x_t\|^2 + 2\delta \|A(u_t - x_t)\|(T_{r_i}^{F_2} - I)Ax_t. \tag{3.3.15}
\]

Since \((x_t)\) and \((u_t)\) are bounded and \(A\) is a bounded linear operator then the net \((A(u_t - x_t))\) is bounded and hence, we may assume that \(l := \sup_{0 < t < 1} \|A(u_t - x_t)\|\). It follows from (3.3.13) and (3.3.15) that

\[
\|x_t - p\|^2 \leq \|u_t - p\|^2 + 2t\gamma\alpha \|x_t - p\|^2 + t\bar{\gamma}^2 \|x_t - p\|^2 + 2t\|\gamma f(p) - Bp\| \|x_t - p\|
\leq \|x_t - p\|^2 - \|u_t - x_t\|^2 + 2\delta \|A(u_t - x_t)\|(T_{r_i}^{F_2} - I)Ax_t + tJ,
\]

where \(J := (2\gamma\alpha + \bar{\gamma}^2)\varrho^2 + 2\|\gamma f(p) - Bp\|\varrho\).
Therefore, from (3.3.14), we obtain
\[
\|x_t - u_t\|^2 \leq 2\delta l\|\left(T^{F_2}_{r_t} - I\right)x_t\| + tJ \rightarrow 0, \quad \text{as} \ t \rightarrow 0.
\] (3.3.16)

Since \(u_t = T^{F_1}_{r_t}(x_t + \delta A^*(T^{F_2}_{r_t} - I)x_t)\), on setting \(d_t := x_t + \delta A^*(T^{F_2}_{r_t} - I)x_t\), we have
\[
\|u_t - d_t\| = \|u_t - (x_t + \delta A^*(T^{F_2}_{r_t} - I)x_t)\| \\
\leq \|u_t - x_t\| + \delta\|A^*(T^{F_2}_{r_t} - I)x_t\| \\
\leq \|u_t - x_t\| + \delta\|A^*\|(T^{F_2}_{r_t} - I)x_t\|.
\] (3.3.17)

It follows from (3.3.14) and (3.3.16) that
\[
\lim_{n \rightarrow \infty} \|u_t - d_t\| = 0.
\] (3.3.18)

Next, we have
\[
\|T(s)x_t - x_t\| \leq \left\|T(s)x_t - T(s)\frac{1}{s_t}\int_0^{s_t} T(s)u_tds\right\| \\
+ \left\|T(s)\frac{1}{s_t}\int_0^{s_t} T(s)u_tds - \frac{1}{s_t}\int_0^{s_t} T(s)u_tds\right\| \\
+ \left\|\frac{1}{s_t}\int_0^{s_t} T(s)u_tds - x_t\right\| \\
\leq \left\|x_t - \frac{1}{s_t}\int_0^{s_t} T(s)u_tds\right\| \\
+ \left\|T(s)\frac{1}{s_t}\int_0^{s_t} T(s)u_tds - \frac{1}{s_t}\int_0^{s_t} T(s)u_tds\right\| \\
+ \left\|\frac{1}{s_t}\int_0^{s_t} T(s)u_tds - x_t\right\| \\
\leq 2\left\|x_t - \frac{1}{s_t}\int_0^{s_t} T(s)u_tds\right\| \\
+ \left\|T(s)\frac{1}{s_t}\int_0^{s_t} T(s)u_tds - \frac{1}{s_t}\int_0^{s_t} T(s)u_tds\right\|.
\] (3.3.19)

Since \((x_t)\) and \((f(x_t))\) are bounded. Let \(K := \{w \in C : \|w - p\| \leq \frac{1}{\gamma - \gamma\alpha}\|\gamma f(p) - Bp\|\}\), then \(K\) is nonempty, bounded, closed and convex subset of \(C\) which is \(T(s)\)-invariant for each \(0 \leq s < \infty\) and contains \((x_t)\). So without loss of generality, we may assume
that $S := \{T(s) : 0 \leq s < \infty\}$ is nonexpansive semigroup on $K$. By Lemma 1.2.4, we have

$$\lim_{s_t \to \infty} \left\| T(s) \frac{1}{s_t} \int_0^{s_t} T(s) u_t ds - \frac{1}{s_t} \int_0^{s_t} T(s) u_t ds \right\| = 0. \tag{3.3.20}$$

Using (3.3.12), (3.3.19) and (3.3.20), we obtain

$$\lim_{t \to 0} \| T(s)x_t - x_t \| = 0. \tag{3.3.21}$$

Let $t, t_0 \in (0, \|B\|^{-1})$. Then, we have

$$\|x_t - x_{t_0}\| = \left\| (t - t_0)\gamma f(x_t) + t_0\gamma (f(x_t) - f(x_{t_0})) + (t_0 - t) \frac{B}{s_t} \int_0^{s_t} T(s) u_t ds \\
+ (I - t_0B) \left[ \frac{1}{s_t} \int_0^{s_t} T(s) u_t ds - \frac{1}{s_{t_0}} \int_0^{s_{t_0}} T(s) u_{t_0} ds \right] \right\| \leq \|t - t_0\| \gamma \|f(x_t) - f(p)\| + t_0\gamma \alpha \|x_t - x_{t_0}\| \\
+ \|t_0 - t\| \|B\| \left\| \frac{1}{s_t} \int_0^{s_t} T(s) u_t ds - p \right\| + \|t_0 - t\| \|B\| \|p\| \\
+ (I - t_0\gamma) \left[ \frac{1}{s_t} \int_0^{s_t} T(s) u_t ds - \frac{1}{s_{t_0}} \int_0^{s_{t_0}} T(s) u_{t_0} ds \right] \right\| \leq \|t - t_0\| \left( \gamma \alpha \|x_t - p\| + \gamma \|f(p)\| + \|B\| \left\| \frac{1}{s_t} \int_0^{s_t} T(s) u_t ds - p \right\| \right) \\
+ \|t_0 - t\| \|B\| \|p\| + t_0\gamma \alpha \|x_t - x_{t_0}\| \\
+ (1 - t_0\gamma) \left[ \frac{1}{s_t} \int_0^{s_t} T(s) u_t ds - \frac{1}{s_t} \int_0^{s_t} T(s) u_{t_0} ds \right] \right\| + (1 - t_0\gamma) \left[ \frac{1}{s_t} \int_0^{s_t} T(s) u_{t_0} ds - \frac{1}{s_{t_0}} \int_0^{s_{t_0}} T(s) u_{t_0} ds \right]. \tag{3.3.22}$$

Since $\left\| \frac{1}{s_t} \int_0^{s_t} T(s) u_t ds - p \right\| \leq \|u_t - p\| \leq \|x_t - p\| \leq \varrho$, and if, we denote

$$M := (\gamma \alpha + \|B\|) \varrho + \gamma \|f(p)\|, \quad 62$$
we obtain

\[
\|x_t - x_{t_0}\| \leq |t - t_0| M + |t_0 - t| \|B\|_s \|p\| + t_0 \gamma \alpha \|x_t - x_{t_0}\| + (1 - t_0 \gamma) \|u_t - u_{t_0}\| \\
+ (1 - t_0 \gamma) \left\| \left( \frac{1}{s_t} \right) \int_0^{s_t} T(s) u_{t_0} ds - \frac{1}{s_{t_0}} \int_0^{s_t} T(s) u_{t_0} ds \right\|
\]

\[
\leq |t - t_0| M + |t_0 - t| \|B\|_s \|p\| + t_0 \gamma \alpha \|x_t - x_{t_0}\| + (1 - t_0 \gamma) \|u_t - u_{t_0}\| \\
+ (1 - t_0 \gamma) \left\| \frac{1}{s_t} \int_0^{s_t} T(s) u_{t_0} ds \right\|.
\]

(3.3.23)

