Chapter 1

Preliminaries

1.1 Introduction

Recently, the theory of variational inequality and equilibrium problems have emerged as an interesting branch of applicable mathematics and become a rich source of inspiration and motivation for the study of a large number of problems arising in economics, optimization, and operations research in a general and unified way.

Variational inequality theory was initiated independently by Fichera [65] and Stampacchia [158] in the early 1960’s to study the problems in the elasticity and potential theory, respectively. The first general theorem for the existence and uniqueness of solution of variational inequality was proved by Lions and Stampacchia [109] in 1967. Since then, from theoretical and practical point of view, the variational inequality problems have a great importance. It is well-known that the variational inequality theory has played a fundamental and important role in the study of a wide range of problems arising in physics, mechanics, elasticity, optimization, control theory, management science, operations research, economics, transportation and other branches of mathematical and engineering sciences, see for example Baiocchi and Capelo [9], Barbu [10], Bensoussan [14], Bensoussan and Lions [16], Brézis [19], Cottle et al. [54], Crank [55], Duvaut and Lions [60], Ekland and Temam [61], Giannessi and Maugeri [69], Glowinski [70], Glowinski et al. [71], Hlaváček et al. [80], Kikuchi and Oden [101], Kinderlehrer and Stampacchia [102], Mosco [119], Nagurney [130], Nečas et al. [138], Panagiotopoulos [143] and the references cited therein.
Equilibrium problems which were initially introduced by Zuhovickii, Poljak and Pri-mak [179], Fan [62,63], perhaps motivated by minimax problems appearing in economic equilibrium. A more general result than that in [63] was established by Brézis, Nirenberg and Stampacchia [20]. But, in 1994, the terminology of equilibrium problem was adopted by Blum and Oettli [18]. They discussed existence theorems and variational principle for equilibrium problems. Since then various generalizations of equilibrium problems considered by Blum and Oettli [18] have been introduced and studied by many authors. It is known that the equilibrium problem has a great impact and influence in the development of several topics of science and engineering. It turned out that the theories of many well known problems could be fitted into the theory of equilibrium problems. It has been shown that the theory of equilibrium problem provides a natural, novel and unified framework for several problems arising in nonlinear analysis, optimization, economics, finance, game theory, physics and engineering. The equilibrium problem includes many mathematical problems as particular cases for examples, mathematical programming problems, complementary problems, variational inequality problems, saddle point problems, Nash equilibrium problems in noncooperative games, minimax inequality problems, minimization problems and fixed point problems, see [18,58,68,91,99,112,122].

The remaining part of this chapter is organized as follows:

In Section 1.2, we review various notations, known definitions and results which are essential for the presentation of the results in subsequent chapters.

In Section 1.3, we give brief survey of some classes of variational inequalities and equilibrium problems. Further, we give brief survey of some iterative methods for solving fixed point problems, variational inequalities and equilibrium problems.

1.2 Some tools of nonlinear functional analysis

Throughout the thesis unless otherwise stated, $H$ denotes a real Hilbert space; $H^*$ denotes the topological dual of $H$, we denote the norm and inner product of $H$ by $\| \cdot \|$, and $\langle \cdot, \cdot \rangle$ respectively. Let $C$ be a nonempty, closed and convex subset of $H$. Let $\{x_n\}$ be any sequence in $H$, then $x_n \to x$ (respectively, $x_n \rightharpoonup x$) will denote strong (respectively,
weak) convergence of the sequence \( \{x_n\} \). \( \mathbb{R} \) denotes the set of all real numbers.

**Definition 1.2.1.** [154] Let \( T : C \to C \) be a mapping. A point \( x_0 \) is called a fixed point of \( T \), if \( Tx_0 = x_0 \), for all \( x_0 \in C \), i.e., a point which remains invariant under the transformation \( T \).

The fixed point problem (in short, FPP) for the mapping \( T \) is to find \( x \in C \) such that

\[
x = Tx.
\]

(1.2.1)

We denote \( \text{Fix}(T) \), the set of solutions of FPP(1.2.1).

**Definition 1.2.2.** [44] Let \( X \) be normed linear space. A mapping \( T : X \to X \) is said to be:

(i) continuous at a arbitrary point \( x_0 \in X \), if for each \( \epsilon > 0 \) there is real number \( \delta > 0 \) such that

\[
x \in X, \|x - x_0\| < \delta \Rightarrow \|T(x) - T(x_0)\| \leq \epsilon, \quad \forall \ x_0 \in X;
\]

(ii) Lipschitz continuous if there exists a real constant \( k > 0 \) such that

\[
\|T(x) - T(y)\| \leq k\|x - y\| \quad \forall \ x, y \in X;
\]

(iii) contraction if it is Lipschitz continuous with \( k \in (0, 1) \);

(iv) nonexpansive if it is Lipschitz continuous with \( k = 1 \).

**Theorem 1.2.1.** [154] (Banach Contraction Theorem) Let \( X \) be a complete normed linear space and \( T : X \to X \) be a contraction mapping on \( X \). Then FPP(1.2.1) has a unique solution in \( X \).

**Remark 1.2.1.** It is well known that every nonexpansive operator \( T : H \to H \) satisfies, for all \( (x, y) \in H \times H \), the inequality

\[
\langle (x - T(x)) - (y - T(y)), T(y) - T(x) \rangle \leq (1/2)\|T(x) - x - (T(y) - y)\|^2 \tag{1.2.2}
\]
and therefore, we get, for all $(x,y) \in H \times \text{Fix}(T)$,

$$
\langle x - T(x), y - T(x) \rangle \leq (1/2)\|T(x) - x\|^2, \quad (1.2.3)
$$

see, e.g., [56], Theorem 2.1 and [57], Theorem 3.1.

**Definition 1.2.3.** [13] Let $T : H \to H$ be a nonlinear mapping. Then $T$ is called:

(i) monotone, if

$$
\langle Tx - Ty, x - y \rangle \geq 0, \quad \forall x, y \in C;
$$

(ii) $\alpha$-strongly monotone, if there exists a constant $\alpha > 0$ such that

$$
\langle Tx - Ty, x - y \rangle \geq \alpha\|x - y\|^2, \quad \forall x, y \in H;
$$

(iii) $\beta$-inverse strongly monotone, if there exists a constant $\beta > 0$ such that

$$
\langle Tx - Ty, x - y \rangle \geq \beta\|Tx - Ty\|^2, \quad \forall x, y \in H;
$$

(iv) firmly nonexpansive, if it is $\beta$-inverse strongly monotone with $\beta = 1$.

It is easy to observe that every $\beta$-inverse strongly monotone mapping $T$ is monotone and $\frac{1}{\beta}$-Lipschitz continuous.

**Definition 1.2.4.** [13] For every point $x \in H$, there exists a unique nearest point in $C$ denoted by $P_C x$ such that

$$
\|x - P_C x\| \leq \|x - y\|, \quad \forall y \in C, \quad (1.2.4)
$$

where $P_C$ is called the metric projection of $H$ onto $C$.

