The present thesis entitled “Iterative methods of solutions for certain classes of equilibrium problems, variational inequality and fixed point problems” is an outcome of the studies made by the author at Department of Mathematics, Aligarh Muslim University, Aligarh, India.

In the last decades, the theory of optimization techniques has been developed very significantly for its wide range of applications in many research areas. It is known that the optimization problems and many others important mathematical problems are closely related to variational inequality problems and equilibrium problems. Hence the theory of variational inequality and equilibrium problems are also been developed and are very interesting topics of current studies. Variational inequality and equilibrium problems unify numerous familiar problems of applied and pure mathematics. Therefore, instead of studying varieties types of problems differently, sometimes it is convenient to study a single problem like equilibrium problem or variational inequality problem, which covers a vast range of problems.

Variational inequality theory was initiated independently by Fichera [22] and Stampacchia [44] in the early 1960’s to study the problems in the elasticity and potential theory, respectively. The first general theorem for the existence and uniqueness of solution of variational inequality was proved by Lions and Stampacchia [29] in 1967. Since then, from theoretical and practical point of view, the variational inequality problems have a great importance. It is well-known that the variational inequality theory has played a fundamental and important role in the study of a wide range of problems arising in physics, mechanics, elasticity, optimization, control theory, management science, operations research, economics, transportation and other branches of mathematical and engineering sciences.

The classical variational inequality problem (in short, VIP) is to find $x \in C$ such that

$$\langle Dx, y - x \rangle \geq 0, \quad \forall y \in C,$$

where $C$ is a nonempty, closed and convex subset of a real Hilbert space $H$; $\langle ., . \rangle$, and $\| \cdot \|$ denote, respectively, inner product and induced norm on $H$, and $D : C \to H$ is a nonlinear mapping. The set of solutions of VIP(1) is denoted by Sol(VIP(1)).

An important generalization of variational inequality is a system of variational inequalities (in short, SVIP). In 1971, Caffarelli [10] studied the system of variational inequalities...
arising in membrane problem. The Nash equilibrium problem [38, 39] for differentiable functions can be formulated in the form of a variational inequality problem over product of sets [6]. A number of problems arising in operation research, economics, game theory, mathematical physics and other areas can also be uniformly modeled as a variational inequality problem over product of sets. In 1985, Pang [41] decomposed the original variational inequality problem defined on the product of sets into a system of variational inequalities, which is easy to solve, to establish some solution methods for variational inequality problem over product of sets. Later it was found that these two problems are equivalent. Since then a number of researchers studied the existence and iterative approximations of solutions of various systems of abstract variational inequalities, see for example [1, 3–5, 17, 24, 26, 33, 49].

A system of variational inequalities (in short, SVIP) is to find \((x, y) \in C \times C\) such that

\[
\begin{aligned}
\langle \rho_1 B_1 y + x - y, z - x \rangle &\geq 0, \quad \forall z \in C, \\
\langle \rho_2 B_2 x + y - x, z - y \rangle &\geq 0, \quad \forall z \in C,
\end{aligned}
\]

where \(B_i : C \subseteq H \rightarrow C\) is a nonlinear mapping and \(\rho_i > 0\) for each \(i = 1, 2\). The set of solutions of \(\text{SVIP}(2)\) is denoted by \(\text{Sol}(\text{SVIP}(2))\).

Several problems in optimization and minimax theory can be written in an abstract framework as follows: Find \(x \in C \subseteq H\), such that

\[
F(x, y) \geq 0, \quad \forall y \in C,
\]

where \(F : C \times C \rightarrow \mathbb{R}\) is a bifunction.

Problems like (3) have a long history starting with Zuhovickii, Poljak and Primak [54], Ky Fan [20, 21], perhaps motivated by minimax problems appearing in economic equilibrium. A more general result than that in [20] was established by Brézis, Nirenberg and Stampacchia [8].

