3.1. Introduction

Numerous GP formulations and their implementations have been established ([11], [12], [21], [37], [46], [53], [63], [72], [77]). The relatively new field of GP—under preemptive priority structure—has actually increased the application potential of GP to many areas of real-world problems. Although GP can successfully be applied for taking decision, the decision maker may often be required the dual of the GP approach, as with the LP, in some realistic situations. In this direction, a step has been taken by Ignizio [37]. An attempt has been made by Dauer and Krueger [18] to improve the previous dual approach in GP.

The content of this chapter is based on one of our papers presented at the 12-th CASAM-CU National Research Symposium (Basu and Pal [6]).
Regarding dual approach of Dauer and Krueger [18], two vital points generally arise: (i) in any realistic situation, a number of goals may appear as equally important. Therefore, according to their model, if distinct priority is given to each goal and goals are successively considered for their attainment without sacrificing the degree of achievement of a already processed goal, then there is a goal dominating possibility; (ii) due to a set of rigid constraints, all the desired goals may not be achieved in reality. Thus, their solution process is usually indicative of the cardinal values of the incommensurable goals rather than establishing a hierarchy of importance among the incompatible goals.

In this chapter, a dual method for GP with preemptive priority structure is developed which can produce the ordinal solution. That is, the decision can be made according to the priority of a goal or a set of goals. To attain the goals according to their priorities, this dual approach is iterative in nature.

3.2. The Procedure

The general GP model with preemptive priority structure can be mathematically expressed as:
minimize \(: P_1(w_{1i}^- d_{1i}^- + w_{1i}^+ d_{1i}^+),\)

minimize \(: P_2(w_{12}^- d_{12}^- + w_{12}^+ d_{12}^+),\)

\[ \vdots \]

\[ \vdots \]

\[ \vdots \]

\[ \vdots \]

\[ \vdots \]

minimize \(: P_k(w_{ik}^- d_{ik}^- + w_{ik}^+ d_{ik}^+),\)

\[ \vdots \]

\[ \vdots \]

\[ \vdots \]

\[ \vdots \]

minimize \(: P_k(w_{ik}^- d_{ik}^- + w_{ik}^+ d_{ik}^+); i = 1,2, \ldots, m; K \leq m\)

subject to :

\[ \sum_{j=1}^{n} a_{ij} x_j + d_{1i}^- - d_{1i}^+ = b_i; i = 1,2, \ldots, m\]

\[ x_j, d_{1i}^-, d_{1i}^+ \geq 0; j = 1,2, \ldots, n\]

\[ i = 1,2, \ldots, m\]

\[ \ldots \quad (3-1)\]

All the notations in (3-1) are defined in (1-1).

In a decision-making situation, the most common desirement of a decision maker is to achieve all the
goals fully for which the value of the function of the deviational variables at each priority level of the objective function in (3-1) is to be zero. Although it may not always be possible to achieve all the goals fully in a realistic situation, the dual formulation is based on this perception of the decision maker.

Now, to develop the dual of the model in (3-1), it is assumed that $P_1, P_2, \ldots, P_K$ are the number of goals at the priority levels $P_1, P_2, \ldots, P_K$, respectively; where $\sum_{k=1}^{K} p_k = m$. The dual procedure is presented below.

At the first stage, the attainment problem for the goals at the first priority level ($P_1$) is considered and transformed to the dual optimization problem $Z_1$ as:

\[
\text{maximize : } Z_1 = -\left( \sum_{i=1}^{R} b_i y_i + \sum_{i=p_1}^{m} b_i^* x_i \right)
\]

subject to:

\[
\sum_{i=1}^{R} a_{ij} y_i + \sum_{i=p_1}^{m} a_{ij}^* x_i \geq 0
\]

\[
y_i \geq 0; \quad i = 1, 2, \ldots, p_1
\]

\[
\ldots \quad (3-2)
\]

where $y_i, i = 1, 2, \ldots, p_1$, are the dual variables correspond
to the respective goals. $\alpha_i, (|\alpha_i| \leq \max (w_{ik}^-, w_{ik}^+))$
at the k-th priority level, $k = 2, 3, \ldots, K$),
i = p_1+1, p_1+2, \ldots, m$, are the parameters correspond to
the respective goals appearing in (3-1) at the successive
priority levels.

The problem in (3-2) is a LP model. By forecasting
the values of the parameters within their given ranges,
this dual problem can offer the values of the actual
decision variables by attaining its optimal solution.

At the second stage of the procedure, the goals at
the second priority level ($P_2$) are to be considered for
their attainment where it is desired not to sacrifice
the degree of achievement of each goal at the first
priority level ($P_1$). Therefore, at this stage, the goals
at the priority level $P_1$ appear as:

$$\sum_{j=1}^{n} ai_j x_j = (b_1 + d_1^+ - d_1^-); i = 1, 2, \ldots, p_1$$

Now, to attain the goals at the priority level $P_2$,
the dual optimization problem $Z_2$ takes the form:

$$\text{maximize } Z_2 = -\left( \sum_{i=1}^{B} (b_1 + d_1^+ - d_1^-) y_1 \right)$$

$$+ \sum_{i=\hat{p}_1}^{\hat{p}_2} b_i y_1 + \sum_{i=\hat{p}_1}^{\hat{p}_1} b_i \alpha_i$$
subject to:

\[
\sum_{i=1}^{n+p_b} a_{ij} y_i + \sum_{i=p_1+p_2}^{m} a_{ij} \alpha_i \geq 0
\]

\[
\sum_{i=1}^{m} \alpha_i > 0
\]

where \( y_i, i = 1, 2, ..., p_1 \), are unrestricted in sign, and \( y_i \geq 0; i = p_1+1, p_1+2, ..., p_1+p_2 \). Also, \( \alpha_i, i = p_1+p_2+1, p_1+p_2+2, ..., m \), can be defined as the problem in (3-2).