**Remark 1.2.2.** [13] It is well known that $P_C$ is nonexpansive mapping and satisfies

$$
\langle x - y, P_C x - P_C y \rangle \geq \|P_C x - P_C y\|^2, \quad \forall x, y \in H. \quad (1.2.5)
$$
Moreover, \( P_C x \) is characterized by the fact \( P_C x \in C \) and

\[
\langle x - P_C x, y - P_C x \rangle \leq 0,
\]

(1.2.6)

and

\[
\|x - y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2, \quad \forall x \in H, \ y \in C.
\]

(1.2.7)

**Definition 1.2.5.** [13] A multi-valued mapping \( M : H \to 2^H \) is called monotone if for all \( x, y \in H \), \( u \in M x \) and \( v \in M y \) such that

\[
\langle x - y, u - v \rangle \geq 0.
\]

**Definition 1.2.6.** [13] A multi-valued monotone mapping \( M : H \to 2^H \) is maximal if the Graph(\( M \)), the graph of \( M \), is not properly contained in the graph of any other monotone mapping.

It is known that a multi-valued monotone mapping \( M \) is maximal if and only if for \( (x, u) \in H \times H \), \( \langle x - y, u - v \rangle \geq 0 \), for every \( (y, v) \in \text{Graph}(M) \) implies that \( u \in M x \).

**Definition 1.2.7.** [13] Let \( M : H \to 2^H \) be a multi-valued maximal monotone mapping. Then, the resolvent mapping \( J^M_\lambda : H \to H \) associated with \( M \), is defined by

\[
J^M_\lambda(x) := (I + \lambda M)^{-1}(x), \quad \forall x \in H,
\]

for some \( \lambda > 0 \), where \( I \) stands identity operator on \( H \).

**Remark 1.2.3.** [13] We note that for all \( \lambda > 0 \), the resolvent operator \( J^M_\lambda \) is single-valued, nonexpansive and firmly nonexpansive.

**Lemma 1.2.1.** [72] (Demiclosedness Principle) Assume that \( T \) is nonexpansive self mapping of a nonempty, closed and convex subset \( C \) of a Hilbert space \( H \). If \( T \) has a fixed point, then \( I - T \) is demiclosed, i.e., whenever \( \{x_n\} \) is a sequence in \( C \) converging weakly to some \( x \in C \) and the sequence \( \{(I - T)x_n\} \) converges strongly to some \( y \), it follows that \( (I - T)x = y \). Here \( I \) is the identity mapping on \( H \).
Definition 1.2.8. [175] A mapping $T : C \to C$ is said to be $k$-strict pseudocontractive, if there exists a constant $0 \leq k < 1$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in C.$$  

Lemma 1.2.2. [175] Let $T : C \to C$ be a $k$-strictly pseudocontractive mapping. Let $\gamma$ and $\delta$ be two positive real numbers. Assume that $(\gamma + \delta)k \leq \gamma$. Then

$$\|\gamma(x - y) + \delta(Tx - Ty)\| \leq (\gamma + \delta)\|x - y\|. \quad (1.2.8)$$

Lemma 1.2.3. [117] Let $T : C \to C$ be a $k$-strict pseudocontractive mapping. Then:

(i) $T$ satisfies the Lipschitz condition

$$\|Tx - Ty\| \leq \frac{1 + k}{1 - k}\|x - y\|, \quad \forall x, y \in C; \quad (1.2.9)$$

(ii) The mapping $I - T$ is demiclosed at 0;

(iii) The set $\text{Fix}(T)$ of $T$ is closed and convex so that the projection $P_{\text{Fix}(T)}$ is well defined.

Definition 1.2.9. [13] A mapping $T : H \to H$ is said to be averaged if and only if it can be written as the average of the identity mapping and a nonexpansive mapping, i.e.,

$$T = (1 - \alpha)I + \alpha S$$

where $\alpha \in (0, 1)$ and $S : H \to H$ is nonexpansive and $I$ is the identity operator on $H$.

We note that the firmly nonexpansive mappings (in particular, projections on nonempty, closed and convex subsets and resolvent operators of maximal monotone operators) are averaged. Obviously, averaged mapping is a nonexpansive mapping.

The following are some key properties of averaged mappings, see for instance [24, 25, 111, 126, 150].
Proposition 1.2.1.  

(i) If \( T = (1 - \alpha)S + \alpha V \), where \( S : H \to H \) is averaged, 
\( V : H \to H \) is nonexpansive and \( \alpha \in (0, 1) \), then \( T \) is averaged; 

(ii) The composite of finitely many averaged mappings is averaged; 

(iii) If the mappings \( \{T_i\}_{i=1}^N \) are averaged and have a nonempty common fixed point, 
then 
\[ \bigcap_{i=1}^N \text{Fix}(T_i) = \text{Fix}(T_1T_2...T_N); \]

(iv) If \( T \) is \( \tau \)-ism, then for \( \gamma > 0 \), \( \gamma T \) is \( \frac{\tau}{\gamma} \)-ism; 

(v) \( T \) is averaged if and only if, its complement \( I - T \) is \( \tau \)-ism for some \( \tau > \frac{1}{2} \). 

Definition 1.2.10. A family \( S := \{T(s) : 0 \leq s < \infty\} \) of mappings from \( C \) into itself 
is called nonexpansive semigroup on \( C \) if it satisfies the following conditions: 

(i) \( T(0)x = x \) for all \( x \in C \); 

(ii) \( T(s+t) = T(s)T(t) \) for all \( s, t \geq 0 \); 

(iii) \( \|T(s)x - T(s)y\| \leq \|x - y\| \) for all \( x, y \in C \) and \( s \geq 0 \); 

(iv) for all \( x \in C \), \( s \mapsto T(s)x \) is continuous.

The set of all the common fixed points of a family \( S \) is denoted by \( \text{Fix}(S) \), i.e., 

\[ \text{Fix}(S) := \{x \in C : T(s)x = x, 0 \leq s < \infty\} = \bigcap_{0 \leq s < \infty} \text{Fix}(T(s)), \quad (1.2.10) \]

where \( \text{Fix}(T(s)) \) is the set of fixed points of \( T(s) \). It is well known that \( \text{Fix}(S) \) is closed 
and convex.

Lemma 1.2.4. [152] Let \( C \) be a nonempty, bounded, closed and convex subset of a 
Hilbert space \( H \) and let \( S := \{T(s) : 0 \leq s < \infty\} \) be a nonexpansive semigroup on \( C \). 
Then for \( t > 0 \) and for every \( 0 \leq h < \infty \), 

\[ \lim_{t \to \infty} \sup_{x \in C} \left\| \frac{1}{t} \int_0^t T(s)xds - T(h) \left( \frac{1}{t} \int_0^t T(s)xds \right) \right\| = 0. \]
Definition 1.2.11. [116] An operator $B : H \to H$ is said to be strongly positive bounded linear operator, if there exists a constant $\bar{\gamma} > 0$ such that
\[
\langle Bx, x \rangle \geq \bar{\gamma} \|x\|^2, \quad \forall x \in H.
\]

Lemma 1.2.5. [116] Assume that $B$ is a strongly positive self-adjoint bounded linear operator on a Hilbert space $H$ with constant $\bar{\gamma} > 0$ and $0 < \rho \leq \|B\|^{-1}$. Then $\|I - \rho B\| \leq 1 - \rho \bar{\gamma}$.

