Chapter 5

Common solutions of split equilibrium problem, variational inequality problem and fixed point problem for a nonexpansive mapping

5.1 Introduction

In 2008, Plubtieng and Punpaeng [148] introduced the general iterative method for approximating a common solution of EP(3.1.4), VIP(2.1.2) and FPP for a nonexpansive mapping. Since then a number of papers appeared related to the study of iterative methods for these problems, see for instance [42, 86, 88, 105–107, 145, 170].

Motivated by the work of Plubtieng and Punpaeng [148] and by the ongoing research in this direction, we suggest and analyze a Halpern-type iterative method for approximating a common solution of $S_{EP}(3.1.4)-(3.1.5)$, VIP(2.1.2) and FPP for a nonexpansive mapping in real Hilbert spaces. Further, we prove that the sequences generated by the iterative scheme converge strongly to a common solution of $S_{EP}(3.1.4)-(3.1.5)$, VIP(2.1.2) and FPP for a nonexpansive mapping. Furthermore, we derive some consequences from our main result. Some of them are new. The results and iterative method presented in this chapter generalize and extend the corresponding result and iterative method due to Takashashi and Toyoda [163].

5.2 Iterative method

We prove a strong convergence theorem based on the Halpern-type iterative method for computing an approximate common solution of $S_{EP}(3.1.4)-(3.1.5)$, VIP(2.1.2) and
FPP for a nonexpansive mapping in real Hilbert spaces.

Assume that \( \text{Sol}(\text{SpEP}(3.1.4)-(3.1.5)) \neq \emptyset \).

**Theorem 5.2.1.** Let \( H_1 \) and \( H_2 \) be two real Hilbert spaces and \( C \subseteq H_1 \) and \( Q \subseteq H_2 \) be nonempty, closed and convex sets. Let \( A : H_1 \to H_2 \) be a bounded linear operator and let \( D : C \to H_1 \) be a \( \tau \)-inverse strongly monotone mapping. Assume that \( F_1 : C \times C \to \mathbb{R} \) and \( F_2 : Q \times Q \to \mathbb{R} \) are bifunctions satisfying Assumption 2.2.1 and \( F_2 \) is upper semicontinuous in first argument. Let \( T : C \to C \) be a nonexpansive mapping such that \( \Upsilon := \text{Fix}(T) \cap \text{Sol}(\text{SpEP}(3.1.4)-(3.1.5)) \cap \text{Sol}(\text{VIP}(2.1.2)) \neq \emptyset \). For a given \( x_0 = v \in C \) arbitrarily, let the iterative sequences \( \{u_n\} \), \( \{x_n\} \) and \( \{y_n\} \) be generated by

\[
\begin{align*}
u_n &= T_{r_n}^{F_1}(x_n + \delta A^*(T_{r_n}^{F_2} - I)Ax_n), \\
y_n &= PC(u_n - \lambda_n Du_n), \\
x_{n+1} &= \alpha_n v + \beta_n x_n + \gamma_n Ty_n,
\end{align*}
\]

where \( r_n \subset (0, \infty) \), \( \lambda_n \in (0, 2\tau) \) and \( \delta \in (0, 1/L) \), \( L \) is the spectral radius of the operator \( A^*A \) and \( A^* \) is the adjoint of \( A \) and \( \{\alpha_n\} \), \( \{\beta_n\} \) and \( \{\gamma_n\} \) are the sequences in \((0, 1)\) satisfying the following conditions:

(i) \( \alpha_n + \beta_n + \gamma_n = 1 \);

(ii) \( \lim_{n \to \infty} \alpha_n = 0 \) and \( \sum_{n=0}^{\infty} \alpha_n = \infty \);

(iii) \( 0 < \lim \inf_{n \to \infty} \beta_n \leq \lim \sup_{n \to \infty} \beta_n < 1 \);

(iv) \( \lim \inf_{n \to \infty} r_n > 0 \), \( \sum_{n=1}^{\infty} |r_{n+1} - r_n| < +\infty \);

(v) \( \lim_{n \to \infty} \left( \frac{\gamma_{n+1}}{1 - \beta_{n+1}} - \frac{\gamma_n}{1 - \beta_n} \right) = 0 \);

(vi) \( 0 < \lim \inf_{n \to \infty} \lambda_n \leq \lim \sup_{n \to \infty} \lambda_n < 2\alpha \) and \( \lim_{n \to \infty} |\lambda_{n+1} - \lambda_n| = 0 \).