But, in 1994, Blum and Oettli [7] called problem (3) an equilibrium problem (in short, EP) and studied its existence theory. The set of solutions of \(\text{EP}(3)\) is denoted by \(\text{Sol}(\text{EP}(3))\). Since then equilibrium problems have been extended and generalized in several directions using novel and innovative techniques both for their own sake and for applications. It is known that the equilibrium problem has a great impact and influence in the development of several topics of science and engineering. It turned out that the theories of many well known problems could be fitted into the theory of equilibrium problems. It has been shown that the theory of equilibrium problem provides a natural, novel and unified frame-
work for several problems arising in nonlinear analysis, optimization, economics, finance, game theory, physics and engineering. The equilibrium problem includes many mathematical problems as particular cases for examples, mathematical programming problems, complementary problems, variational inequality problems, saddle point problems, Nash equilibrium problems in noncooperative games, minimax inequality problems, minimization problems and fixed point problems, see [7,19,23,25,27,32].

In 1999, Moudafi and Théra [35] introduced the following generalization of EP(3): Find \( x \in C \subseteq H \) such that
\[
F(x, y) + \langle Ax, y - x \rangle \geq 0, \quad \forall y \in C,
\]
where \( A : C \rightarrow H \) be a nonlinear mapping. This problem is called mixed equilibrium problem (in short, MEP). The solution set of MEP(4) is denoted by \( \text{Sol}(\text{MEP}(4)) \).

In recent years, much attention has been given for developing efficient and implementable iterative methods including projection method and its variant forms, extragradient method, linear approximation, auxiliary principle method, descent and Newton methods for VIPs and EPs.

On the other hand, many problems arising in different areas of mathematics such as optimization, variational analysis and differential equations, can be modeled by fixed point problem (FPP) consisting of finding \( x \in C \) such that \( x = Tx \), where \( T \) is a nonlinear self map on \( C \). \( \text{Fix}(T) \) denotes the set of fixed points of \( T \). The study of iterative methods, viz. Halpern iterative method, Mann and Ishikawa iterative methods, and viscosity approximation method for FPP has yielded a host of works in the last decades.

There is a substantial number of iterative methods for studying separately, the fixed point problems for nonlinear mappings, variational inequalities, and equilibrium problems. It is of further interest to develop and study the iterative methods to approximate the common solutions of these problems. In this direction, Takahashi and Toyoda [48] in 2003, considered the problem of approximating a common solution of a fixed point problem for nonexpansive mapping \( T \) on \( C \) and variational inequality with \( \alpha \)-inverse strongly monotone mappings and developed the following Mann-type iterative method:

\[
\begin{align*}
x_0 & \in C, \\
x_{n+1} &= \alpha_n x_n + (1 - \alpha_n)TP_C(x_n - \lambda_n Dx_n),
\end{align*}
\]

where \( \{\alpha_n\} \) and \( \{\lambda_n\} \) are sequences of real numbers and \( P_C \) is the projection of \( H \) onto \( C \). They proved that under certain appropriate conditions on \( \{\alpha_n\} \) and \( \{\lambda_n\} \), the sequences \( \{x_n\} \) generated by (5) converge strongly to \( z \in \text{Fix}(T) \cap \text{Sol}(\text{VIP}(1)) \).
Takahashi [37] extended the iterative method (5) to the extragradient method.

In 2008, Ceng et al. [13] introduced and studied the following iterative method so called relaxed extragradient method for approximating a common solutions of SVIP(2) with α-inverse strongly monotone mappings and fixed point problem for a nonexpansive mapping $T$:

$$
\begin{align*}
  x_0 &\in C, \\
  y_n &= P_C(x_n - \mu B_2 x_n) \\
  x_{n+1} &= \alpha_n x_0 + \beta_n x_n + \gamma_n T P_C(y_n - \lambda B_1 y_n),
\end{align*}
$$

(6)

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are sequences in $[0, 1]$ and $\lambda, \mu > 0$.