Lemma 1.2.6. [116] Let $C$ be a nonempty, closed and convex subset of a real Hilbert space $H$, let $f : H \to H$ be an $\alpha$-contraction mapping and let $B$ be a strongly positive self-adjoint bounded linear operator with constant $\bar{\gamma}$. Then for every $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$, $(B - \gamma f)$ is strongly monotone with constant $(\bar{\gamma} - \gamma \alpha)$, i.e.,
\[
\langle x - y, (B - \gamma f)x - (B - \gamma f)y \rangle \geq (\bar{\gamma} - \gamma \alpha)\|x - y\|^2.
\]

Definition 1.2.12. [13] Let $C$ be a nonempty subset of a Hilbert space $H$ and let $\{x_n\}$ be a sequence in $H$. Then $\{x_n\}$ is Fejer monotone with respect to $C$ if
\[
\|x_{n+1} - x\| \leq \|x_n - x\|, \quad \forall x \in C.
\]

Lemma 1.2.7. [13, 72, 140] In real Hilbert space $H$, the following hold:

(i)
\[
\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle, \quad \forall x, y \in H; \quad (1.2.11)
\]

(ii)
\[
\|\lambda x + (1 - \lambda)y\|^2 = \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|^2, \quad (1.2.12)
\]

for all $x, y \in H$ and $\lambda \in (0, 1)$;

(iii) (Opial’s condition) For any sequence $\{x_n\}$ with $x_n \rightharpoonup x$ the inequality
\[
\lim_{n \to \infty} \inf \|x_n - x\| < \lim_{n \to \infty} \inf \|x_n - y\| \quad (1.2.13)
\]
holds for every $y \in H$ with $y \neq x$;

(iv) (Kadec-Klee property) If $\{x^n\}$ be a sequence in $H$ which satisfies $x^n \rightharpoonup x$ and $\|x^n\| \to \|x\|$ as $n \to \infty$, then $\|x^n - x\| \to 0$ as $n \to \infty$.

**Lemma 1.2.8.** [159] (Suzuki Lemma) Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space $X$ and $\{\beta_n\}$ be a sequence in $[0, 1]$ with $0 < \lim \inf_{n \to \infty} \beta_n \leq \lim \sup_{n \to \infty} \beta_n < 1$. Suppose $x_{n+1} = (1 - \beta_n)y_n + \beta_n x_n$, for all integers $n \geq 0$ and $\lim sup_{n \to \infty} (\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then $\lim_{n \to \infty} \|y_n - x_n\| = 0$.

**Lemma 1.2.9.** [142] Let $(E, \langle \cdot, \cdot \rangle)$ be an inner product space, then for all $x, y \in E$ and $\alpha, \beta, \gamma \in [0, 1]$ with $\alpha + \beta + \gamma = 1$, we have

$$\|\alpha x + \beta y + \gamma z\|^2 = \alpha \|x\|^2 + \beta \|y\|^2 + \gamma \|z\|^2 - \alpha \beta \|x - y\|^2 - \alpha \gamma \|x - z\|^2 - \beta \gamma \|y - z\|^2.$$ 

**Definition 1.2.13.** [154] Let $C$ be a nonempty and convex subset of $H$. A functional $f : C \to \mathbb{R}$ is said to be

(i) convex, if for any $x, y \in C$ and $0 \leq \alpha \leq 1$,

$$f(\alpha x + (1 - \alpha)y) \leq \alpha f(x) + (1 - \alpha)f(y);$$

(ii) lower semicontinuous on $C$, if for every $\alpha \in \mathbb{R}$, the set

$$\{x \in C : f(x) \leq \alpha\}$$

is closed in $C$;

(iii) concave, if $-f$ is convex;

(iv) upper semicontinuous on $C$, if $-f$ is lower semicontinuous on $C$.

**Definition 1.2.14.** [154] Let $C$ be a subset of a Banach space $B$, then the mapping $T : C \to B^*$ is hemicontinuous, if for any $x \in C$, $y \in B$ and any sequence $\{t_n\} \in \mathbb{R}^+$, $T(x + t_ny) \rightharpoonup Tx$ as $t_n \to 0$ and $n \to \infty$.

The following result is a special case of Theorem 3.9.3 of Chang [41].

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Lemma 1.2.10. [113] Let \( C \) be a closed and convex subset of a Hausdorff topological vector space \( E \), \( F : C \times C \to \mathbb{R} \) be a bifunction. Assume that the following conditions hold:

(i) \( F(x, x) \geq 0, \ \forall x \in C; \)

(ii) \( F \) is monotone, i.e., \( F(x, y) + F(y, x) \leq 0, \ \forall x \in C; \)

(iii) For each \( y \in C \) fixed, the function \( x \to F(x, y) \) is upper-hemicontinuous, i.e.,

\[
\limsup_{t \to 0} F(tz + (1-t)x, y) \leq F(x, y), \ \forall x, y, z \in C, \ t \in [0, 1];
\]

(iv) For each \( x \in C \) fixed, the function \( y \to F(x, y) \) is convex and lower semicontinuous;

(v) There exists a compact subset \( D \) of \( E \) and \( y_0 \in C \cap D \) such that \( F(x, y_0) < 0, \ \forall y \in C \setminus D. \)

Then the set \( \{ x^* \in C : F(x^*, y) \geq 0, \ \forall y \in C \} \) is nonempty, convex and compact.

Definition 1.2.15. A bifunction \( \phi : H \times H \to \mathbb{R} \) is said to be skew-symmetric if

\[
\phi(x, x) - \phi(x, y) - \phi(y, x) + \phi(y, y) \geq 0, \ \forall x, y \in H.
\]

The skew symmetric bifunctions have the properties which can be considered an analog of monotonicity of gradient and nonnegativity of second derivative for the convex function. For properties and applications of the skew symmetric bifunction, we refer to see [7].

Lemma 1.2.11. [171] Let \( \{a_n\} \) be a sequence of nonnegative real numbers such that

\[
a_{n+1} \leq (1 - \alpha_n)a_n + \delta_n, \ \ n \geq 0,
\]

where \( \{a_n\} \) is a sequence in \((0, 1)\) and \( \{\delta_n\} \) is a sequence in \( \mathbb{R} \) such that

(i) \( \sum_{n=1}^{\infty} \alpha_n = \infty; \)
(ii) \( \limsup_{n \to \infty} \frac{\delta_n}{\alpha_n} \leq 0 \) or \( \sum_{n=1}^{\infty} |\delta_n| < \infty \).