Then the sequence \( \{x_n\} \) converges strongly to \( z \in \Upsilon \), where \( z = P_{\Upsilon}v \).

**Proof.** Let \( p \in \Upsilon \), we have \( p = T_{r_n}^{F_1}p \) and \( Ap = T_{r_n}^{F_2}Ap \). As estimated (3.3.8), we obtain

\[
||u_n - p||^2 \leq ||x_n - p||^2 + \delta (L\delta - 1) \| (T_{r_n}^{F_2} - I)Ax_n \|^2.
\]
Since $\delta \in (0, \frac{1}{L})$, we obtain

$$\|u_n - p\|^2 \leq \|x_n - p\|^2. \quad (5.2.3)$$

Now, we can easily estimate

$$\|y_n - p\|^2 \leq \|x_n - p\|^2. \quad (5.2.4)$$

Further, we estimate

$$\|x_{n+1} - p\| = \|\alpha_n v + \beta_n x_n + \gamma_n Ty_n - p\|$$

$$\leq \alpha_n \|v - p\| + \beta_n \|x_n - p\| + \gamma_n \|Ty_n - p\|$$

$$\leq \alpha_n \|v - p\| + \beta_n \|x_n - p\| + \gamma_n \|y_n - p\|$$

$$\leq \alpha_n \|v - p\| + \beta_n \|x_n - p\| + \gamma_n \|x_n - p\|$$

$$\leq \alpha_n \|v - p\| + (1 - \alpha_n) \|x_n - p\|$$

$$\leq \max \{\|v - p\|, \|x_0 - p\|\} = \|v - p\|. \quad (5.2.5)$$

Hence $\{x_n\}$ is bounded and consequently, we deduce that $\{u_n\}$, $\{y_n\}$ and $\{Ty_n\}$ are bounded. On the other hand, from the nonexpansivity of the mapping $(I - \lambda_n D)$, we have

$$\|y_{n+1} - y_n\| \leq \|u_{n+1} - u_n\| + |\lambda_{n+1} - \lambda_n| \|Du_n\|. \quad (5.2.6)$$

As estimated (3.3.24), we obtain

$$\|u_{n+1} - u_n\| \leq \|x_{n+1} - x_n\| + \delta \|A\| \sigma_n + \delta_n, \quad (5.2.7)$$

where

$$\sigma_n = \left|1 - \frac{r_{n+1}}{r_n}\right| \|T_{r_n}^F Ax_n - Ax_n\|$$

$$\delta_n = \left|1 - \frac{r_{n+1}}{r_n}\right| \|T_{r_n}^F (x_n + \delta A^*(T_{r_n}^F - I) Ax_n) - (x_n + \delta A^*(T_{r_n}^F - I) Ax_n)\|. \quad 103$$
It follows from (5.2.6) and (5.2.7)

\[ \|y_{n+1} - y_n\| \leq \|x_{n+1} - x_n\| + \delta \|A\| \sigma_n + \delta_n + |\lambda_{n+1} - \lambda_n| \|Du_n\|. \] (5.2.8)

Setting \( x_{n+1} = \beta_n x_n + (1 - \beta_n)e_n \), which implies from (5.2.1) that

\[ e_n = \frac{x_{n+1} - \beta_n x_n}{1 - \beta_n} = \frac{\alpha_n v + \gamma_n T y_n}{1 - \beta_n}. \]

Further, it follows that

\[ e_{n+1} - e_n = \frac{\alpha_n v + \gamma_{n+1} T y_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_n v + \gamma_n T y_n}{1 - \beta_n} = \left( \frac{\alpha_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_n}{1 - \beta_n} \right) v + \frac{\gamma_{n+1}}{1 - \beta_{n+1}} (T y_{n+1} - T y_n) + \left( \frac{\gamma_{n+1}}{1 - \beta_{n+1}} - \frac{\gamma_n}{1 - \beta_n} \right) T y_n. \]