In 2006, by combining a hybrid iterative method with an extragradient method, Nadezhkina and Takahashi [36] introduced the following iterative method:

$$
\begin{align*}
  x_1 &= x \in C, \\
  y_n &= P_C(x_n - \lambda_n A x_n), \\
  z_n &= \beta_n x_n + (1 - \beta_n) T P_C(x_n - \lambda_n A y_n), \\
  C_n &= \{z \in C : \|z_n - z\| \leq \|x_n - z\|\}, \\
  Q_n &= \{z \in C : \langle x_n - z, x - x_n \rangle \geq 0\}, \\
  x_{n+1} &= P_{C_n \cap Q_n} x,
\end{align*}
$$

(7)

for every $n = 1, 2, \ldots$ . They proved that under certain appropriate conditions on $\{\beta_n\}$ and $\{\lambda_n\}$, the sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}$ generated by (7) converge strongly to $z \in \text{Fix}(T) \cap \text{Sol}(\text{VIP}(1))$. Ceng et al. [12] introduced the extragradient-like iterative method, an extension of method given by Nadezhkina and Takahashi [36, 37], for approximating a common solution of FPP for a nonexpansive mapping $T$ and VIP(1) for a monotone, Lipschitz-continuous mapping.

On the other hand, Takahashi and Takahashi [47] in 2007, proposed an iterative method based on viscosity approximation method which improves the result of Moudafi [31] for approximating a common solution of EP(3) and FPP for a nonexpansive mapping $T$ in Hilbert space:

$$
\begin{align*}
  F(u_n, y) + \frac{1}{r_n}(y - u_n, u_n - x_n) &\geq 0 \quad \forall y \in C, \\
  x_{n+1} &= \alpha_n f(x_n) + (1 - \alpha_n) T u_n.
\end{align*}
$$

(8)

They proved that under certain appropriate conditions on $\{\alpha_n\}$, $\{r_n\}$ and $\{\lambda_n\}$, the sequences $\{x_n\}$ and $\{u_n\}$ generated by (8) converge strongly to $z \in \text{Fix}(T) \cap \text{Sol}(\text{EP}(3))$, where $z = P_{\text{Fix}(T) \cap \text{Sol}(\text{EP}(3))} f(z)$. Further related work in Banach space, see for instance Agarwal et al. [2].
Further, Tada and Takahashi [46] introduced a hybrid method for finding a common solution of EP(3) and FPP for a nonexpansive mapping $T$ in a Hilbert space. Starting with an arbitrary $x_1 \in H$, define sequences $\{x_n\}$ and $\{u_n\}$ by

$$\begin{align*}
F(u_n, y) + \frac{1}{r_n} (y - u_n, u_n - x_n) & \geq 0 \quad \forall y \in C,
\w_n = (1 - \alpha_n)x_n + \alpha_n Tu_n,
C_n = \{z \in H : \|w_n - z\| \leq \|x_n - z\|\},
Q_n = \{z \in H : \langle x_n - z, x - x_n \rangle \geq 0\},
x_{n+1} = P_{C_n \cap Q_n} x.
\end{align*}$$

They proved that under certain appropriate conditions on $\{\alpha_n\}$ and $\{r_n\}$, the sequences $\{x_n\}$ and $\{u_n\}$ generated by (9) converge strongly to a common solution $z = P_{\text{Fix}(T) \cap \text{Sol}(\text{EP}(3))}(x)$ of EP(3) and FPP.

Using the idea of Takahashi and Takahashi [47], Plubtieng and Punpaeng [43] introduced and studied the general iterative method for finding a common solution of EP(3), VIP(1) and FPP for a nonexpansive mapping $T$.

The objective of this thesis is to develop and study some iterative methods for approximating the common solutions of variational inequality problems, system of variational inequalities, split equilibrium problems, split variational inclusion problems and fixed point problems for a (family of) nonlinear mapping(s).

The thesis comprises of eight chapters.

In Chapter 1, we review various notations, known definitions and results which are required in carrying out the research work presented in the thesis. Further, we give brief survey of some classes of variational inequalities and equilibrium problems. Furthermore, we give brief survey of some iterative methods for solving fixed point problems, variational inequalities and equilibrium problems.