Then \( \lim_{n \to \infty} a_n = 0 \).

1.3 Variational inequalities, equilibrium problems and iterative methods

In this section, we give brief survey of some classes of variational inequalities and equilibrium problems. Further, we give brief survey of some iterative methods for solving fixed point problems, variational inequalities and equilibrium problems.

1.3.1 Variational inequalities

Let \( a(\cdot, \cdot) : H \times H \to \mathbb{R} \) be a bilinear form.

**Problem 1.3.1.** For given \( f \in H^* \), find \( x \in C \) such that

\[
a(x, y - x) \geq \langle f, y - x \rangle, \quad \forall y \in C.
\]

The inequality (1.3.1) is termed as variational inequality which characterizes the classical Signorini problem of elasto-statistics, that is, the analysis of a linear elastic body in contact with a rigid frictionless foundation. This problem was investigated and studied by Lions and Stampacchia [109] by using the projection technique.

If the bilinear form is continuous, then by Riesz-Fréchet theorem, we have

\[
a(x, y) = \langle A(x), y \rangle, \quad \forall x, y \in H,
\]

where \( A : H \to H^* \) is a continuous linear operator. Then Problem 1.3.1 is equivalent to the following problem:

**Problem 1.3.2.** Find \( x \in C \) such that

\[
\langle A(x), y - x \rangle \geq \langle f, y - x \rangle, \quad \forall y \in C.
\]
If \( f \equiv 0 \in H^* \), then (1.3.2) reduces to the following classical variational inequality problem introduced by Hartmann and Stampacchia [76].

**Problem 1.3.3.** Find \( x \in C \) such that

\[
\langle A(x), y - x \rangle \geq 0, \ \forall y \in C.
\] (1.3.3)

The solution set of variational inequality problem VIP(1.3.3) is denoted by \( \text{Sol(VIP (1.3.3))} \). In the variational inequality formulation, the underlying convex set \( C \) does not depend upon the solution. In many important applications, the convex set \( C \) also depends implicitly on the solution. In this case, VIP(1.3.3) is known as *quasi-variational inequality* which arises in the study of impulse control theory and decision science, see for example [14, 16]. Quasi-variational inequality was introduced and studied by Bensoussan, Goursat and Lions [15]. To be more precise, given a multi-valued mapping \( C : x \rightarrow C(x) \), which associates a nonempty, closed and convex subset \( C(x) \) of \( H \) for each \( x \in H \), a typical quasi-variational inequality problem is:

**Problem 1.3.4.** Find \( x \in C(x) \) such that

\[
a(x, y - x) \geq \langle f, y - x \rangle, \ \forall y \in C(x).
\] (1.3.4)

In many important applications, see for example Baiocchi and Capelo [9], Bensoussan and Lions [16] and Mosco [119], the underlying set \( C(x) \) is of the following form:

\[
C(x) = K + m(x),
\]

where \( m : H \rightarrow H \) is a nonlinear mapping and \( K \) is a nonempty, closed and convex subset in \( H \). Note that if the \( m \) is a zero mapping, then Problem 1.3.4 is same as Problem 1.3.1.

Further, since the general problem of equilibrium of elastic bodies in contact with rigid foundation on which frictional forces are developed is one of the most difficult problems in solid mechanics. Duvaut and Lions [60] investigated the following variational
inequality problem with friction:

**Problem 1.3.5.** For $f \in H^*$, find $x \in C$ such that

$$a(x, y - x) + \psi(y) - \psi(x) \geq \langle f, y - x \rangle, \quad \forall y \in C,$$

(1.3.5)

where $\psi : H \to \mathbb{R} \cup \{+\infty\}$ is a proper, convex and lower semicontinuous functional. Problem 1.3.5 characterizes the *classical Signorini problem with frictional force*. The existence of its solution has been proved by Glowinski *et al.* [71] and Nečas *et al.* [138].

The complete study of boundary value problem arising in the formulation of Signorini problem with friction is an interesting problem both in mechanics and mathematical points of view. A generalization of the Problem 1.3.5 is the following:

**Problem 1.3.6.** Given $f \in H^*$, find $x \in C$ such that

$$a(x, y - x) + \phi(x, y) - \phi(x, x) \geq \langle f, y - x \rangle, \quad \forall y \in C,$$

(1.3.6)

where $\phi : H \times H \to \mathbb{R}$ is an appropriate nonlinear form. This type of problems have been studied in Duvaut and Lions [60], Kikuchi and Oden [101]. The Problem 1.3.6 characterizes the *fluid flow through porous media and Signorini problems with non-local frictions*. For physical and mathematical formulation of the inequality (1.3.6), see for example Oden and Pires [139]. For related work, see also Baiocchi and Capelo [9] and Crank [55].

Since then various generalizations of the above mentioned variational inequalities have been introduced and studied by a number of authors. Some of them are given below:

In 1975, Noor [134] extended the Problem 1.3.2 to study a class of mildly nonlinear elliptic boundary value problems having constraints. Given nonlinear operators $T, A : H \to H^*$, Noor [134] considered the following problem:

**Problem 1.3.7.** Find $x \in K$ such that

$$\langle T(x), y - x \rangle \geq \langle A(x), y - x \rangle, \quad \forall y \in K.$$

(1.3.7)
Then inequality (1.3.7) is known as mildly nonlinear variational inequality.

**Problem 1.3.8.** Find \( x \in C \) such that

\[
\langle T(x), y - x \rangle + \psi(y) - \psi(x) \geq \langle A(x), y - x \rangle, \quad \forall y \in C.
\] (1.3.8)

Problem 1.3.8 has been studied by Siddiqi *et al.* [153] in the setting of Banach space.

**Problem 1.3.9.** Find \( x \in C \) such that

\[
\langle T(x), y - x \rangle + \phi(x, y) - \phi(x, x) \geq 0, \quad \forall y \in C.
\] (1.3.9)

Problem 1.3.9 has been studied by Kikuchi and Oden [101], Noor [135].

### 1.3.2 Systems of variational inequalities

The system of variational inequalities is an important generalization of variational inequality. In 1971, Caffarelli [29] studied the system of variational inequalities arising in membrane problem. Later, Frehse [67], Ural’teeva [164], Yamada [172], Hayasida and Nagase [78] studied the systems of elliptic variational inequalities. The Nash equilibrium problem [132, 133] for differentiable functions can be formulated in the form of a variational inequality problem over product of sets [8]. A number of problems arising in operation research, economics, game theory, mathematical physics and other areas can also be uniformly modelled as a variational inequality problem over product of sets. In 1985, Pang [144] decomposed the original variational inequality problem defined on the product of sets into a system of variational inequalities, which is easy to solve, to establish some solution methods for variational inequality problem over product of sets. Later, it was found that these two problems are equivalent. Since then a number of researchers studied the existence and iterative approximations of solutions of various systems of abstract variational inequalities, see for example [3–6, 46, 89, 94, 124, 165].