Using (5.2.8), we have

\[ \|e_{n+1} - e_n\| \leq \left| \frac{\alpha_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_n}{1 - \beta_n} \right| \|v\| + \frac{\gamma_{n+1}}{1 - \beta_{n+1}} \|y_{n+1} - y_n\| + \left| \frac{\gamma_{n+1}}{1 - \beta_{n+1}} - \frac{\gamma_n}{1 - \beta_n} \right| \|T y_n\| \]

\[ \leq \left| \frac{\alpha_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_n}{1 - \beta_n} \right| \|v\| + \frac{\gamma_{n+1}}{1 - \beta_{n+1}} \left( \|x_{n+1} - x_n\| + \delta \|A\| \sigma_n + \delta_n \right) + \left| \frac{\gamma_{n+1}}{1 - \beta_{n+1}} - \frac{\gamma_n}{1 - \beta_n} \right| \|T y_n\| \]

\[ \leq \left| \frac{\alpha_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_n}{1 - \beta_n} \right| \|v\| + (1 - \alpha_{n+1}) \left( \|x_{n+1} - x_n\| + \delta \|A\| \sigma_n + \delta_n \right) + \left| \frac{\gamma_{n+1}}{1 - \beta_{n+1}} - \frac{\gamma_n}{1 - \beta_n} \right| \|T y_n\| \]

\[ \leq \left| \frac{\alpha_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_n}{1 - \beta_n} \right| \|v\| + \|x_{n+1} - x_n\| + \delta \|A\| \sigma_n + \delta_n \]

\[ + |\lambda_{n+1} - \lambda_n| \|Du_n\| + \left| \frac{\gamma_{n+1}}{1 - \beta_{n+1}} - \frac{\gamma_n}{1 - \beta_n} \right| \|T y_n\|. \]

It follows that
\[\|e_{n+1} - e_n\| \leq \left| \frac{\alpha_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_n}{1 - \beta_n} \right| \|v\| + \|x_{n+1} - x_n\| + \|A\|\|\sigma_n + \delta_n + |\lambda_{n+1} - \lambda_n|\|Du_n\|
+ \left| \frac{\gamma_{n+1}}{1 - \beta_{n+1}} - \frac{\gamma_n}{1 - \beta_n} \right| \|Ty_n\|,\]

which implies that
\[\|e_{n+1} - e_n\| - \|x_{n+1} - x_n\| \leq \left| \frac{\alpha_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_n}{1 - \beta_n} \right| \|v\| + \delta \|A\|\|\sigma_n + \delta_n + |\lambda_{n+1} - \lambda_n|\|Du_n\|
+ \left| \frac{\gamma_{n+1}}{1 - \beta_{n+1}} - \frac{\gamma_n}{1 - \beta_n} \right| \|Ty_n\|,\]

Hence, it follows by conditions (ii)-(vi) that
\[\limsup_{n \to \infty} \left[ \|e_{n+1} - e_n\| - \|x_{n+1} - x_n\| \right] \leq 0. \quad (5.2.9)\]

From Lemma 1.2.8, we get \(\lim_{n \to \infty} \|e_n - x_n\| = 0\) and
\[\lim_{n \to \infty} \|x_{n+1} - x_n\| = \lim_{n \to \infty} (1 - \beta_n)\|e_n - x_n\| = 0. \quad (5.2.10)\]

Now,
\[x_{n+1} - x_n = \alpha_n v + \beta_n x_n + \gamma_n Ty_n - x_n\]
\[= \alpha_n (v - x_n) + \gamma_n (Ty_n - x_n).\]

Since \(\|x_{n+1} - x_n\| \to 0\) and \(\alpha_n \to 0\) as \(n \to \infty\), we obtain \(\|Ty_n - x_n\| \to 0\) as \(n \to \infty\).