In Chapter 2, we develop an iterative method based on relaxed extragradient and viscosity approximation methods for approximating a common solution of VIP(1), SVIP(2), MEP(4) and FPP for a strictly pseudocontractive mapping in a real Hilbert space. We establish a strong convergence theorem for the sequence generated by the proposed iterative method. Further, we derive some consequences from the strong convergence theorem. The result and iterative method presented here extend and generalize the work given in [11, 13, 50, 52].

In Chapter 3, we consider the split equilibrium problem (SFP) due to Moudafi [34],
an another important generalization of split variational inequality [14] and EP(3), which consisting of finding $x^* \in C$ such that

$$F_1(x^*, x) \geq 0, \forall x \in C,$$

and such that

$$y^* = Ax^* \in Q \text{ solves } F_2(y^*, y) \geq 0, \forall y \in Q,$$

where $F_1 : C \times C \to \mathbb{R}, F_2 : Q \times Q \to \mathbb{R}$ are nonlinear bifunctions; $A : H_1 \to H_2$ is a bounded linear operator, and $C, Q$ be nonempty, closed and convex subsets of real Hilbert spaces $H_1, H_2$, respectively.

We suggest and analyze implicit and explicit iterative methods for approximating a common solution of $\text{S}_P\text{EP}(10-11)$ and FPP for a nonexpansive semigroup in real Hilbert spaces. Further, based on these methods, we prove the strong convergence theorem for approximating a common solution of $\text{S}_P\text{EP}(10-11)$ and FPP for a nonexpansive semigroup. Some consequences from these theorems are also derived. Furthermore, we justify our main results through a numerical example. The results and iterative methods presented here extend and generalize the corresponding results and methods given in [18, 28, 42].

In Chapter 4, we consider $\text{S}_P\text{EP}(10-11)$ with $F_i = f_i + h_i$ for $i = 1, 2$, and called it split generalized equilibrium problem (in short, $\text{S}_P\text{GEP}$). Further, we suggest and analyze an iterative method for approximating a common solution of $\text{S}_P\text{GEP}$ and FPP for a nonexpansive semigroup in real Hilbert spaces. Furthermore, we prove the strong convergence theorem based on this method for approximating a common solution of these problems. Some consequences from these theorems are also derived. The result and iterative method presented here extend and generalize the work given in [16, 18, 42, 45].

In Chapter 5, we suggest and analyze a Halpern-type iterative method for approximating a common solution of $\text{S}_P\text{EP}(10-11)$, VIP(1) and FPP for a nonexpansive mapping in real Hilbert spaces. Further, we prove that the sequences generated by the iterative scheme converge strongly to a common solution of $\text{S}_P\text{EP}(10-11)$, VIP(1) and FPP for a nonexpansive mapping. Some consequences from the main result are also derived. The result and iterative method presented here generalize the work given in [43, 48].

In Chapter 6, we consider the split variational inclusion problem ($\text{S}_P\text{VI}$): Find $x \in H_1$ such that

$$0 \in B_1(x),$$

and such that

$$y^* = Ax^* \in Q \text{ solves } F_2(y^*, y) \geq 0, \forall y \in Q,$$
and such that
\[ y = Ax \in H_2 \text{ solves } 0 \in B_2(y), \] (13)
where \( A : H_1 \to H_2 \) is a bounded linear operator, \( B_1 : H_1 \to 2^{H_1} \) and \( B_2 : H_2 \to 2^{H_2} \) are multi-valued maximal monotone mappings. Based on viscosity approximation method, we suggest and analyze implicit and explicit iterative methods for approximating a common solution of \( \text{SpVI}(12-13) \) and \( \text{FPP} \) of a nonexpansive mapping in Hilbert spaces. Further, we prove the strong convergence theorems for the nets and sequences generated by these iterative methods. Some consequences from these theorems are also derived. These theorems and iterative methods generalize the corresponding result and iterative method of Byrne \textit{et al.} \[9\] and Xu \[51\].