Since the mapping \(T_{r_1}^F (I + \delta A^*(T_{r_2}^F - I) A_{x_t})\) is averaged and hence nonexpansive. Further, since \(u_t = T_{r_1}^F (x_t + \delta A^*(T_{r_2}^F - I) A_{x_t})\) and \(u_{t_0} = T_{r_{t_0}}^F (x_{t_0} + \delta A^*(T_{r_{t_0}}^F - I) A_{x_{t_0}})\), it follows from Lemma 3.2.1 that

\[
\|u_t - u_{t_0}\|
\]

\[
\leq \|T_{r_1}^F (x_t + \delta A^*(T_{r_2}^F - I) A_{x_t}) - T_{r_{t_0}}^F (x_{t_0} + \delta A^*(T_{r_{t_0}}^F - I) A_{x_{t_0}})\|
\]

\[
+ \|T_{r_1}^F (x_{t_0} + \delta A^*(T_{r_2}^F - I) A_{x_{t_0}}) - T_{r_{t_0}}^F (x_{t_0} + \delta A^*(T_{r_{t_0}}^F - I) A_{x_{t_0}})\|
\]

\[
\leq \|x_t - x_{t_0}\| + \|T_{r_{t_0}} (x_{t_0} + \delta A^*(T_{r_2}^F - I) A_{x_{t_0}}) - (x_{t_0} + \delta A^*(T_{r_{t_0}}^F - I) A_{x_{t_0}})\|
\]

\[
+ \left(1 - \frac{r_t}{r_{t_0}}\right) \|T_{r_{t_0}} (x_{t_0} + \delta A^*(T_{r_2}^F - I) A_{x_{t_0}}) - (x_{t_0} + \delta A^*(T_{r_{t_0}}^F - I) A_{x_{t_0}})\|
\]

\[
\leq \|x_t - x_{t_0}\| + \delta \|A\| \|T_{r_{t_0}}^F A_{x_{t_0}} - T_{r_{t_0}}^F A_{x_{t_0}}\| + \delta_n
\]

\[
\leq \|x_t - x_{t_0}\| + \delta \|A\| \left(1 - \frac{r_t}{r_{t_0}}\right) \|T_{r_{t_0}}^F A_{x_{t_0}} - A_{x_{t_0}}\| + \delta_t
\]

\[
= \|x_t - x_{t_0}\| + \delta \|A\| \sigma_t + \delta_t,
\]

(3.3.24)

where

\[
\sigma_t = \left(1 - \frac{r_t}{r_{t_0}}\right) \|T_{r_{t_0}}^F A_{x_{t_0}} - A_{x_{t_0}}\|
\]

\[
\delta_t = \left(1 - \frac{r_t}{r_{t_0}}\right) \|T_{r_{t_0}}^F (x_{t_0} + \delta A^*(T_{r_2}^F - I) A_{x_{t_0}}) - (x_{t_0} + \delta A^*(T_{r_{t_0}}^F - I) A_{x_{t_0}})\|.
\]

It follows from (3.3.14) that the net \((T_{r_2}^F A_{x_t} - A_{x_t})\) is convergent and hence bounded.
Therefore, we may assume $M_1 := \sup_{0 < t < 1} \| T_{r_i}^{F_2} A x_t - A x_t \|$. Further, we can observe that the net $(x_t + \delta A^* (T_{r_i}^{F_2} - I) A x_t)$ is also bounded and hence, we may assume that $M_2 := \sup_{0 < t < 1} \| T_{r_i}^{F_2} (x_t + \delta A^* (T_{r_i}^{F_2} - I) A x_t) - (x_t + \delta A^* (T_{r_i}^{F_2} - I) A x_t)\|
$. Moreover, since $(r_t)$ is a continuous net of positive real numbers, we can choose a neighborhood $U_{t_0}$ and a positive number $c$ in such a way that $c < r_t$ for $t \in U_{t_0}$, we have

\[
\|u_t - u_{t_0}\| \leq \|x_t - x_{t_0}\| + \left[ \delta \|A\| \frac{M_1}{c} + \frac{M_2}{c} \right] |r_t - r_{t_0}|. \tag{3.3.25}
\]

It follows from (3.3.23) and (3.3.25) that

\[
\|x_t - x_{t_0}\| \leq |t - t_0|M + |t_0 - t| \frac{\|B\|}{s_t} \|p\| + t_0 \gamma \alpha \|x_t - x_{t_0}\| + (1 - t_0 \gamma) \|x_t - x_{t_0}\|
\]

\[
+ (1 - t_0 \gamma) \left[ \frac{1}{s_t} - \frac{1}{s_{t_0}} \right] s_t (\gamma + \|p\|) + (1 - t_0 \gamma) \left[ \frac{1}{s_{t_0}} \int_{s_t}^{s_{t_0}} T(s) u_{t_0} ds \right]
\]

\[
+ (1 - t_0 \gamma) \left[ \gamma \Delta A \left( \frac{M_1}{c} + \frac{M_2}{c} \right) \right] |r_t - r_{t_0}|
\]

\[
\leq \frac{1}{\gamma - \gamma \alpha} \left[ |t - t_0|M + |t_0 - t| \frac{\|B\|}{s_t} \|p\| + \frac{1}{s_t} - \frac{1}{s_{t_0}} s_t (\gamma + \|p\|) + (1 - t_0 \gamma) \left[ \gamma \Delta A \left( \frac{M_1}{c} + \frac{M_2}{c} \right) \right] |r_t - r_{t_0}|.
\]

The continuity of $(r_t)$ and $(s_t)$ shows that $(x_t)$ is a continuous curve. The continuity of $(u_t)$ is followed by (3.3.25).

Let \{t_n\} be a sequence in $(0, 1)$ such that $t_n \to 0$ as $n \to \infty$. Setting $x_n := x_{t_n}$, $u_n := u_{t_n}$, $s_n := s_{t_n}$, $r_n := r_{t_n}$. Since \{x_n\} is a bounded sequence, there is a subsequence \{x_{n_j}\} of \{x_n\} which converges weakly to $w \in C$. It follows from (3.3.21) and Lemma 1.2.1 that $w \in \text{Fix}(S)$. Further, we show that $x_{n_j} \to w$ as $j \to \infty$. Indeed, for each $n$, we have

\[
\|x_n - w\|^2 = \langle x_n \gamma f(x_n), x_n - w \rangle + \left( (1 - t_n \gamma) \frac{1}{s_n} \int_0^{s_n} T(s) u_n ds - w, x_n - w \right)
\]

\[
\leq \langle x_n \gamma f(x_n) - Bw, x_n - w \rangle + (1 - t_n \gamma) \left\| \frac{1}{s_n} \int_0^{s_n} T(s) u_n ds - w \right\| \|x_n - w\|
\]

\[
\leq \langle x_n \gamma f(x_n) - Bw, x_n - w \rangle + (1 - t_n \gamma) \|x_n - w\|^2
\]

\[
\leq t_n \gamma \alpha \|x_n - w\|^2 + t_n \gamma f(w) - Bw, x_n - w \rangle + (1 - t_n \gamma) \|x_n - w\|^2
\]

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\[ \leq [1 - t_n(\bar{\gamma} - \gamma \alpha)]\|x_n - w\|^2 + t_n(\gamma f(w) - Bw, x_n - w) \]
\[ \leq \frac{1}{\bar{\gamma} - \gamma \alpha} (\gamma f(w) - Bw, x_n - w). \]

In particular, we have
\[ \|x_{n_j} - w\|^2 \leq \frac{1}{\bar{\gamma} - \gamma \alpha} (\gamma f(w) - Bw, x_{n_j} - w). \quad (3.3.26) \]

Since \( x_{n_j} \to w \), it follows from (3.3.26) that \( x_{n_j} \to w \) as \( j \to \infty \).

Next, we show that \( w \in \text{Sol}(\text{EP}(3.1.4)) \). Since \( u_t = T_{r_t}^{F_1} d_t \) then, we have \( u_{n_j} = T_{r_{n_j}}^{F_1} d_{n_j} \) and
\[ F_1(u_{n_j}, y) + \frac{1}{r_{n_j}} \langle y - u_{n_j}, u_{n_j} - d_{n_j} \rangle \geq 0, \quad \forall y \in C. \]

It follows from monotonicity of \( F_1 \) that
\[ \frac{1}{r_{n_j}} \langle y - u_{n_j}, u_{n_j} - d_{n_j} \rangle \geq F_1(y, u_{n_j}) \]
and hence
\[ \left\langle y - u_{n_j}, \frac{u_{n_j} - d_{n_j}}{r_{n_j}} \right\rangle \geq F_1(y, u_{n_j}). \]

It follows from (3.3.16) and (3.3.18) that \( \|u_{n_j} - x_{n_j}\| \to 0, \|u_{n_j} - d_{n_j}\| \to 0 \). Since \( x_{n_j} \to w \), we get \( u_{n_j} \to w \) and \( d_{n_j} \to w \). Further, since \( \liminf_{t \to 0} r_t = r > 0 \) then \( \frac{u_{n_j} - d_{n_j}}{r_{n_j}} \to 0 \). It follows by Assumption 2.2.1 (iv) that \( 0 \geq F_1(y, w), \forall w \in C \). For \( \tau \) with \( 0 < \tau \leq 1 \) and \( y \in C \), let \( y_{\tau} = \tau y + (1 - \tau)w \). Since \( y \in C, w \in C \), we get \( y_{\tau} \in C \) and hence \( F_1(y_{\tau}, w) \leq 0 \). So from Assumption 2.2.1 (i) and (iv) we have
\[ 0 = F_1(y_{\tau}, y_{\tau}) \leq \tau F_1(y_{\tau}, y) + (1 - \tau)F_1(y_{\tau}, w) \leq \tau F_1(y_{\tau}, y). \]

Therefore \( 0 \leq F_1(y_{\tau}, y) \). From Assumption 2.2.1 (iii), we have \( 0 \leq F_1(w, y) \). This implies that \( w \in \text{Sol}(\text{EP}(3.1.4)) \).