It follows from (5.2.2) and Lemma 1.2.9 that
\[\|x_{n+1} - p\|^2 \leq \alpha_n\|v - p\|^2 + \beta_n\|x_n - p\|^2 + \gamma_n\|Ty_n - p\|^2 \leq \alpha_n\|v - p\|^2 + \beta_n\|x_n - p\|^2 + \gamma_n\|y_n - p\|^2 \leq \alpha_n\|v - p\|^2 + \beta_n\|x_n - p\|^2 + \gamma_n\|u_n - p\|^2. \quad (5.2.11)\]
\[
\leq \alpha_n \|v - p\|^2 + \beta_n \|x_n - p\|^2 + \gamma_n \left(\|x_n - p\|^2 + \delta(L\delta - 1)\|T_{r_n}^{F_2} - I\|A\|x_n\|^2\right)
\]
\[
\leq \alpha_n \|v - p\|^2 + (1 - \alpha_n)\|x_n - p\|^2 + \delta(L\delta - 1)\|T_{r_n}^{F_2} - I\|A\|x_n\|^2
\]
\[
\leq \alpha_n \|v - p\|^2 + \|x_n - p\|^2 + \delta(L\delta - 1)\|T_{r_n}^{F_2} - I\|A\|x_n\|^2.
\]

Therefore,
\[
\delta(1 - L\delta)\|T_{r_n}^{F_2} - I\|A\|x_n\|^2 \leq \alpha_n \|v - p\|^2 + \left(\|x_n - p\|^2 - \|x_{n+1} - p\|^2\right)
\]
\[
\leq \alpha_n \|v - p\|^2 + (\|x_n - p\| + \|x_{n+1} - p\|)\|x_n - x_{n+1}\|.
\]

Since \(\delta(1 - L\delta) > 0\), \(\alpha_n \to 0\), and \(\|x_{n+1} - x_n\| \to 0\) as \(n \to \infty\), we obtain
\[
\lim_{n \to \infty} \|T_{r_n}^{F_2} - I\|A\|x_n\| = 0. \quad (5.2.12)
\]

As estimated (3.3.15), we obtain
\[
\|u_n - p\|^2 \leq \|x_n - p\|^2 - \|u_n - x_n\|^2 + 2\delta\|A(u_n - x_n)\|\|T_{r_n}^{F_2} - I\|A\|x_n\| \quad (5.2.13)
\]