In Chapter 7, we consider the following system of unrelated mixed equilibrium problems (in short, SUMEP): For each \( i = 1, 2, \ldots, N \), let \( K_i \) be a nonempty, closed and convex subset of a real Hilbert space \( H \) with \( \bigcap_{i=1}^{N} K_i \neq \emptyset \); let \( F_i : K_i \times K_i \to \mathbb{R} \) be a bifunction such that \( F_i(x_i, y_i) = 0, \forall x_i \in K_i \) and let \( A_i : H \to H \) be a nonlinear mapping. Then SUMEP is to find \( x \in \bigcap_{i=1}^{N} K_i \) such that
\[ F_i(x, y_i) + \langle A_i x, y_i - x \rangle \geq 0, \forall y_i \in K_i, \quad i = 1, 2, \ldots, N. \] (14)

We note that for each \( i = 1, 2, \ldots, N \), the mixed equilibrium problem (MEP) \[35\] is to find \( x_i \in K_i \) such that
\[ F_i(x_i, y_i) + \langle A_i x_i, y_i - x_i \rangle \geq 0, \quad \forall y_i \in K_i, \quad i = 1, 2, \ldots, N. \] (15)

We denote by \( \text{Sol}(\text{MEP}(15)) \), the set of solutions of MEP(15) corresponding to the mappings \( F_i, A_i \) and the set \( K_i \) for each \( i \). Then the set of solutions of SUMEP(14) is given by \( \bigcap_{i=1}^{N} \text{Sol}(\text{MEP}(15)) \).

We introduce an iterative method based on hybrid method, extragradient method and convex approximation method for approximating a common solution of system of unrelated mixed equilibrium problems and the common fixed point problem (CFPP) for a family of nonexpansive mappings. We call it hybrid-extragradient-convex approximation method. We define the notion of 2-monotone bifunction which is a natural extension of monotone bifunction and 2-cyclically monotone operator. Further, we obtain a strong convergence theorem for the sequences generated by the proposed iterative scheme. Furthermore, we derive some consequences from our main result, some of them are new. The result and iterative method presented here generalize the work given by Censor \textit{et al.} \[15\].
and Nadezhkina and Takahashi [36].

In Chapter 8, we consider the following generalized mixed equilibrium problem (in short, GMEP): Let $F : C \times C \to \mathbb{R}$, $\phi : H \times H \to \mathbb{R}$ be nonlinear bifunctions and let $A : C \to H$ be a nonlinear mapping. Then GMEP is to find $x \in C$ such that

$$F(x, y) + \langle Ax, y - x \rangle + \phi(x, y) - \phi(x, x) \geq 0, \quad \forall y \in C. \quad (16)$$

We extend the iterative methods given in [30,40,53] to GMEP(16) in general real Hilbert space. We prove the existence and uniqueness of solution of an auxiliary problem of GMEP(16) which will be helpful in construction of an resolvent-projection iterative algorithm for GMEP(16) which consists of a resolvent mapping technique step followed by a suitable orthogonal projection onto a moving hyperplane. Further, we prove that the sequences generated by iterative algorithm converge weakly to a solution of GMEP(16).

In addition, based on extragradient method, we also study the convergence analysis of two iterative algorithms for GMEP(16) for the monotone ($\theta$-pseudo monotone) mapping. Using the auxiliary principle, we define a class of resolvent mappings. Further, using fixed point and resolvent methods, we give some iterative algorithms for solving GMEP(16). Furthermore, we prove that the sequences generated by iterative algorithms converge weakly to the solution of GMEP(16). The results and iterative methods presented in this paper improve and extend the iterative methods given in Noor [40] for a mixed variational inequality problem in finite dimensional space, and given in Moudafi [30] for MEP(4).

A comprehensive list of references of books, monographs, proceedings and research papers is provided at the end of the thesis.

The published research papers based on the work of this thesis are as follows:


Some results of this thesis have been presented in the following National and International Conferences:


3. *International Conference on Analysis and its Applications*, Under the aegis of UGC-DRS Programme, organized by Department of Mathematics, Aligarh Muslim University, Aligarh, India (19th - 21st November, 2011).
References