Next, we show that \( Aw \in \text{Sol}(\text{EP}(3.1.5)) \). Since \( x_{n_j} \to w \) and \( A \) is a bounded linear
operator so that $Ax_{n_j} \to Aw$.

Now setting $v_{n_j} = Ax_{n_j} - T_{r_{n_j}}^{F_2} Ax_{n_j}$. It follows that from (3.3.14) that $\lim_{j \to \infty} v_{n_j} = 0$ and $Ax_{n_j} - v_{n_j} = T_{r_{n_j}}^{F_2} Ax_{n_j}$.

Therefore from Lemma 2.2.2, we have

$$F_2(Ax_{n_j} - v_{n_j}, z) + \frac{1}{r_{n_j}} \langle z - (Ax_{n_j} - v_{n_j}), (Ax_{n_j} - v_{n_j}) - Ax_{n_j} \rangle \geq 0, \forall z \in Q.$$

Since $F_2$ is upper semicontinuous in first argument, taking $\lim\inf$ to above inequality as $j \to \infty$ and using $\lim\inf_{t \to 0} r_t = r > 0$, we obtain

$$F_2(Aw, z) \geq 0, \forall z \in Q,$$

which means that $Aw \in \text{Sol}(\text{EP}(3.1.5))$ and hence $w \in \text{Sol}(\text{SP}_p\text{EP}(3.1.4)-(3.1.5))$.

Next, we show that $w \in \text{Fix}(S) \cap \text{Sol}(\text{SP}_p\text{EP}(3.1.4)-(3.1.5))$ solves the variational inequality (3.3.3). Since $x_t$ is the unique solution of fixed point equation (3.3.2), we have

$$(B - \gamma f)x_t = -\frac{1}{t} (I - tB) \left[ x_t - \frac{1}{s_t} \int_{0}^{s_t} T(s)u_t ds \right],$$

Then for any $q \in \text{Fix}(S) \cap \text{Sol}(\text{SP}_p\text{EP}(3.1.4)-(3.1.5))$, we obtain

$$\langle (B - \gamma f)x_t, x_t - q \rangle$$

$$= -\frac{1}{t} \left\langle (I - tB) \left[ x_t - \frac{1}{s_t} \int_{0}^{s_t} T(s)T_{r_t}^{F_1}(I + \delta A^*(T_{r_t}^{F_2} - I)A)x_t ds \right], x_t - q \right\rangle$$

$$= -\frac{1}{t} \left[ \frac{1}{s_t} \int_{0}^{s_t} \left\langle 
\left( I - T(s)T_{r_t}^{F_1}(I + \delta A^*(T_{r_t}^{F_2} - I)A) \right) x_t 
- \left( I - T(s)T_{r_t}^{F_1}(I + \delta A^*(T_{r_t}^{F_2} - I)A) \right) q, x_t - q \right\rangle ds \right]$$

$$+ \frac{1}{s_t} \left\langle B \int_{0}^{s_t} [x_t - T(s)u_t] ds, x_t - q \right\rangle.$$  \hspace{1cm} (3.3.27)

Since the mapping $U := T(s)T_{r_t}^{F_1}(I + \delta A^*(T_{r_t}^{F_2} - I)A)$ is nonexpansive then $(I - U)$ is monotone and hence
\[
\frac{1}{s_t} \int_0^{s_t} \left( (I - T(s)T^F_{r_i}(I + \delta A^*(T^F_{r_i} - I)A)x_t \\
- (I - T(s)T^F_{r_i}(I + \delta A^*(T^F_{r_i} - I)A)q, x_t - q \right) ds \geq 0.
\]

This together with (3.3.27), we have

\[
\langle (B - \gamma f)x_t, x_t - q \rangle \leq \langle Bx_t - \frac{B}{s_t} \int_0^{s_t} T(s)u_t ds, x_t - q \rangle.
\]

From (3.3.2), we have

\[
Bx_t - \frac{B}{s_t} \int_0^{s_t} T(s)u_t ds = tB \left( \gamma f(x_t) - \frac{B}{s_t} \int_0^{s_t} T(s)u_t ds \right).
\]

Hence, we have

\[
\langle (B - \gamma f)x_t, x_t - q \rangle \leq t \left( B \left( \gamma f(x_t) - \frac{B}{s_t} \int_0^{s_t} T(s)u_t ds \right), x_t - q \right).
\]

Since the nets \((x_t), (z_t), (u_t)\) and \((f(x_t))\) are bounded, on taking the limit \(t := t_{n_j} \to 0\), we obtain

\[
\langle (B - \gamma f)w, w - q \rangle = \lim_{j \to \infty} \langle (B - \gamma f)x_{n_j}, x_{n_j} - q \rangle \leq 0,
\]

which implies \(w = P_\Delta(I + \gamma f - B)\).

To show that the net \((x_t)\) converges strongly to \(w\), we assume that there is a sequence \(\{s_n\} \subset (0,1)\) such that \(x_{s_n} \to q\) when \(s_n \to 0\) as \(n \to \infty\). Following the same steps of the proof given above, we can prove \(q \in \text{Fix}(S) \cap \text{Sol}(S_P\text{EP}(3.1.4)-(3.1.5))\). Hence, it follows from (3.3.28) that

\[
\langle (B - \gamma f)q, q - w \rangle \leq 0.
\]

Interchanging the role of \(w\) and \(z\), we obtain

\[
\langle (B - \gamma f)w, w - q \rangle \leq 0.
\]
Adding (3.3.29) and (3.3.30) yields
\[
(\bar{\gamma} - \gamma \alpha) \|w - q\|^2 \leq \langle w - q, (B - \gamma f)w - (B - \gamma f)q \rangle \leq 0.
\]
By Lemma 1.2.6, we have \(w = q\) and therefore \(x_t \to q\).
Thus, we have shown that each cluster point of \((x_t)\) equals \(w\) as \(t \to 0\). Therefore
\(x_t \to w\) and \(u_t \to w\) as \(t \to 0\), where \(w \in \text{Fix}(S) \cap \text{Sol}(\text{SpEP}(3.1.4)-(3.1.5))\) is the unique solution of the variational inequality (3.3.2). This completes the proof. \(\square\)

### 3.4 Explicit iterative method

We prove a strong convergence theorem based on the explicit iterative discretization of the implicit iterative method (3.3.1)-(3.3.1) for computing an approximate common solution of \(\text{SpEP}(3.1.4)-(3.1.5)\) and FPP for a nonexpansive semigroup \(S\) in real Hilbert spaces.

**Theorem 3.4.1.** Let \(H_1\) and \(H_2\) be two real Hilbert spaces and let \(C \subseteq H_1\) and \(Q \subseteq H_2\) be nonempty, closed and convex sets. Let \(A : H_1 \to H_2\) be a bounded linear operator. Assume that \(F_1 : C \times C \to \mathbb{R}\) and \(F_2 : Q \times Q \to \mathbb{R}\) are bifunctions satisfying Assumption 2.2.1 and \(F_2\) is upper semicontinuous in first argument. Let \(S = \{T(s) : 0 \leq s < \infty\}\) be a nonexpansive semigroup on \(C\) such that \(\Gamma := \text{Fix}(S) \cap \text{Sol}(\text{SpEP}(3.1.4)-(3.1.5)) \neq \emptyset\).

Let \(f : H_1 \to H_1\) be a contraction mapping with constant \(\alpha \in (0,1)\) and \(B\) be a strongly positive self-adjoint bounded linear operator on \(H_1\) with constant \(\bar{\gamma} > 0\) such that \(0 < \gamma < \frac{\bar{\gamma}}{\alpha} < \gamma + \frac{1}{\alpha}\).