It follows from (5.2.11) and (5.2.12) that
\[
\|x_{n+1} - p\|^2 \leq \alpha_n \|v - p\|^2 + \beta_n \|x_n - p\|^2 + \gamma_n \|x_n - p\|^2
\]
\[
\leq \alpha_n \|v - p\|^2 + \beta_n \|x_n - p\|^2 + \gamma_n \left[\|x_n - p\|^2 - \|u_n - x_n\|^2 \right]
\]
\[
+ 2\delta\|A(u_n - x_n)\|\|T_{r_n}^{F_2} - I\|A\|x_n\|
\]
\[
\leq \alpha_n \|v - p\|^2 + (1 - \alpha_n)\|x_n - p\|^2 - \gamma_n \|u_n - x_n\|^2
\]
\[
+ 2\gamma_n\delta\|A(u_n - x_n)\|\|T_{r_n}^{F_2} - I\|A\|x_n\|
\]
\[
\leq \alpha_n \|v - p\|^2 + \|x_n - p\|^2 - \gamma_n \|u_n - x_n\|^2
\]
\[
+ 2\delta\|A(u_n - x_n)\|\|T_{r_n}^{F_2} - I\|A\|x_n\|. \quad (5.2.14)
\]
Therefore,
\[
\gamma_n \|u_n - x_n\|^2 \leq \alpha_n \|v - p\|^2 + (\|x_n - p\|^2 - \|x_{n+1} - p\|^2)
\]
\[
+ 2\delta \|A(u_n - x_n)\| \| (T_{r_n}^{F_2} - I)Ax_n \|
\]
\[
\leq \alpha_n \|v - p\|^2 + (\|x_n - p\| + \|x_{n+1} - p\|)\|x_n - x_{n+1}\|
\]
\[
+ 2\delta \|A(u_n - x_n)\| \| (T_{r_n}^{F_2} - I)Ax_n \|.
\]
Since \(\alpha_n \to 0\), \(\| (T_{r_n}^{F_2} - I)Ax_n \| \to 0\) and \(\|x_{n+1} - x_n\| \to 0\) as \(n \to \infty\), we obtain
\[
\lim_{n \to \infty} \|u_n - x_n\| = 0. \quad (5.2.15)
\]
Next, we have
\[
\|x_{n+1} - p\|^2
\]
\[
\leq \alpha_n \|v - p\|^2 + \beta_n \|x_n - p\|^2 + \gamma_n \|Ty_n - p\|^2
\]
\[
\leq \alpha_n \|v - p\|^2 + \beta_n \|x_n - p\|^2 + \gamma_n \|y_n - p\|^2
\]
\[
\leq \alpha_n \|v - p\|^2 + \beta_n \|x_n - p\|^2 + \gamma_n \left\{ \|P_C(u_n - \lambda_n Du_n) - P_C(p - \lambda_n Dp)\|^2 \right\}
\]
\[
\leq \alpha_n \|v - p\|^2 + \beta_n \|x_n - p\|^2 + \gamma_n \left\{ \|u_n - p\|^2 + \lambda_n (\lambda_n - 2\tau) \|Du_n - Dp\|^2 \right\}
\]
\[
\leq \alpha_n \|v - p\|^2 + \beta_n \|x_n - p\|^2 + \gamma_n \left\{ \|x_n - p\|^2 + \lambda_n (\lambda_n - 2\tau) \|Du_n - Dp\|^2 \right\}
\]
\[
\leq \alpha_n \|v - p\|^2 + \lambda_n \lambda_n (\lambda_n - 2\tau) \|Du_n - Dp\|^2
\]
which yields
\[
\gamma_n \lambda_n (\lambda_n - 2\tau) \|Du_n - Dp\|^2 \leq \alpha_n \|v - p\|^2 + \|x_n - p\|^2 - \|x_{n+1} - p\|^2
\]
\[
\leq \alpha_n \|v - p\|^2 + (\|x_n - p\| + \|x_{n+1} - p\|)\|x_n - x_{n+1}\|.
\]
Since \(\|x_{n+1} - x_n\| \to 0\), \(\alpha_n \to 0\) as \(n \to \infty\), we obtain \(\lim_{n \to \infty} \|Du_n - Dp\| = 0\).
Furthermore, we observe that

\[ \|y_n - p\|^2 = \|PC(u_n - \lambda_n Du_n) - PC(p - \lambda_n Dp)\|^2 \]
\[ \leq \langle y_n - p, (u_n - \lambda_n Du_n) - (p - \lambda_n Dp) \rangle \]
\[ \leq \frac{1}{2} \left\{ \|y_n - p\|^2 + \|(u_n - \lambda_n Du_n) - (p - \lambda_n Dp)\|^2 \right. \\
- \left. \|y_n - u_n\| + \lambda_n(Du_n - Dp)\|^2 \right\} \]
\[ \leq \frac{1}{2} \left\{ \|y_n - p\|^2 + \|u_n - p\|^2 - \|y_n - u_n + \lambda_n(Du_n - Dp)\|^2 \right\}. \]

Hence,

\[ \|y_n - p\|^2 \leq \|u_n - p\|^2 - \|y_n - u_n\|^2 - \lambda_n^2\|Du_n - Dp\|^2 + 2\lambda_n \langle y_n - u_n, Du_n - Dp \rangle \]
\[ \leq \|u_n - p\|^2 - \|y_n - u_n\|^2 + 2\lambda_n\|y_n - u_n\||Du_n - Dp| \]
\[ \leq \|x_n - p\|^2 - \|y_n - u_n\|^2 + 2\lambda_n\|y_n - u_n\||Du_n - Dp|. \]