Here, we give some classes of systems of variational inequalities. Let \( T : C \times C \to H \) be a nonlinear mapping.
Problem 1.3.10. Find $x, y \in C$ such that

\[
\begin{cases}
\langle \rho_1 T(y, x) + x - y, z - x \rangle \geq 0, \quad \forall z \in C, \\
\langle \rho_2 T(x, y) + y - x, z - y \rangle \geq 0, \quad \forall z \in C,
\end{cases}
\]  

(1.3.10)

where $\rho_i > 0$, for $i = 1, 2$ are some constants. Problem 1.3.10 is called system of variational inequalities which has been introduced and studied by Verma [165–167].

Ceng et al. [32] considered and studied the following system of variational inequalities (in short, SVIP):

Problem 1.3.11. Find $(x, y) \in C \times C$ such that

\[
\begin{cases}
\langle \rho_1 B_1 y + x - y, z - x \rangle \geq 0, \quad \forall z \in C, \\
\langle \rho_2 B_2 x + y - x, z - y \rangle \geq 0, \quad \forall z \in C,
\end{cases}
\]  

(1.3.11)

where $B_i : C \rightarrow C$ is a nonlinear mapping and $\rho_i > 0$ for each $i = 1, 2$. The set of solutions of SVIP(1.3.11) is denoted by $\text{Sol(SVIP(1.3.11))}$.

For each $i = 1, 2$, let $K_i$ be a nonempty, closed and convex subset of Hilbert space $H_i$; $T_i : H_1 \times H_2 \rightarrow H_i$ be a nonlinear mapping, and let $\phi_i : H_i \times H_i \rightarrow \mathbb{R}$ be a bifunction, which is not necessarily differentiable.

Problem 1.3.12. Find $(x_1, x_2) \in K_1 \times K_2$ such that

\[
\begin{cases}
\langle T_1(x_1, x_2), y_1 - x_1 \rangle \geq 0, \quad \forall y_1 \in K_1, \\
\langle T_2(x_1, x_2), y_2 - x_2 \rangle \geq 0, \quad \forall y_2 \in K_2.
\end{cases}
\]  

(1.3.12)

Problem 1.3.12 has been introduced and studied by Kassay [89].

Problem 1.3.13. Find $(x_1, x_2) \in H_1 \times H_2$ such that

\[
\begin{cases}
\langle T_1(x_1, x_2), y_1 - x_1 \rangle + \phi_1(x_1, y_1) - \phi_1(x_1, x_1) \geq 0, \quad \forall y_1 \in H_1, \\
\langle T_2(x_1, x_2), y_2 - x_2 \rangle + \phi_2(x_2, y_2) - \phi_2(x_2, x_2) \geq 0, \quad \forall y_2 \in H_2.
\end{cases}
\]  

(1.3.13)

Similar to Problem 1.3.13 has been introduced and studied by Kazmi and Khan [94].
1.3.3 Equilibrium problems

In 1994, the terminology of equilibrium problem was adopted by Blum and Oettli [18]. They introduced the following abstract equilibrium problem (in short, EP):

**Problem 1.3.14.** Find $x \in C$ such that

$$F(x, y) \geq 0, \quad \forall x, y \in C.$$  

(1.3.14)

The solution set of EP(1.3.14) is denoted by $\text{Sol}(\text{EP}(1.3.14))$.

They discussed some existence theorems and variational principle for equilibrium problems. Since then various generalizations of equilibrium problem considered by Blum and Oettli [18] have been introduced and studied by many authors. One of the useful generalizations of EP(1.3.14) is vector equilibrium problem which has wide range of applications in multi objective optimizations. For the existence theory of various types of vector equilibrium problems, see for instance [68, 91, 92, 98].

In 1999, Moudafi and Théra [127] introduced and studied the following mixed equilibrium problem (in short, MEP):

**Problem 1.3.15.** Find $x \in C$ such that

$$F(x, y) + \langle Ax, y - x \rangle \geq 0, \quad \forall y \in C,$$

(1.3.15)

where $A : C \to H$ be a nonlinear maping. The solution set of MEP(1.3.15) is denoted by $\text{Sol}(\text{MEP}(1.3.15))$.

EP(1.3.14) and MEP(1.3.15) have been generalized by many authors. Some generalizations of these problems are given below.

Equilibrium problems have potential and useful applications in nonlinear analysis and mathematical economics. For example, if we set $F(x, y) = \sup_{\zeta \in M_x} \langle \zeta, y - x \rangle$ with $M : C \to 2^C$ a multi-valued maximal monotone operator. Then MEP(1.3.15) reduces to the following basic class of variational inclusions:
(1) **Variational inclusion.** Find \( x \in C \) such that

\[
0 \in A(x) + M(x), \quad \forall \ y \in C.
\] (1.3.16)

For further related work, see [1, 64, 90, 95]. Set \( F(x, y) = \psi(y) - \psi(x) \), then MEP(1.3.15) reduces to the following problem.

(2) **Mixed variational inequality.** Find \( x \in C \) such that

\[
\langle A(x), y - x \rangle + \psi(y) - \psi(x) \geq 0, \quad \forall \ y \in C,
\] (1.3.17)

which has been studied by Noor [136]. Set \( F(x, y) = \psi(y) - \psi(x) \), \( \forall \ x, y \in C \), where \( \psi : C \to \mathbb{R} \) is a real function and \( A = 0 \), then MEP(1.3.15) reduces to the following minimization problem subject to the implicit constraints.

(3) **Optimization.** Let \( \psi : C \to \mathbb{R} \), then problem is to find \( \bar{x} \in C \) such that

\[
\psi(\bar{x}) \leq \psi(y), \quad \forall y \in C.
\] (1.3.18)

We write \( \min\{\psi(x) : x \in C\} \) for this problem. Set \( F(x, y) = \psi(y) - \psi(x) \) then problem (1.3.15) coincides with (1.3.14).

(4) **Saddle point problem.** Let \( \phi : C_1 \times C_2 \to \mathbb{R} \). Then \((\bar{x}_1, \bar{x}_2)\) is called **saddle point** of \( \phi \) if and only if

\[
(\bar{x}_1, \bar{x}_2) \in C_1 \times C_2, \quad \phi(\bar{x}_1, y_2) \leq \phi(y_1, \bar{x}_2), \quad \forall \ (y_1, y_2) \in C_1 \times C_2.
\] (1.3.19)

Set \( C = C_1 \times C_2 \) and define \( F : C \times C \to \mathbb{R} \) by \( f((x_1, x_2), (y_1, y_2)) = \phi(y_1, x_2) - \phi(x_1, y_2) \) then \( \bar{x} = (\bar{x}_1, \bar{x}_2) \) is a solution of (1.3.19) if and only if \((\bar{x}_1, \bar{x}_2)\) satisfies (1.3.14).