\(\{s_n\}\) is a positive real sequence which diverges to \(+\infty\). For a given \(x_0 \in C\) arbitrarily, let the iterative sequences \(\{u_n\}\) and \(\{x_n\}\) be generated by

\[
\begin{align*}
\text{for } n = 0, 1, 2, \ldots, \\
\{u_n = T_{r_n}^{F_1}(x_n + \delta A^*(T_{r_n}^{F_2} - I)Ax_n), \\
x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n B)\frac{1}{s_n} \int_0^{s_n} T(s)u_n ds,\)
\end{align*}
\]

(3.4.1)

where \(r_n \subset (0, \infty)\) and \(\delta \in (0,1/L)\), \(L\) is the spectral radius of the operator \(A^*A\) and \(A^*\) is the adjoint of \(A\) and \(\{\alpha_n\}\) and \(\{\beta_n\}\) are the sequences in \((0,1)\) satisfying the following conditions:
\( \lim_{n \to \infty} \alpha_n = 0, \sum_{n=0}^{\infty} \alpha_n = \infty; \) and \( \sum |\alpha_{n+1} - \alpha_n| < \infty; \)

(ii) \( \liminf r_n > 0 \) and \( \sum_{n=0}^{\infty} |r_{n+1} - r_n| < \infty; \)

(iii) \( \lim_{n \to \infty} \frac{s_{n+1} - s_n}{s_{n+1}} = 0. \)

Then the sequence \( \{x_n\} \) converges strongly to \( z \in \Gamma \), where \( z = P_{\Gamma}(I - B + \gamma f)z. \)

**Proof.** Note that from condition (i), we may assume without loss of generality that \( \alpha_n \leq (1 - \beta_n)\|B\|^{-1} \) for all \( n \). From Lemma 1.2.5, we know that if \( 0 < \rho \leq \|B\|^{-1} \), then \( \|I - \rho B\| \leq 1 - \rho \bar{\gamma}. \) We will assume that \( \|I - B\| \leq 1 - \bar{\gamma}. \)

Since \( B \) is strongly positive self-adjoint bounded linear operator on \( H_1 \), then

\[
\|B\| = \sup\{|\langle Bu, u \rangle| : u \in H_1, \|u\| = 1\}.
\]

Observe that

\[
\langle (I - \alpha_n B)u, u \rangle = 1 - \alpha_n \langle Bu, u \rangle \\
\geq 1 - \alpha_n \|B\| \geq 0,
\]

which implies that \( (1 - \alpha_n B) \) is positive. It follows that

\[
\|(I - \alpha_n B)\| = \sup\{|\langle (1 - \alpha_n B)u, u \rangle| : u \in H_1, \|u\| = 1\} \\
= \sup\{1 - \alpha_n \langle Bu, u \rangle : u \in H_1, \|u\| = 1\} \\
\leq 1 - \alpha_n \bar{\gamma}.
\]

Let \( q = P_{\Gamma} \). Since \( f \) is a contraction mapping with constant \( \alpha \in (0, 1) \). It follows that

\[
\|q(I - B + \gamma f)(x) - q(I - B + \gamma f)(y)\| \leq \|(I - B + \gamma f)(x) - (I - B + \gamma f)(y)\| \\
\leq \|I - B\|\|x - y\| + \gamma \|f(x) - f(y)\| \\
\leq (1 - \bar{\gamma})\|x - y\| + \gamma \alpha \|x - y\| \\
\leq (1 - (\bar{\gamma} - \gamma \alpha))\|x - y\|,
\]

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for all \( x, y \in H_1 \). Therefore, the mapping \( q(I - B + \gamma f) \) is a contraction mapping from \( H_1 \) into itself. It follows from Banach contraction principle that there exists an element \( z \in H_1 \) such that \( z = q(I - B + \gamma f)z = P_f(I - B + \gamma f)(z) \).

Let \( p \in \Gamma \), i.e., \( p \in \text{Sol}(\text{S}_{P} \text{EP}(3.1.4)-(3.1.5)) \), we have \( p = T_{r_n}^{F_1}p \) and \( Ap = T_{r_n}^{F_2}(Ap) \).

As estimated (3.3.8), we obtain

\[
\|u_n - p\|^2 \leq \|x_n - p\|^2 + \delta(L\delta - 1)\|(T_{r_n}^{F_2} - I)Ax_n\|^2. \tag{3.4.2}
\]

Since, \( \delta \in (0, \frac{1}{L}) \), we obtain

\[
\|u_n - p\|^2 \leq \|x_n - p\|^2. \tag{3.4.3}
\]

Set \( t_n := \frac{1}{s_n} \int_{0}^{s_n} T(s)u_n ds \). As estimated (3.3.10), we obtain

\[
\|t_n - p\| \leq \|x_n - p\|. \tag{3.4.4}
\]

Further, we estimate

\[
\|x_{n+1} - p\| = \|\alpha_n \gamma f(x_n) + (I - \alpha_n B)t_n - p\|
\]

\[
= \|\alpha_n \gamma f(x_n) - Bp + (I - \alpha_n B)(t_n - p)\|
\]

\[
\leq \alpha_n \|\gamma f(x_n) - Bp\| + (1 - \alpha_n \gamma)\|t_n - p\|
\]

\[
\leq \alpha_n \gamma \|f(x_n) - f(p)\| + \alpha_n \|\gamma f(p) - Bp\| + (1 - \alpha_n \gamma)\|x_n - p\|
\]

\[
\leq \alpha_n \gamma \alpha \|x_n - p\| + \alpha_n \|\gamma f(p) - Bp\| + (1 - \alpha_n \gamma)\|x_n - p\|
\]

\[
= (1 - (\gamma - \gamma \alpha)\alpha_n)\|x_n - p\| + \alpha_n \|\gamma f(p) - Bp\|
\]

\[
\leq \max \left\{ \|x_n - p\|, \frac{1}{\gamma - \gamma \alpha} \|\gamma f(p) - Bp\| \right\}, \quad n \geq 0
\]

\[
\leq \max \left\{ \|x_0 - p\|, \frac{1}{\gamma - \gamma \alpha} \|\gamma f(p) - Bp\| \right\}. \tag{3.4.5}
\]

Hence \( \{x_n\} \) is bounded and consequently, we deduce that \( \{u_n\} \), \( \{t_n\} \) and \( \{f(x_n)\} \) are bounded.
Next, we estimate

\[
\|t_{n+1} - t_n\| = \left\| \frac{1}{s_{n+1}} \int_0^{s_{n+1}} T(s)u_{n+1}ds - \frac{1}{s_n} \int_0^{s_n} T(s)u_n ds \right\|
\]

\[
= \left\| \frac{1}{s_{n+1}} \int_0^{s_{n+1}} [T(s)u_{n+1} - T(s)u_n]ds + \left( \frac{1}{s_{n+1}} - \frac{1}{s_n} \right) \times \right. \\
\left. \int_0^{s_n} T(s)u_n ds + \frac{1}{s_{n+1}} \int_{s_n}^{s_{n+1}} T(s)u_n ds \right\|
\]

\[
\leq \|u_{n+1} - u_n\| + \frac{|s_{n+1} - s_n| s_n}{(s_{n+1}) s_n} \|u_n - p\| + \frac{|s_{n+1} - s_n| s_n}{s_{n+1}} \|u_n - p\| 
\]

\[
\leq \|u_{n+1} - u_n\| + 2 \frac{|s_{n+1} - s_n| s_n}{s_{n+1}} \|u_n - p\|. 
\]

(3.4.6)

As estimated (3.3.24), we obtain

\[
\|u_{n+1} - u_n\| \leq \|x_{n+1} - x_n\| + \delta \|A\| \sigma + \delta_n, 
\]

(3.4.7)

where

\[
\sigma_n = \left| 1 - \frac{r_n}{r_{n+1}} \right| \|T_{r_{n+1}}^F A x_{n+1} - A x_{n+1}\|
\]

and

\[
\delta_n = \left| 1 - \frac{r_n}{r_{n+1}} \right| \|T_{r_{n+1}}^F d_{n+1} - d_{n+1}\|,
\]

where \(d_{n+1} = (x_{n+1} + \delta A^* (T_{r_{n+1}}^F - I) A x_{n+1})\).