It follows that

\[ \|x_{n+1} - p\|^2 \leq \alpha_n\|v - p\|^2 + \beta_n\|x_n - p\|^2 + \gamma_n\|T y_n - p\|^2 \]
\[ \leq \alpha_n\|v - p\|^2 + \beta_n\|x_n - p\|^2 + \gamma_n\|y_n - p\|^2 \]
\[ \leq \alpha_n\|v - p\|^2 + \beta_n\|x_n - p\|^2 + \gamma_n \left[ \|x_n - p\|^2 - \|y_n - u_n\|^2 \right. \\
+ 2\lambda_n\|y_n - u_n\||Du_n - Dp| \right] \]
\[ \leq \alpha_n\|v - p\|^2 + (1 - \alpha_n)\|x_n - p\|^2 - \gamma_n\|y_n - u_n\|^2 \\
+ 2\gamma_n\lambda_n\|y_n - u_n\||Du_n - Dp| \]
\[ \leq \alpha_n\|v - p\|^2 + \|x_n - p\|^2 - \gamma_n\|y_n - u_n\|^2 \\
+ 2\gamma_n\lambda_n\|y_n - u_n\||Du_n - Dp|. \]

Therefore, we obtain

\[ \gamma_n\|y_n - u_n\|^2 \leq \alpha_n\|v - p\|^2 + \|x_n - p\|^2 - \|x_{n+1} - p\|^2 \]
\[ + 2\gamma_n\lambda_n\|y_n - u_n\||Du_n - Dp| \]

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\[
\alpha_n \|v - p\|^2 + (\|x_n - p\| + \|x_{n+1} - p\|)\|x_n - x_{n+1}\|
+ 2\gamma_n \lambda_n \|y_n - u_n\|\|Du_n - Dp\|.
\]

Since \(\|x_{n+1} - x_n\| \to 0\), \(\alpha_n \to 0\) as \(n \to \infty\) and \(\lim_{n \to \infty} \|Du_n - Dp\| = 0\), we obtain
\[
\lim_{n \to \infty} \|y_n - u_n\| = 0.
\]

Since, we can write
\[
\|Ty_n - y_n\| \leq \|Ty_n - x_n\| + \|x_n - u_n\| + \|u_n - y_n\|
\]
\[
\to 0 \text{ as } n \to \infty.
\]

Next, we show that \(\limsup_{n \to \infty} \langle v - z, x_n - z \rangle \leq 0\), where \(z = P_\Upsilon v\). To show this inequality, we choose a subsequence \(\{y_{n_i}\}\) of \(\{y_n\}\) such that
\[
\limsup_{n \to \infty} \langle v - z, T y_n - z \rangle = \limsup_{i \to \infty} \langle v - z, T y_{n_i} - z \rangle.
\]

Since \(\{y_{n_i}\}\) is bounded, there exists a subsequence \(\{y_{n_{i_j}}\}\) of \(\{y_{n_i}\}\) which converges weakly to some \(w \in C\). Without loss of generality, we can assume that \(y_{n_i} \rightharpoonup w\). Further, from \(\|Ty_n - y_n\| \to 0\), we obtain \(T y_{n_i} \rightharpoonup w\) as \(i \to \infty\).

Now, we see that \(w \in \text{Fix}(T) \cap \text{Sol}(S_P \text{EP}(3.1.4)-(3.1.5)) \cap \text{Sol}(VIP(2.1.2))\), see proof of Theorem 2.3.1 and Theorem 3.3.1.

Next, we claim that \(\limsup_{n \to \infty} \langle v - z, x_n - z \rangle \leq 0\), where \(z = P_\Upsilon v\). Now from (1.2.6), we have
\[
\limsup_{n \to \infty} \langle v - z, x_n - z \rangle = \limsup_{n \to \infty} \langle v - z, T y_n - z \rangle
= \limsup_{i \to \infty} \langle v - z, T y_{n_i} - z \rangle
= \langle v - z, w - z \rangle
\leq 0.
\]
Finally, we show that $x_n \to z$.