(5) **Nash equilibria.** Let \( I \) be the index set (the set of players). For every \( i \in I \) let there be given a set \( C_i \) (the strategy set of \( i^{th} \) player). Let \( C = \prod_{i \in I} C_i \). For
every \( i \in I \) let there be given a function \( f_i : C \to \mathbb{R} \) (the loss function of the \( i^{th} \) player depending on the strategies of all players). For \( x = (x_i)_{i \in I} \in C \), we define \( x^i := (x_j)_{j \in I, j \neq i} \). The point \( \bar{x} = (\bar{x}_i)_{i \in I} \in C \) is called a Nash equilibrium if and only if, for all \( i \in I \), there holds
\[
f_i(\bar{x}_i) \leq f_i(\bar{x}^i, y_i), \quad \forall y_i \in C_i,
\]
(i.e., no player can reduce his loss by varying his strategy alone). Define \( F : C \times C \to \mathbb{R} \) by
\[
F(x, y) = \sum_{i \in I} (f_i(x^i, y_i) - f_i(x)).
\]
Then \( \bar{x} \in C \) is a Nash equilibrium if and only if \( \bar{x} \) fulfills (1.3.14). Indeed if (1.3.20) holds for all \( i \in I \) we choose \( y \in C \) in such a way that \( \bar{x}^i = y^i \), then \( F(\bar{x}, y) = f_i(\bar{x}^i, y_i) - f_i(\bar{x}) \). Hence (1.3.14) implies (1.3.20) for all \( i \in I \).

(6) Fixed point problem. Let \( T : C \to C \) be a given mapping. The problem is to find \( \bar{x} \in C \) such that
\[
\bar{x} = T\bar{x}.
\]
(1.3.21)

Set \( F(x, y) = \langle x - Tx, y - x \rangle \). Then \( \bar{x} \) solves (1.3.14) if and only if \( \bar{x} \) is a solution of (1.3.21). Indeed (1.3.21) \( \Rightarrow \) (1.3.14) is obvious. And if (1.3.14) is satisfied then \( \bar{y} = T\bar{x} \) to obtain \( 0 \leq F(\bar{x}, \bar{y}) = -\|\bar{x} - T\bar{x}\|^2 \). Hence \( \bar{x} = T\bar{x} \), so (1.3.14) \( \Rightarrow \) (1.3.21).

(7) Variational inequality. Let \( T : C \to H^* \) be a given mapping. The problem is to find \( \bar{x} \in H \) such that \( \bar{x} \in C \),
\[
\langle T\bar{x}, y - \bar{x} \rangle \geq 0 \quad \forall \ y \in C.
\]
(1.3.22)

We set \( F(x, y) = \langle Tx, y - x \rangle \). Clearly (1.3.22) \( \Rightarrow \) (1.3.14).

(8) Complementarity problem. This is special case of previous example. Let \( C \) be a closed and convex cone with \( C^* = \{x^* \in H^* : \langle x^*, y \rangle \geq 0, \ \forall \ y \in C \} \) denotes the polar cone. Let \( T : C \to H^* \) be a given mapping. The problem is to find \( \bar{x} \in H \)
such that $T\bar{x} \in C^*$,

$$\langle T\bar{x}, \bar{x} \rangle = 0.$$ \hspace{1cm} (1.3.23)

It is easily seen that (1.3.23) is equivalent with (1.3.22). Indeed (1.3.23) $\Rightarrow$ (1.3.22) is obvious. If (1.3.22) holds then setting $y = 2\bar{x}$ and $y = 0$, we obtain from (1.3.23) that $\langle T\bar{x}, \bar{x} \rangle = 0$ then by $\langle T\bar{x}, y \rangle \geq 0$, $\forall y \in C$. Hence (1.3.22) $\Rightarrow$ (1.3.23).

**Problem 1.3.16.** Find $x \in C$ such that

$$F(x, y) + \psi(y) - \psi(x) \geq 0 \geq 0, \quad \forall y \in C,$$ \hspace{1cm} (1.3.24)

where $\psi : H \to \mathbb{R} \cup \{+\infty\}$ is a nonlinear functional. EP(1.3.24) has been studied by Ceng and Yao [33].

**Problem 1.3.17.** Find $x \in C$ such that

$$F(x, y) + \langle Ax, y - x \rangle + \psi(y) - \psi(x) \geq 0, \quad \forall y \in C,$$ \hspace{1cm} (1.3.25)

where $A : C \to H$ is a nonlinear mapping. EP(1.3.25) has been studied by Peng and Yao [145].

Let $F : C \times C \to \mathbb{R}$, $\phi : H \times H \to \mathbb{R} \cup \{+\infty\}$ be nonlinear bifunctions. The *generalized equilibrium problem* (in short, GEP) is:

**Problem 1.3.18.** Find $x \in C$ such that

$$F(x, y) + \phi(y, x) - \phi(x, x) \geq 0, \quad \forall y \in C,$$ \hspace{1cm} (1.3.26)

which has been studied by Noor [137].

### 1.3.4 Iterative methods

Here, we give brief survey of some iterative methods for solving fixed point problems, variational inequalities and equilibrium problems.
Let $T$ be a self-mapping defined on a nonempty, closed and convex subset $C$ of a Banach space $X$. Since 1922, we know that if $T$ is a contraction defined on a complete metric space $X$, the Banach contraction principle sets up that, for any $x \in X$, Picard iteration $\{T_n x\}$ converges strongly to the unique fixed point of $T$. If the mapping $T$ is nonexpansive, we must assume additional conditions to ensure the existence of fixed points of $T$ and, even when a fixed point exists, the sequences of iterates in general do not converge to a fixed point. In the particular case when $T$ is firmly nonexpansive, Picard iteration does converge assuming the existence of a fixed point (see, for instance, [73]). The study of iterative methods for approximating fixed points of a nonexpansive mapping $T$ has yielded a host of works in the last decades. The most relevant progresses are mainly based on two types of iterative algorithms: Mann and Halpern iterative algorithms. Both algorithms have extensively been studied for decades.

**Mann iterative method.** Mann iterative algorithm, initially due to Mann [115], is essentially an averaged algorithm which generates a sequence recursively

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) Tx_n, \quad n \geq 0,$$  \hspace{1cm} (1.3.27)

where the initial guess $x_0 \in C$ and $\{\alpha_n\}$ is a sequence in $(0, 1)$. Later, Krasnosel’skiı̆ [104] studied the iterative algorithm (1.3.27) in the particular case when $\alpha_n = \lambda$.

In 1974, Ishikawa [83] enlarged and improved Mann iterative algorithm to a new iterative algorithm which generates the sequence $\{x_n\}$ defined by

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T[\beta_n x_n + (l - \beta_n)Tx_n],$$  \hspace{1cm} (1.3.28)

where $0 \leq \alpha_n \leq \beta_n \leq 1$, $\lim_{n \to \infty} \beta_n = 0$, $\sum_{n \geq 1} \alpha_n \beta_n = \infty$. However, the iterative algorithms of both mann and Ishikawa converge weakly in Banach space. As a matter of fact, Mann’s iterative algorithm may fail to converge while Ishikawa iterative algorithm can still converge for a Lipschitz pseudocontractive mapping in a Hilbert space.