Using (3.4.6) and (3.4.7), we have

\[
\|t_{n+1} - t_n\| \leq \|x_{n+1} - x_n\| + \delta \|A\| \sigma + \delta_n + 2 \frac{|s_{n+1} - s_n| s_n}{s_{n+1}} \|u_n - p\|. 
\]

(3.4.8)

Next, we show that the sequence \(\{x_n\}\) is asymptotically regular, i.e., \(\|x_{n+1} - x_n\| \rightarrow 0\) as \(n \rightarrow \infty\).

\[
\|x_{n+1} - x_n\| = \|\alpha_n f(x_n) + (1 - \alpha_n B) t_n - [\alpha_n - 1] f(x_{n-1}) + (1 - \alpha_{n-1} B) t_{n-1} \|
\]

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\[\begin{align*}
= \|\alpha_n^2 f(x_n) - \alpha_n f(x_{n-1}) + \alpha_n f(x_{n-1}) - \alpha_{n-1} f(x_{n-1})
+ (1 - \alpha_n B) t_n - (1 - \alpha_n B) t_{n-1} + (1 - \alpha_{n-1} B) t_{n-1}\| \\
\leq \alpha_n^2 \alpha \|x_n - x_{n-1}\| + \gamma \alpha_n - \alpha_{n-1} \|f(x_{n-1})\| + \|I - \alpha_n B\| \|t_n - t_{n-1}\| \\
+ \|\alpha_n - \alpha_{n-1}\| \|B t_{n-1}\|,
\end{align*}\]

where \(K_1 = \sup \{\|f(x_n)\| + \|B t_n\| : n \in N\} < \infty\). It follows from (3.4.9) that

\[\begin{align*}
\|x_{n+1} - x_n\| \leq \alpha_n^2 \alpha \|x_n - x_{n-1}\| + (1 + \gamma) \|\alpha_n - \alpha_{n-1}\| K_1 + (1 - \alpha_n \bar{\gamma}) \left[\|x_n - x_{n-1}\| \\
+ \delta \|A\| \|\sigma_n + \delta_n + 2 \frac{s_n - s_{n-1}}{s_n} \|u_{n-1} - p\|\right],
\end{align*}\]

where \(K_2 = \sup \{\|(T_{r_n}^E - I) A x_n\| : n \in N\}\), \(K_3 = \sup \{\|(T_{r_n}^F - I) d_n\| : n \in N\}\) and \(K_4 = \sup \{\|u_n - p\| : n \in N\}\). Setting \(K = \max \{K_1, K_2, K_3, K_4\}\). It follows from (3.4.10) that

\[\begin{align*}
\|x_{n+1} - x_n\| \leq \left(1 - \alpha_n (\bar{\gamma} - \gamma)\right) \|x_n - x_{n-1}\| \\
+ K \left[\|x_n - x_{n-1}\| + K_2 + \|s_n - s_{n-1}\| K_4, \right.
\end{align*}\]

Hence by Lemma 1.2.11 and conditions (i)-(iii), we have

\[\lim_{n \to \infty} \|x_{n+1} - x_n\| = 0.\]
Now,

\[ x_{n+1} - x_n = \alpha_n \gamma f(x_n) + (I - \alpha_n B)t_n - x_n \]
\[ = \alpha_n (\gamma f(x_n) - x_n) + (I - \alpha_n B)(t_n - x_n). \]

Since \( \|x_{n+1} - x_n\| \to 0 \) and \( \alpha_n \to 0 \) as \( n \to \infty \), we obtain

\[ \lim_{n \to \infty} \|t_n - x_n\| = 0. \quad (3.4.13) \]

Since \( \{x_n\} \) and \( \{f(x_n)\} \) are bounded. Let \( K := \{w \in C : \|w - p\| \leq \max\{\|x_0 - p\|, \frac{1}{\gamma - \gamma_n} \|\gamma f(p) - Bp\|\}\} \), then \( K \) is nonempty, bounded, closed and convex subset of \( C \) which is \( T(s) \)-invariant for each \( 0 \leq s < \infty \) and contains \( \{x_n\} \). So without loss of generality, we may assume that \( S := \{T(s) : 0 \leq s < \infty\} \) is nonexpansive semigroup on \( K \). Now as estimated (3.3.21), we obtain

\[ \lim_{n \to \infty} \|T(s)x_n - x_n\| = 0. \quad (3.4.14) \]

It follows from (3.4.2) and Lemma 1.2.7(i) that

\[ \|x_{n+1} - p\|^2 = \|\alpha_n \gamma f(x_n) + (I - \alpha_n B)t_n - p\|^2 \]
\[ = \|\alpha_n (\gamma f(x_n) - Bp) + (I - \alpha_n B)(t_n - p)\|^2 \]
\[ \leq \|(I - \alpha_n B)(t_n - p)\|^2 + 2\alpha_n \|\gamma f(x_n) - Bp, x_{n+1} - p\| \]
\[ \leq (1 - \alpha_n \tilde{\gamma})^2 \|t_n - p\|^2 + 2\alpha_n \|\gamma f(x_n) - Bp\| \|x_{n+1} - p\| \]
\[ = (I - \alpha_n \tilde{\gamma})^2 \|u_n - p\|^2 + 2\alpha_n \|\gamma f(x_n) - Bp\| \|x_{n+1} - p\| \]
\[ \leq (1 - \alpha_n \tilde{\gamma})^2 \|x_n - p\|^2 + \delta(L\delta - 1)\|(T_{r_n}^{F_2} - I)Ax_n\|^2 \]
\[ + 2\alpha_n \|\gamma f(x_n) - Bp\| \|x_{n+1} - p\| \]
\[ \leq \left[ 1 - 2\alpha_n \tilde{\gamma} + (\alpha_n \tilde{\gamma})^2 \right] \|x_n - p\|^2 + (1 - \alpha_n \tilde{\gamma})^2 \delta(L\delta - 1)\|(T_{r_n}^{F_2} - I)Ax_n\|^2 \]
\[ + 2\alpha_n \|\gamma f(x_n) - Bp\| \|x_{n+1} - p\| \]
\[ \leq \|x_n - p\|^2 + \alpha_n \tilde{\gamma}^2 \|x_n - p\|^2 + (1 - \alpha_n \tilde{\gamma})^2 \delta(L\delta - 1)\|(T_{r_n}^{F_2} - I)Ax_n\|^2 \]
\[ + 2\alpha_n \|\gamma f(x_n) - Bp\| \|x_{n+1} - p\|. \]

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Therefore,

\[
(1 - \alpha_n \bar{\gamma})^2 \delta (1 - L \delta) \| (T_{r_n}^{F_2} - I) A x_n \|^2 \\
\leq \| x_n - p \|^2 - \| x_{n+1} - p \|^2 + \alpha_n \bar{\gamma}^2 \| x_n - p \|^2 \\
+ 2 \alpha_n \| \gamma f(x_n) - B p \| \| x_{n+1} - p \| \\
\leq (\| x_n - p \|^2 + \| x_{n+1} - p \|) \| x_n - x_{n+1} \| + \alpha_n \bar{\gamma}^2 \| x_n - p \|^2 \\
+ 2 \alpha_n \| \gamma f(x_n) - B p \| \| x_{n+1} - p \|.
\]

Since \( \delta (1 - L \delta) > 0, \alpha_n \to 0, \| x_n - t_n \| \to 0 \) and \( \| x_{n+1} - x_n \| \to 0 \) as \( n \to \infty \), we obtain

\[
\lim_{n \to \infty} \| (T_{r_n}^{F_2} - I) A x_n \| = 0. \tag{3.4.16}
\]

As estimated (3.3.15), we obtain

\[
\| u_n - p \|^2 \leq \| x_n - p \|^2 - \| u_n - x_n \|^2 + 2 \delta \| A(u_n - x_n) \| \| (T_{r_n}^{F_2} - I) A x_n \|. \tag{3.4.17}
\]

It follows from (3.4.15) and (3.4.16) that

\[
\| x_{n+1} - p \|^2 \leq (1 - \alpha_n \bar{\gamma})^2 \left[ \| x_n - p \|^2 - \| u_n - x_n \|^2 \\
+ 2 \delta \| A(u_n - x_n) \| \| (T_{r_n}^{F_2} - I) A x_n \| \\
+ 2 \alpha_n \| \gamma f(x_n) - B p \| \| x_{n+1} - p \| \\
\right. \\
\left. \leq \left( 1 - 2 \alpha_n \bar{\gamma} + (\alpha_n \bar{\gamma})^2 \right) \| x_n - p \|^2 - (1 - \alpha_n \bar{\gamma})^2 \| u_n - x_n \|^2 \\
+ 2(1 - \alpha_n \bar{\gamma})^2 \delta \| A(u_n - x_n) \| \| (T_{r_n}^{F_2} - I) A x_n \| \\
+ 2 \alpha_n \| \gamma f(x_n) - B p \| \| x_{n+1} - p \| \\
\right. \\
\left. \leq \| x_n - p \|^2 + \alpha_n \bar{\gamma}^2 \| x_n - p \|^2 - (1 - \alpha_n \bar{\gamma})^2 \| u_n - x_n \|^2 \\
+ 2(1 - \alpha_n \bar{\gamma})^2 \delta \| A(u_n - x_n) \| \| (T_{r_n}^{F_2} - I) A x_n \| \\
+ 2 \alpha_n \| \gamma f(x_n) - B p \| \| x_{n+1} - p \|.
\]
Therefore,

\[(1 - \alpha_n \bar{\gamma})^2 \|u - x\|^2 \leq \|x - p\|^2 - \|x_{n+1} - p\|^2 + 2(1 - \alpha_n \bar{\gamma})^2 \delta \|A(u_n - x_n)\|\|(T_{F_n}^2 - I)Ax_n\| + \alpha_n \bar{\gamma}^2 \|x_n - p\|^2 + 2(1 - \alpha_n \bar{\gamma}) \beta_n \|u_n - p\| \|x_n - t_n\| + 2\alpha_n \|f(x_n) - Bp\| \|x_{n+1} - p\| \]