\[
\|x_{n+1} - z\|^2 = \left(\alpha_n v + \beta_n x_n + \gamma_n T y_n - z, x_{n+1} - z\right)
\]
\[
\quad = \alpha_n \langle v - z, x_{n+1} - z \rangle + \beta_n \langle x_n - z, x_{n+1} - z \rangle
\]
\[
\quad + \gamma_n \langle T y_n - z, x_{n+1} - z \rangle
\]
\[
\leq \frac{\beta_n}{2} \left\{\|x_n - z\|^2 + \|x_{n+1} - z\|^2\right\} + \frac{\gamma_n}{2} \left\{\|T y_n - z\|^2 + \|x_{n+1} - z\|^2\right\}
\]
\[
\quad + \alpha_n \langle v - z, x_{n+1} - z \rangle
\]
\[
\leq \frac{\beta_n}{2} \left\{\|x_n - z\|^2 + \|x_{n+1} - z\|^2\right\} + \frac{\gamma_n}{2} \left\{\|y_n - z\|^2 + \|x_{n+1} - z\|^2\right\}
\]
\[
\quad + \alpha_n \langle v - z, x_{n+1} - z \rangle
\]
\[
\leq \frac{\beta_n}{2} \left\{\|x_n - z\|^2 + \|x_{n+1} - z\|^2\right\} + \frac{\gamma_n}{2} \left\{\|x_n - z\|^2 + \|x_{n+1} - z\|^2\right\}
\]
\[
\quad + \alpha_n \langle v - z, x_{n+1} - z \rangle
\]
\[
\leq \frac{1 - \alpha_n}{2} \left\{\|x_n - z\|^2 + \|x_{n+1} - z\|^2\right\} + \alpha_n \langle v - z, x_{n+1} - z \rangle
\]
\[
\leq \frac{1}{2} \left\{\|x_n - z\|^2 + \|x_{n+1} - z\|^2\right\} + \alpha_n \langle v - z, x_{n+1} - z \rangle.
\]

This implies that

\[
\|x_{n+1} - z\|^2 \leq (1 - \alpha_n)\|x_n - z\|^2 + 2\alpha_n \langle v - z, x_{n+1} - z \rangle.
\]

Finally, by using (5.2.17) and Lemma 1.2.11, we deduce that $x_n \to z$. This completes the proof.

\[
\square
\]

### 5.3 Consequences

As the consequences of Theorem 5.2.1, we have the following strong convergence results for computing an approximate common solution of EP(2.1.4), VIP(2.1.2) and FPP for a nonexpansive mapping in real Hilbert space. Some of them are new. The following is a special case of Theorem 3.1 of Plubtieng and Punpaeng [148].

**Corollary 5.3.1.** [148] Let $H_1$ be a real Hilbert space and $C \subseteq H_1$ be nonempty, closed and convex set. Let $D : C \to H_1$ be a $\tau$-inverse strongly monotone mapping. Assume
that $F_1 : C \times C \to \mathbb{R}$ is a bifunction satisfying Assumption 2.2.1. Let $T : C \to C$ be a nonexpansive mapping such that $\Upsilon_1 := \text{Fix}(T) \cap \text{Sol}(\text{EP}(3.1.4)) \cap \text{Sol}(\text{VIP}(2.1.2)) \neq \emptyset$. For a given $x_0 = v \in C$ arbitrarily, let the iterative sequences $\{u_n\}$, $\{x_n\}$ and $\{y_n\}$ be generated by

$$
\begin{align*}
&u_n = T_{r_n}F_1 x_n, \\
y_n = \text{P}_C(u_n - \lambda_n Du_n), \\
x_{n+1} = \alpha_n v + \beta_n x_n + \gamma_n T y_n,
\end{align*}
$$

where $r_n \in (0, \infty)$, $\lambda_n \in (0, 2\tau)$ and $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are the sequences in $(0, 1)$ satisfying the conditions (i)-(vi) of Theorem 5.2.1. Then the sequence $\{x_n\}$ converges strongly to $z \in \Upsilon_1$, where $z = \text{P}_{\Upsilon_1} v$.