**Halpern implicit iteration method.** An iterative approach for solving the problem of approximating a fixed point of a mapping $T$, which may have multiple solutions, is
to replace it by a family of perturbed problems admitting a unique solution, and then to get a particular original solution as the limit of these perturbed solutions as the perturbation vanishes. For example, given a nonempty, closed and convex set $C \subseteq H$, $T : C \to C$, $u \in C$ and $t \in (0,1)$, Browder [21–23] studied the approximating curve $\{x_t\}$ defined by

$$x_t = tu + (1 - t)Tx_t, \quad (1.3.29)$$

that is, $x_t$ is the unique fixed point of the contraction $tu + (1 - t)T$. He proved that if the underlying space $H$ is Hilbert, $\{x_t\}$ converges strongly to the fixed point of $T$ closest to $u$ as $t \to 0$.

**Halpern explicit iterative method.** Halpern [75] was the first in introducing the explicit iterative algorithm which generates a sequence via the recursive formula

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)Tx_n, \quad n \geq 0, \quad (1.3.30)$$

where the initial guess $x_0 \in C$ and anchor $u \in C$ are arbitrary (but fixed) and the sequence $\{\alpha_n\}$ is contained in $[0,1]$, for finding a fixed point of a nonexpansive mapping $T : C \to C$ with $\text{Fix}(T) \neq \emptyset$, where $C$ is a nonempty, closed and convex subset of a Hilbert space $H$. This iterative method is now commonly known as Halpern iterative method although Halpern initially considered the case where $C$ is the unit closed ball and $u = 0$.

**Viscosity approximation method.** Given a nonexpansive self-mapping $T$ on a nonempty, closed and convex subset $C$, a real number $t \in (0,1]$ and a contraction mapping $f$ on $C$, define the mapping $T_t : C \to C$ by

$$T_t x = tf(x) + (1 - t)Tx, \quad x \in C.$$

It is easily seen that $T_t$ is a contraction; hence $T_t$ has a unique fixed point which is denoted by $x_t$. That is, $x_t$ is the unique solution to the fixed point equation

$$x_t = tf(x_t) + (1 - t)Tx_t, \quad t \in (0,1]. \quad (1.3.31)$$
The explicit iterative discretization of (1.3.31) is
\[ x_{n+1} = \alpha_n f(x_n) + (1 - t)Tx_n, \ n \geq 0, \] (1.3.32)
where \( \{\alpha_n\} \subset [0, 1] \). Note that these two iterative processes (1.3.31) and (1.3.32) generalize the results of Browder [22] and Halpern [75] in another direction. The viscosity approximation method of selecting a particular fixed point of a given nonexpansive mapping was proposed by Moudafi [120] in the framework of a Hilbert space. The convergence of the implicit (1.3.31) and explicit (1.3.32) algorithms has been the subject of many papers because under suitable conditions these iterations converge strongly to the unique solution \( q \in \text{Fix}(T) \) of the variational inequality
\[ \langle (I - f)q, x - q \rangle \geq 0, \ \forall x \in \text{Fix}(T). \] (1.3.33)

This fact allows us to apply this method to convex optimization, linear programming and monotone inclusions. In 2004, Xu [171] extended the result of Moudafi to uniformly smooth Banach spaces and obtained strong convergence theorem. For related work, see [17, 44, 118, 173].

**Hybrid iterative method.** The hybrid iterative method is also known as outer-approximation method. This type of method was originally introduced by Haugazeau [77] in 1968 and was successfully generalized and extended by Bauschke and Combettes [11, 12], Combettes [52], Nakajo and Takahashi [131], Kikkawa and Takahashi [100].

In 2004, Nakajo and Takahashi [131] introduced and studied the following iterative method for a nonexpansive mapping \( T \) over a Hilbert space:

\[
\begin{align*}
x_0 &= x \in C \subseteq H, \\
w_n &= \alpha_n x_n + (1 - \alpha_n)Tx_n, \\
C_n &= \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\
Q_n &= \{z \in C : \langle x_n - z, x - x_n \rangle \geq 0\}, \\
x_{n+1} &= P_{C_n \cap Q_n} x.
\end{align*}
\] (1.3.34)
They proved that the sequence \( \{x_n\} \) generated by (1.3.34) converges strongly to \( P_{\text{Fix}(T)}x_0 \), where \( P_{\text{Fix}(T)} \) denotes the metric projection from \( H \) onto \( \text{Fix}(T) \). For further related work, see [28, 82].

In 1967, Lions and Stampacchia [109] proved the first general theorem for the existence and uniqueness of solution of variational inequality problem in Hilbert space using projection mapping. The variational inequality of finding \( x \in C \) such that

\[
\langle Tx, y - x \rangle \geq 0, \quad \forall y \in C,
\]

where \( T : C \to H \) be a nonlinear mapping, is equivalent to finding a fixed point \( x \) of the equation

\[
x = P_C(x - \lambda T(x)),
\]

where \( \lambda > 0 \). Using this fixed point formulation, one can have an iterative algorithm which generates the sequence \( \{x_n\} \) given by

\[
x_{n+1} = P_C(x_n - \lambda T(x_n)),
\]

where \( x_0 \in C \) is given and \( \lambda > 0 \), see Baiocchi and Capelo [9], Glowinski, Lions and Tremolieres [71].

In 1976, Korpelevich [103] proposed an extragradient method with iterative scheme

\[
\begin{align*}
x_1 &= x \in C \\
y_n &= P_C(x_n - \lambda Fx_n), \\
x_{n+1} &= P_C(x_n - \lambda Fy_n),
\end{align*}
\]

for every \( n = 0, 1, 2, ..., \) and \( \lambda > 0 \), where \( P_C \) is an orthogonal projection onto \( C \) in the finite dimensional Euclidean space. The idea of the extragradient iterative process introduced by Korpelevich was successfully generalized and extended not only in Euclidean but also in Hilbert and Banach spaces; see, e.g., He et al. [79], iusem and Svaiter [84], Otero and Iusem [141], Solodov [155], Solodov and Svaiter [156, 157], Wang et al. [168].
It is worth mentioning that the scope of the projection method and its variants is quite limited due to the fact that it is very difficult to find the projection of the space into the convex set except in very simple cases. Secondly, the projection method cannot be applied for other classes of variational inequalities of type (1.3.6). These facts motivated Glowinski et al. [71] to develop alternative methods (called Auxiliary principle method) to study the existence of solution of variational inequalities. This method provides a general framework to suggest and analyze an iterative algorithm for computing the solution of variational inequalities. Further related work see for instance, Chidume et al. [45], Huang and Deng [81], Kazmi and Khan [94], Kazmi et al. [93, 97] and the references therein.