Since \(\alpha_n \to 0\), \(\|x_n - t_n\| \to 0\), \(\|(T_{F_n}^2 - I)Ax_n\| \to 0\) and \(\|x_{n+1} - x_n\| \to 0\) as \(n \to \infty\), we obtain

\[\lim_{n \to \infty} \|u_n - x_n\| = 0. \quad (3.4.18)\]

Since \(u_n = T_{F_n}^{F_1}(x_n + \delta A^*(T_{F_n}^2 - I)Ax_n)\), on setting \(d_n := x_n + \delta A^*(T_{F_n}^2 - I)Ax_n\), we have

\[\|u_n - d_n\| = \|u_n - (x_n + \delta A^*(T_{F_n}^2 - I)Ax_n)\| \leq \|u_n - x_n\| + \delta \|A^*(T_{F_n}^2 - I)Ax_n\| \leq \|u_n - x_n\| + \delta \|A^*\| \|(T_{F_n}^2 - I)Ax_n\|. \quad (3.4.19)\]

It follows from (3.4.16) and (3.4.18) that

\[\lim_{n \to \infty} \|u_n - d_n\| = 0. \quad (3.4.20)\]

Since, we can write

\[\|T(s)t_n - x_n\| \leq \|T(s)t_n - T(s)x_n\| + \|T(s)x_n - x_n\| \leq \|t_n - x_n\| + \|T(s)x_n - x_n\| \to 0 \text{ as } n \to \infty.\]
Also, we have
\[
\|T(s) t_n - t_n \| \leq \|T(s) t_n - T(s) x_n \| + \|T(s) x_n - x_n \| + \|x_n - t_n \|
\]
\[
\leq \| t_n - x_n \| + \|T(s) x_n - x_n \| + \|x_n - t_n \|
\to 0 \text{ as } n \to \infty.
\]

Next, we show that \( \limsup_{n \to \infty} \langle (B - \gamma f) z, t_n - z \rangle \leq 0 \), where \( z = P_T(I - B + \gamma f) z \). To show this inequality, we choose a subsequence \( \{t_{n_i}\} \subseteq \{t_n\} \) such that

\[
\limsup_{n \to \infty} \langle (B - \gamma f) z, t_n - z \rangle = \limsup_{i \to \infty} \langle (B - \gamma f) z, t_{n_i} - z \rangle.
\]

Since \( \{t_{n_i}\} \) is bounded, there exists a subsequence \( \{t_{n_{i_j}}\} \) of \( \{t_{n_i}\} \) which converges weakly to some \( w \in C \). Without loss of generality, we can assume that \( t_{n_i} \rightharpoonup w \).

Now, we prove that \( w \in \Gamma \). Let us first show that \( w \in \text{Fix}(S) \). Assume that \( w \notin \text{Fix}(S) \). Since \( t_{n_i} \rightharpoonup w \) and \( T(s) w \neq w \). From Opial’s condition (1.2.13), we have

\[
\liminf_{i \to \infty} \| t_{n_i} - w \| < \liminf_{i \to \infty} \| t_{n_i} - T(s) w \|
\]
\[
\leq \liminf_{i \to \infty} \{ \| t_{n_i} - T(s) t_{n_i} \| + \|T(s) t_{n_i} - T(s) w \| \}
\]
\[
\leq \liminf_{i \to \infty} \| t_{n_i} - w \|,
\]

which is a contradiction. Thus, we obtain \( w \in \text{Fix}(S) \).

It is proved in Theorem 3.3.1 that \( w \in \text{Sol}(\text{EP}(3.1.5)) \) and \( Aw \in \text{Sol}(\text{EP}(3.1.5)) \) and hence \( w \in \text{Sol}(S_p \text{EP}(3.1.4)-(3.1.5)) \).

Next, we claim that \( \limsup_{n \to \infty} \langle f(z) - z, x_n - z \rangle \leq 0 \), where \( z = P_T(I - B + \gamma f) z \). Now from (1.2.6) and (3.4.13), we have

\[
\limsup_{n \to \infty} \langle (B - \gamma f) z - z, x_n - z \rangle = \limsup_{n \to \infty} \langle (B - \gamma f) z - z, t_n - z \rangle
\]
\[
= \limsup_{i \to \infty} \langle (B - \gamma f) z - z, t_{n_i} - z \rangle
\]
\[
= \langle (B - \gamma f) z - z, w - z \rangle \leq 0. \quad (3.4.21)
\]

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Finally, we show that $x_n \to z$. 

$$\|x_{n+1} - z\|^2 = \|\alpha_n f(x_n) + ((I - \alpha_n B)t_n - z\|^2$$

$$= \|\alpha_n (f(x_n) - Bz) + (I - \alpha_n B)(t_n - z)\|^2$$

$$\leq \|(I - \alpha_n B)(t_n - z)\|^2 + 2\alpha_n \langle f(x_n) - Bz, x_{n+1} - z \rangle$$

$$\leq (1 - \alpha_n \bar{\gamma})^2 \|t_n - z\|^2 + 2\alpha_n \bar{\gamma} \langle f(x_n) - f(z), x_{n+1} - z \rangle$$

$$+ 2\alpha_n \langle f(z) - Bz, x_{n+1} - z \rangle$$

$$\leq (1 - \alpha_n \bar{\gamma})^2 \|x_n - z\|^2 + 2\alpha_n \gamma \alpha \|x_n - z\| \|x_{n+1} - z\|$$

$$+ 2\alpha_n \langle f(z) - Bz, x_{n+1} - z \rangle$$

$$\leq (1 - \alpha_n \bar{\gamma})^2 \|x_n - z\|^2 + \alpha_n \gamma \alpha \|x_n - z\|^2 + \gamma \alpha \|x_{n+1} - z\|^2$$

$$+ 2\alpha_n \langle f(z) - Bz, x_{n+1} - z \rangle.$$

This implies that

$$\|x_{n+1} - z\|^2 \leq \frac{1 - 2\alpha_n \bar{\gamma} + (\alpha_n \bar{\gamma})^2 + \alpha_n \gamma \alpha}{1 - \gamma \alpha} \|x_n - z\|^2$$

$$+ \frac{2\alpha_n}{1 - \gamma \alpha} \langle f(z) - Bz, x_{n+1} - z \rangle$$

$$= \left[1 - \frac{2(\bar{\gamma} - \gamma \alpha)\alpha_n}{1 - \gamma \alpha}\right] \|x_n - z\|^2 + \frac{(\alpha_n \bar{\gamma})^2}{1 - \gamma \alpha} \|x_n - z\|^2$$

$$+ \frac{2\alpha_n}{1 - \gamma \alpha} \langle f(z) - Bz, x_{n+1} - z \rangle$$

$$\leq \left[1 - \frac{2(\bar{\gamma} - \gamma \alpha)\alpha_n}{1 - \gamma \alpha}\right] \|x_n - z\|^2 + \frac{2(\bar{\gamma} - \gamma \alpha)\alpha_n}{1 - \gamma \alpha}$$

$$\times \left\{\frac{(\alpha_n \bar{\gamma})^2 M}{2(\bar{\gamma} - \gamma \alpha)} + \frac{1}{\bar{\gamma} - \gamma \alpha} \langle f(z) - Bz, x_{n+1} - z \rangle\right\}$$

$$= (1 - \delta_n) \|x_n - z\|^2 + \delta_n \sigma_n, \quad (3.4.22)$$

where $M := \sup\{\|x_n - z\|^2 : n \geq 1\}$, $\delta_n = \frac{2(\bar{\gamma} - \gamma \alpha)\alpha_n}{1 - \gamma \alpha}$ and $\sigma_n = \frac{(\alpha_n \bar{\gamma})^2 M}{2(\bar{\gamma} - \gamma \alpha)} + \frac{1}{\bar{\gamma} - \gamma \alpha} \langle f(z) - Bz, x_{n+1} - z \rangle$. Since $\lim_{n \to \infty} \alpha_n = 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$, it is easy to see that

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\[ \lim_{n \to \infty} \delta_n = 0, \sum_{n=0}^{\infty} \delta_n = \infty \text{ and } \limsup_{n \to \infty} \sigma_n \leq 0. \] Hence from (3.4.21), (3.4.22) and Lemma 1.2.11, we deduce that \( x_n \to z \). This completes the proof.