**Proof.** Taking $T_{r_n}^{F_2} = I$ in Theorem 5.2.1 then the conclusion of Corollary 5.3.1 is obtained.

**Corollary 5.3.2.** Let $H_1$ be a real Hilbert space and $C \subseteq H_1$ be nonempty, closed and convex set. Let $D : C \to H_1$ be a $\tau$-inverse strongly monotone mapping. Let $T : C \to C$ be a nonexpansive mapping such that $\Upsilon_2 := \text{Fix}(T) \cap \text{Sol}(\text{VIP}(2.1.2)) \neq \emptyset$. For a given $x_0 = v \in C$ arbitrarily, let the iterative sequences $\{x_n\}$ and $\{y_n\}$ be generated by

$$
\begin{align*}
&y_n = \text{P}_C(u_n - \lambda_n Du_n), \\
x_{n+1} = \alpha_n v + \beta_n x_n + \gamma_n T y_n,
\end{align*}
$$

where $\lambda_n \in (0, 2\tau)$ and $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are the sequences in $(0, 1)$ satisfying the conditions (i)-(iii) and (v)-(vi) of Theorem 5.2.1. Then the sequence $\{x_n\}$ converges strongly to $z \in \Upsilon_2$, where $z = \text{P}_{\Upsilon_2} v$.

**Proof.** Taking $F_1 = F_2 = 0$ in Theorem 5.2.1 then the conclusion of Corollary 5.3.2 is obtained.

**Corollary 5.3.3.** Let $H_1$ be a real Hilbert space and $C \subseteq H_1$ be nonempty, closed and convex set. Assume that $F_1 : C \times C \to \mathbb{R}$ is a bifunction satisfying Assumption 2.2.1. Let $T : C \to C$ be a nonexpansive mapping such that $\Upsilon_3 := \text{Fix}(T) \cap \text{Sol}(\text{EP}(3.1.4)) \neq \emptyset$. For a given $x_0 = v \in C$ arbitrarily, let the iterative sequences $\{u_n\}$ and $\{x_n\}$ be generated

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by

\[
\begin{align*}
    u_n &= T_{r_n}^{F_1} x_n, \\
    x_{n+1} &= \alpha_n v + \beta_n x_n + \gamma_n T u_n,
\end{align*}
\]

where \( r_n \subset (0, \infty) \) and \( \{\alpha_n\}, \{\beta_n\} \) and \( \{\gamma_n\} \) are the sequences in \( (0,1) \) satisfying the conditions (i)-(v) of Theorem 5.2.1. Then the sequence \( \{x_n\} \) converges strongly to \( z \in \Upsilon_3 \), where \( z = P_{\Upsilon_3} v \).

**Proof.** Taking \( T_{r_n}^{F_2} = I \) and \( D = 0 \) in Theorem 5.2.1 then the conclusion of Corollary 5.3.3 is obtained.

The following Corollary is due to Takahashi and Toyoda [163].

**Corollary 5.3.4.** [163] Let \( H_1 \) be a real Hilbert space and \( C \subseteq H_1 \) be nonempty, closed and convex set. Let \( D : C \to H_1 \) be a \( \tau \)-inverse strongly monotone mapping. Let \( T : C \to C \) be a nonexpansive mapping such that \( \Upsilon_4 := \text{Fix}(T) \cap \text{Sol}(\text{VIP}(2.1.2)) \neq \emptyset \). For a given \( x_0 = v \in C \) arbitrarily, let the iterative sequences \( \{x_n\} \) and \( \{y_n\} \) be generated by

\[
\begin{align*}
    y_n &= P_C(x_n - \lambda_n Dx_n), \\
    x_{n+1} &= \beta_n x_n + (1 - \beta_n) T y_n,
\end{align*}
\]

where \( \lambda_n \in (0,2\tau) \) and \( \{\beta_n\} \) is a sequence in \( (0,1) \) satisfying the conditions (i), (iii), (v) and (vi) of Theorem 5.2.1. Then the sequence \( \{x_n\} \) converges strongly to \( z \in \Upsilon_4 \), where \( z = P_{\Upsilon_4} v \).

**Proof.** Taking \( F_1 = F_2 = 0 \) and \( \alpha_n = 0 \) in Theorem 5.2.1 then the conclusion of Corollary 5.3.4 is obtained.