Equilibrium problems which were initially introduced by Zuhovickii et al. [179], Fan [63]; Brezis et al. [20], perhaps motivated by mini-max problems that appeared in economic equilibrium. But it was Blum and Oettli [18], who understood equilibrium problem. Since then extensive research has been started on equilibrium problems.

In 1997, Flam and Antipin [66] introduced and studied some proximal type iterative methods for equilibrium problems. Then Moudafi and Thera [127], Moudafi [121, 122] introduced and studied proximal type methods for some classes of equilibrium problems. Further, Combettes and Hirstoaga [53] introduced and studied an iterative method for finding the best approximation to the initial data when \( \text{Sol}(\text{EP}(1.3.14)) \neq \emptyset \) and proved a strong convergence theorem. For related work, see for instance, Kazmi and Khan [96], Ding [59] and references therein.

There is a substantial number of iterative methods for studying separately, the fixed point problems for nonlinear mappings, variational inequalities, and equilibrium problems. It is of further interest to develop and study the iterative methods to approximate the common solutions of these problems. In this direction, Takahashi and Toyoda [163] in 2003, considered the problem of finding a common solution of a fixed point problem for nonexpansive mapping \( T \) on \( C \) and variational inequality with \( \alpha \)-inverse strongly monotone mappings and developed the following Mann-type iterative method:
\[
\begin{align*}
\{ x_n \} & \in C, \\
x_{n+1} & = \alpha_n x_n + (1 - \alpha_n) TP_C (x_n - \lambda_n D x_n), 
\end{align*}
\]  
\tag{1.3.36}

where \( \{ \alpha_n \} \) and \( \{ \lambda_n \} \) are sequences of real numbers. They proved that under certain appropriate conditions on \( \{ \alpha_n \} \) and \( \{ \lambda_n \} \), the sequence \( \{ x_n \} \) generated by (1.3.36) converges strongly to \( z \in \text{Fix}(T) \cap \text{Sol}(\text{VIP}(1.3.3)) \). Nadezhkina and Takahashi [129] extended the iterative method (1.3.36) to the extragradient method. For further related work, see Ceng et al. [30, 34, 35], Plubtieng et al. [147], Zeng et al. [177].

In 2008, Ceng et al. [32] introduced and studied the following iterative method so called relaxed extragradient method for approximating a common solution of \( \text{SVIP}(1.3.11) \) with \( \alpha \)-inverse strongly monotone mappings and \( \text{FPP}(1.2.1) \) for a nonexpansive mapping \( T \):

\[
\begin{align*}
x_0 & \in C, \\
y_n & = P_C (x_n - \mu B_2 x_n), \\
x_{n+1} & = \alpha_n x_0 + \beta_n x_n + \gamma_n TP_C (y_n - \lambda B_1 y_n), 
\end{align*}
\]  
\tag{1.3.37}

where \( \{ \alpha_n \}, \{ \beta_n \}, \{ \gamma_n \} \) are sequences in \([0, 1]\), and \( \lambda, \mu > 0 \). For further related work, see Ceng et al. [30], Yao et al. [175].

In 2006, by combining a hybrid iterative method with an extragradient method, Nadezhkina and Takahashi [128] introduced the following iterative method:

\[
\begin{align*}
x_0 & = x \in C, \\
y_n & = P_C (x_n - \lambda_n A x_n), \\
z_n & = \beta_n x_n + (1 - \beta_n) TP_C (x_n - \lambda_n A y_n), \\
C_n & = \{ z \in C : \| z_n - z \|^2 \leq \| x_n - z \|^2 \}, \\
Q_n & = \{ z \in C : \langle x_n - z, x - x_n \rangle \geq 0 \}, \\
x_{n+1} & = P_{C_n \cap Q_n} x, 
\end{align*}
\]  
\tag{1.3.38}

for every \( n = 0, 1, 2, \ldots \) . They proved that under certain appropriate conditions on \( \{ \beta_n \} \) and \( \{ \lambda_n \} \), the sequences \( \{ x_n \}, \{ y_n \} \) and \( \{ z_n \} \) generated by (1.3.38) converge strongly to \( z \in \text{Fix}(T) \cap \text{Sol}(\text{VIP}(1.3.3)) \). Ceng et al. [31] introduced the extragradient-like iterative method, an extension of method given by Nadezhkina and Takahashi [128,129],
for approximating common solution of FPP(1.2.1) for a nonexpansive mapping $T$ and VIP(1.3.3) for a monotone, Lipschitz-continuous mapping.

On the other hand, Takahashi and Takahashi [161] in 2007, proposed an iterative method based on viscosity approximation method which improves the result of Moudafi [120] for approximating the common solution of EP(1.3.14) and FPP(1.2.1) for a nonexpansive mapping $T$ in Hilbert space.

\[
\begin{aligned}
F(u_n, y) + \frac{1}{r_n} (y - u_n, u_n - x_n) & \geq 0 \quad \forall y \in C, \\
x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) Tu_n.
\end{aligned}
\]  

They proved that under certain appropriate conditions on $\{\alpha_n\}$, $\{r_n\}$ and $\{\lambda_n\}$, the sequences $\{x_n\}$ and $\{u_n\}$ generated by (1.3.39) converge strongly to $z \in \text{Fix}(T) \cap \text{Sol}(\text{EP}(1.3.14))$, where $z = P_{\text{Fix}(T) \cap \text{Sol}(\text{EP}(1.3.14))} f(z)$. For related work, see [2, 50, 85, 114, 148, 149, 176]. Further, Tada and Takahashi [160] introduced a hybrid method for approximating a common solution of EP(1.3.14) and FPP(1.2.1) for a nonexpansive mapping in a Hilbert space. Starting with an arbitrary $x_1 \in H$, define sequences $\{x_n\}$ and $\{u_n\}$ by

\[
\begin{aligned}
F(u_n, y) + \frac{1}{r_n} (y - u_n, u_n - x_n) & \geq 0 \quad \forall y \in C, \\
w_n = (1 - \alpha_n)x_n + \alpha_n Tu_n, \\
C_n = \{z \in H : \|w_n - z\| \leq \|x_n - z\|\}, \\
Q_n = \{z \in H : \langle x_n - z, x - x_n \rangle \geq 0\}, \\
x_{n+1} = P_{C_n \cap Q_n} x.
\end{aligned}
\]  

They proved that under certain appropriate conditions on $\{\alpha_n\}$ and $\{r_n\}$, the sequences $\{x_n\}$ and $\{u_n\}$ generated by (1.3.40) converge strongly to $P_{\text{Fix}(T) \cap \text{Sol}(\text{EP}(1.3.14))} f(x)$.

Using the idea of Takahashi and Takahashi [161], Plubtieng and Punpaeng [148] introduced the general iterative method for finding a common solution of EP(1.3.14), VIP(1.3.3) and FPP for a nonexpansive mapping $T$. For further related work, see for instance [42, 49, 86, 88, 105–107, 110, 145, 151, 169, 170] and the relevant references cited therein.