### 3.5 Consequences

As the consequences of Theorem 3.3.1, we have the following strong convergence results for computing the approximate common solution of EP(3.1.4) and FPP for a nonexpansive semigroup \( S \) in real Hilbert spaces.

**Corollary 3.5.1.** [47]. Let \( F_1 : C \times C \to \mathbb{R} \) be a bifunction such that Assumption 2.2.1 hold. Let \( S = \{ T(s) : 0 \leq s < \infty \} \) be a nonexpansive semigroup on \( C \) such that \( \Delta_1 := \text{Fix}(S) \cap \text{Sol}((3.1.4)) \neq \emptyset \). Let \( f : H_1 \to H_1 \) be a contraction mapping with constant \( \alpha \in (0, 1) \) and \( B \) be a strongly positive self-adjoint bounded linear operator on \( H_1 \) with constant \( \bar{\gamma} > 0 \), such that \( 0 < \gamma < \frac{\bar{\gamma}}{\alpha} < \gamma + \frac{1}{\alpha} \). Assume \( (r_t) \) and \( (s_t) \) are the continuous nets of positive real numbers such that \( \liminf_{t \to 0} r_t = r > 0 \) and \( \lim_{t \to 0} s_t = +\infty \). Let the nets \( (u_t) \) and \( (x_t) \) be implicitly generated by

\[
\begin{aligned}
\begin{cases}
u_t &= T_{r_t}F_1r_t, \\
x_t &= t\gamma f(x_t) + (1 - tB)\frac{1}{s_t} \int_{0}^{s_t} T(s)u_t ds.
\end{cases}
\end{aligned}
\]

Then \( x_t \) and \( u_t \) converges strongly to \( z \in \Delta_1 \), where \( z = P_{\Delta_1}(I + \gamma f - B) \), which is the unique solution of the variational inequality

\[
\langle (\gamma f - B)z, x^* - z \rangle \leq 0, \quad \forall x^* \in \text{Fix}(S) \cap \text{Sol}((3.1.4)).
\]

**Proof.** Taking \( H_2 = H_1, A = 0 \) and \( B = I \) in Theorem 3.3.1 then the conclusion of Corollary 3.5.1 is obtained.

**Corollary 3.5.2.** [147]. Let \( S = \{ T(s) : 0 \leq s < \infty \} \) be a nonexpansive semigroup on \( C \) such that \( \text{Fix}(S) \neq \emptyset \). Let \( f : H_1 \to H_1 \) be a contraction mapping with constant \( \alpha \in (0, 1) \). Assume \( (s_t) \) be a continuous net of positive real number such that \( \lim_{t \to 0} s_t = +\infty \).
Let the net \((x_t)\) be implicitly generated by
\[
x_t = tf(x_t) + (1 - t) \frac{1}{st} \int_0^{st} T(s)x_t ds.
\]
Then \(x_t\) converges strongly to \(z \in \text{Fix}(S)\), where \(z = P_{\text{Fix}(S)}f(z)\), which is the unique solution of the variational inequality
\[
\langle (I - f)z, x^* - z \rangle \geq 0, \quad \forall x^* \in \text{Fix}(S).
\]

**Proof.** Taking \(H_2 = H_1, u_t = x_t\) and \(F_1 = F_2 = 0\) in Theorem 3.3.1 then the conclusion of Corollary 3.5.2 is obtained.

We have the following consequences of Theorem 3.4.1.

**Corollary 3.5.3.** [47] Assume that \(F_1 : C \times C \to \mathbb{R}\) be a bifunction satisfying Assumption 2.2.1. Let \(S = \{T(s) : 0 \leq s < \infty\}\) be a nonexpansive semigroup on \(C\) such that \(\Gamma_1 := \text{Fix}(S) \cap \text{Sol}(\text{EP}(3.1.4)) = \emptyset\). Let \(f : H_1 \to H_1\) be a contraction mapping with constant \(\alpha \in (0, 1)\) and \(B\) be a strongly positive self-adjoint bounded linear operator on \(H_1\) with constant \(\bar{\gamma} > 0\) such that \(0 < \gamma < \frac{\gamma}{\alpha} < \gamma + \frac{1}{\alpha}\). Let \(\{s_n\}\) is a positive real sequence which diverges to \(+\infty\). For a given \(x_0 \in C\) arbitrarily, let the iterative sequences \(\{u_n\}\) and \(\{x_n\}\) be generated by
\[
\begin{align*}
\left\{
\begin{array}{l}
u_n = TF_{r_n}x_n, \\
x_{n+1} = \alpha_n \gamma f(x_n) + (1 - \alpha_n B) \frac{1}{s_n} \int_0^{s_n} T(s)u_n ds,
\end{array}\right.
\end{align*}
\]
where \(r_n \subset (0, \infty)\) and \(\{\alpha_n\}\) is a sequences in \((0, 1)\) satisfying
\[
(i) \lim_{n \to \infty} \alpha_n = 0 \text{ and } \sum_{n=0}^{\infty} \alpha_n = \infty;
\]
\[
(ii) \lim \inf_{n \to \infty} r_n > 0, \sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty;
\]
\[
(iii) \lim_{n \to \infty} \frac{|s_{n+1} - s_n|}{s_{n+1}} = 0.
\]
Then the sequence \(\{x_n\}\) converges strongly to \(z \in \Gamma_1\), where \(z = P_{\Gamma_1}(I - B + \gamma f)z\).
Proof. Taking $F_2 = 0$, $H_2 = H_1$ and $A = 0$ in Theorem 3.4.1 then the conclusion of Corollary 3.5.3 is obtained.

**Corollary 3.5.4.** [108] Let $S = \{T(s) : 0 \leq s < \infty\}$ be a nonexpansive semigroup on $C$ such that $\Gamma_2 := \text{Fix}(S) \neq \emptyset$. Let $f : H_1 \to H_1$ be a contraction mapping with constant $\alpha \in (0, 1)$ and $B$ be a strongly positive self-adjoint bounded linear operator on $H_1$ with constant $\gamma > 0$ such that $0 < \gamma < \frac{\bar{\gamma}}{\alpha} < \gamma + \frac{1}{\alpha}$. Let $\{s_n\}$ is a positive real sequence which diverges to $+\infty$. For a given $x_0 \in C$ arbitrarily, let the iterative sequence $\{x_n\}$ be generated by

$$x_{n+1} = \alpha_n \gamma f(x_n) + (1 - \alpha_n B) \frac{1}{s_n} \int_0^{s_n} T(s)x_n ds,$$

where $\{\alpha_n\}$ be a sequence in $(0, 1)$ satisfying the following conditions:

1. $\lim_{n \to \infty} \alpha_n = 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$;
2. $\lim_{n \to \infty} \frac{|s_{n+1} - s_n|}{s_{n+1}} = 0$.

Then the sequence $\{x_n\}$ converges strongly to $z \in \Gamma_2$, where $z = P_{\Gamma_2} f(z)$.

Proof. Taking $H_2 = H_1$, $u_n = x_n$, and $F_1 = F_2 = 0$ in Theorem 3.4.1 then the conclusion of Corollary 3.5.4 is obtained.

**Corollary 3.5.5.** Let $A : H_1 \to H_2$ be a bounded linear operator. Assume that $F_1 : C \times C \to \mathbb{R}$ and $F_2 : Q \times Q \to \mathbb{R}$ are the bifunctions satisfying Assumption 2.2.1 and $F_2$ is upper semicontinuous in first argument. Let $T$ be a nonexpansive mapping on $H$ such that $\Gamma_3 := \text{Fix}(T) \cap \text{Sol}(S_PEP(3.1.4)-(3.1.5)) \neq \emptyset$. Let $f : H_1 \to H_1$ be a contraction mapping with constant $\alpha \in (0, 1)$ and $B$ be a strongly positive self-adjoint bounded linear operator on $H_1$ with constant $\bar{\gamma} > 0$ such that $0 < \gamma < \frac{\bar{\gamma}}{\alpha} < \gamma + \frac{1}{\alpha}$. For a given $x_0 \in C$ arbitrarily, let the iterative sequences $\{u_n\}$ and $\{x_n\}$ be generated by

$$\begin{cases} u_n = T_{r_n}^{F_1}(x_n + \delta A^* (T_{r_n}^{F_2} - I)Ax_n), \\ x_{n+1} = \alpha_n \gamma f(x_n) + (1 - \alpha_n B)Tu_n, \end{cases}$$
where \( r_n \subset (0, \infty) \) and \( \delta \in (0, 1/L) \), \( L \) is the spectral radius of the operator \( A^*A \) and \( A^* \) is the adjoint of \( A \) and \( \{\alpha_n\} \) is a sequence in \((0, 1)\) satisfying the following conditions:

(i) \( \lim_{n \to \infty} \alpha_n = 0 \) and \( \sum_{n=0}^{\infty} \alpha_n = \infty \); and \( \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty \);

(ii) \( \liminf_{n \to \infty} r_n > 0 \), \( \sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty \).

Then the sequence \( \{x_n\} \) converges strongly to \( z \in \Gamma_3 \), where \( z = P_{\Gamma_3}(I - B + \gamma f)z \).

Proof. Taking \( T(s) := T \) for all \( s > 0 \), a nonexpansive mapping in Theorem 3.4.1 then the conclusion of Corollary 3.5.5 is obtained. \( \square \)

### 3.6 Numerical example

Now, we give a numerical example which justify Theorem 3.3.1 based on implicit iterative method.

**Example 3.6.1.** Let \( H_1 = H_2 = \mathbb{R} \) with the inner product defined by \( \langle x, y \rangle = xy \), \( \forall x, y \in \mathbb{R} \), and induced usual norm \( |.| \). Let \( C = [0, +\infty) \) and \( Q = (-\infty, 0] \); let \( F_1 : C \times C \to \mathbb{R} \) and \( F_2 : Q \times Q \to \mathbb{R} \) be defined by \( F_1(x, y) = (x - 2)(y - x) \), \( \forall x, y \in C \) and \( F_2(u, v) = (u + 4)(v - u) \), \( \forall u, v \in Q \); let for each \( x \in \mathbb{R} \), we define \( f(x) = \frac{1}{8}x \), \( A(x) = -2x \), \( B(x) = 2x \), and let, for each \( x \in C \), \( T(x) = x \). Let \( \{t_n\} \) be a sequence in \((0, 1)\) such that \( t_n \to 0 \) as \( n \to \infty \). Setting \( x_n := x_{t_n}, u_n := u_{t_n}, z_n := z_{t_n}, r_n := r_{t_n} = 1 \).

Then there exist unique sequences \( \{x_n\} \subset \mathbb{R}, \{u_n\} \subset C, \) and \( \{z_n\} \subset Q \) generated by the iterative schemes

\[
\begin{align*}
z_n &= T_{r_n}^{F_2}(Ax_n); \\
\quad u_n &= T_{r_n}^{F_1}\left[x_n + \frac{1}{8}A^*(z_n - Ax_n)\right]; \\
x_n &= \frac{1}{n+2} (2) \left(\frac{1}{8}x_n\right) + \left(I - \frac{1}{n+2}B\right)Tu_n, 
\end{align*}
\]

where \( t_n = \frac{1}{n+2} \) and \( r_n = 1 \). Then \( \{x_n\} \) converges strongly to \( 2 \in \text{Fix}(T) \cap \Omega \).

Proof. It is easy to prove that the bifunctions \( F_1 \) and \( F_2 \) satisfy the Assumption 2.2.1 and \( F_2 \) is upper semicontinuous. \( A \) is a bounded linear operator on \( \mathbb{R} \) with adjoint operator \( A^* \) and \( \|A\| = \|A^*\| = 2 \). Hence \( \delta \in (0, \frac{1}{4}) \), so we can choose \( \delta = \frac{1}{8} \). Further,
\( f \) is contraction mapping with constant \( \alpha = \frac{1}{5} \) and \( B \) is a strongly positive self-adjoint bounded linear operator with constant \( \bar{\gamma} = 1 \) on \( \mathbb{R} \). Therefore, we can choose \( \gamma = 2 \) which satisfies \( 0 < \gamma < \frac{\bar{\gamma}}{\alpha} < \gamma + \frac{1}{\alpha} \).

Furthermore, it is easy to observe that \( \text{Fix}(T) = (0, \infty), \quad \text{Sol}(\text{EP}(3.1.4)) = \{2\}, \quad \text{Sol}(\text{EP}(3.1.5)) = \{-4\}. \) Hence, \( \text{Sol}(S_p \text{EP}(3.1.4)-(3.1.5)) := \{ p \in \text{Sol}(\text{EP}(3.1.4)) : Ap \in \text{Sol}(\text{EP}(3.1.5)) \} = \{2\}. \) Consequently, \( \text{Fix}(T) \cap \text{Sol}(S_p \text{EP}(3.1.4)-(3.1.5)) = \{2\} \neq \emptyset. \)

After simplification, schemes (3.6.1)-(3.6.2) reduce to

\[
z_n = -(x_n + 2); \quad u_n = \frac{1}{8}(3x_n + 10); \quad (3.6.3)
\]

\[
x_n = \frac{1}{4(n + 2)} x_n + \left(1 - \frac{2}{n + 2}\right) u_n, \quad (3.6.4)
\]

which reduce to the following scheme:

\[
x_n = \frac{5}{2} \left[ \frac{1}{2} - \frac{1}{n + 2} \right].
\]

Following the proof of Theorem 3.3.1, we obtain that \( \{z_n\} \) converges strongly to \(-4 \in \text{Sol}(\text{EP}(3.1.5)) \) and \( \{x_n\}, \{u_n\} \) converge strongly to \( w = 2 \in \text{Fix}(T) \cap \text{Sol}(S_p \text{EP}(3.1.4)-(3.1.5)) \) as \( n \to \infty. \)

Next, using the software Matlab 7.0, we have Figure 3.6.1 which shows that \( \{x_n\} \) converges strongly to 2.

\[\square\]

![Figure 3.6.1: Convergence \( \{x_n\} \) using implicit iterative method](image)

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Now, we justify Theorem 3.4.1 based on explicit iterative method.

**Example 3.6.2.** Let $H_1, H_2, C, Q, F_1, F_2, f, A, B, T$ be same as in Example 3.6.1. Then there exist unique sequences $\{x_n\} \subset \mathbb{R}$, $\{u_n\} \subset C$, and $\{z_n\} \subset Q$ generated by the explicit iterative schemes

\[
z_n = T_{r_n}^{F_2}(Ax_n); \quad u_n = T_{r_n}^{F_1}\left[ x_n + \frac{1}{8}A^*(z_n - Ax_n) \right]; \quad (3.6.5)
\]

\[
x_{n+1} = \frac{1}{4(n+2)}x_n + \left( I - \frac{1}{n+2}B \right) Tu_n, \quad (3.6.6)
\]

where $\alpha_n = \frac{1}{n+2}$ and $r_n = 1$, $n \in N$. Then \( \{x_n\} \) converges strongly to $2 \in \text{Fix}(T) \cap \text{Sol}(\text{EP}(3.14)-(3.15))$.

**Proof.** As discussed in Example 3.6.1, we have $\text{Fix}(T) \cap \text{Sol}(\text{EP}(3.14)-(3.15)) = \{2\} \neq \emptyset$. After simplification, schemes (3.6.5)-(3.6.6) reduce to

\[
z_n = -(x_n + 2); \quad u_n = \frac{1}{8}(3x_n + 10); \quad (3.6.7)
\]

\[
x_{n+1} = \frac{1}{4}\left[ \frac{1}{(n+2)}x_n + \frac{3}{2}x_n - \frac{3}{(n+2)}x_n + 5\left(1 - \frac{2}{n+2}\right) \right], \quad (3.6.8)
\]

which reduce to the following scheme:

\[
x_{n+1} = \frac{1}{4}\left[ \left( \frac{3}{2} - \frac{2}{n+2} \right)x_n + 5\left(1 - \frac{2}{n+2}\right) \right].
\]

Following the proof of Theorem 3.4.1, we obtain that $\{x_n\}$, $\{u_n\}$ converge strongly to $w = 2 \in \text{Fix}(T) \cap \text{Sol}(\text{EP}(3.14)-(3.15))$, and $\{z_n\}$ converges strongly to $-4 \in \text{Sol}(\text{EP}(3.15))$ as $n \to \infty$.

Next, using the software Matlab 7.0, we have Figure 3.6.2 which shows that $\{x_n\}$ converges strongly to 2.
Figure 3.6.2: Convergence of $\{x_n\}$ using explicit iterative method

This completes the proof. 

\[ \square \]