The dual problem in (3-4) can be analysed in an analogous to the problem in (3-2). The newly amended actual decision variables must satisfy the goals at the priority level \( P_2 \) as well as the goals at the priority level \( P_1 \).

Proceeding inductively it follows that the attainment problem for the goals at the priority level \( P_{K-1} \) can be transformed to the dual optimization problem \( Z_{K-1} \) as:

\[
\text{maximize} : Z_{K-1} = - \left( \sum_{i=1}^{p_1+p_2+...+p_{K-2}} (b_i + d_i^+ - d_i^-) y_i \right) + \sum_{i=p_1+p_2+...+p_{K-1}}^{m} b_i \alpha_i
\]
subject to:

\[
\begin{align*}
& \sum_{i=1}^{P_1 + P_2 + \cdots + P_{K-1}} a_{ij} y_i \\
& + \sum_{i=P_{K-1} + \cdots + P_{K-1}^2}^{m} a_{ij} \alpha_i \geq 0
\end{align*}
\]

where \( y_{i_1}, i = 1, 2, \ldots, p_1 + p_2 + \cdots + p_{K-2}, \) are unrestricted in sign, and \( y_1 \geq 0, i = p_1 + p_2 + \cdots + p_{K-1}^1, p_1 + p_2 + \cdots + p_{K-2}^2, \ldots, p_1 + p_2 + \cdots + p_{K-1}^i \) Also, \( \alpha_i, i = p_1 + p_2 + \cdots + p_{K-2}^i + 2, \ldots, p_1 + p_2 + \cdots + p_{K-1}^i, \ldots, m, \) can be defined as the problem in (3-2).

The amended actual decision variables of the original problem in (3-1), derived from the dual problem in (3-5), satisfy the goals at the priority level \( P_{K-1} \) as well as the goals concerning all the higher priority levels. At this stage, it is to be perceived that the solution for satisfying the goals of all but the last priority level \( (P_K) \) must also be the solution for the question of attainment of the goals at the last priority level, because it is
desired not to sacrifice the degree of attainment of the goals of all higher priority levels. Thus, it can be forecast that this is the final stage of the dual procedure, and the solution reflects the degree of achievement of each goal within the given priority structure of the goals.

At each stage of the procedure, it is to be noted that the terms containing the parameters appear as a part of the objective function of the dual problem. In practice, they can be ignored in the objective function because, in attaining the goals, the value of the objective function is not of much significance — the important question is to what extent the various goals are attained within the given priority structure of the goals.

3.2.1. **The Iterative Dual Algorithm**

The general procedure used to construct the iterative dual approach to GP may be summarized via the steps given below.

**Step 1:** Set \( k = 1 \) (where \( k \) is used to represent the priority level \( (P_k) \) under consideration).
Step 2: Establish the dual mathematical formulation for priority level $P_k$. Such a formulation is equivalent to the LP model for $k = 1$ (see (3-2)) only.

Step 3: Solve the LP problem. Let $X_k = (x_1, x_2, \ldots, x_n)$ be the feasible solution of the primal problem (problem in (3-1)) corresponding to the optimal solution of the dual problem $Z_k$.

Step 4: Set $k = k + 1$. If $k \geq K - 1$ (where $K$ is used to represent total priority levels), go to Step 7.

Step 5: Recall the goals up to the priority level $k - 1$ by imposing the values of their corresponding deviational variables (see (3-3)) and pass on to the next priority level (level $P_k$).

Step 6: Go to Step 2.

Step 7: The solution $X_{K-1}^* = (x_1^*, x_2^*, \ldots, x_n^*)$ associated with the dual problem $Z_{K-1}$ is the optimal solution of the original problem.
3.3. An Illustrative Example

To demonstrate the implementation of the proposed dual method, the following problem is considered:

minimize : $P_1(d_1^+ + d_2^+)$,

minimize : $P_2(d_4^-)$,

minimize : $P_3(d_3^-)$

subject to:

$x_1 + x_2 + d_1^- - d_1^+ = 400$

$2x_1 + x_2 + d_2^- - d_2^+ = 500$

$0.4x_1 + 0.3x_2 + d_3^- - d_3^+ = 240$

$x_1 + d_4^- - d_4^+ = 300$

$x_j, d_i^-, d_i^+ \geq 0; \ j = 1, 2; \ i = 1, 2, 3, 4.$

In order to solve this problem by the proposed dual method, the following sub-problems are to be executed:

Problem 1: For the first priority level ($P_1$)

The dual problem can be presented as:
maximize : \( z_1 = -(400y_1 + 500y_2 + 300\alpha_3 + 240\alpha_4) \)

subject to:

\[ y_1 + 2y_2 + 0.3\alpha_3 > 0 \]

\[ y_1 + y_2 + 0.4\alpha_4 > 0 \]

\[ y_1, y_2 \geq 0; |\alpha_i| \leq 1; i = 3, 4. \]

Without loss of generality, considering \( \alpha_3 = -1 = \alpha_4 \)
and using the usual dual-simplex procedure, the problem

gives \( x_1 = 250, x_2 = 0 \). With this result, there is to
proceed to the second priority level (P2).

Problem 2: For the second priority level (P2).

In this problem, the goals at the first priority
level appear as:

\[ x_1 + x_2 = 250 \]

and \( 2x_1 + x_2 = 500 \), since \( d_1^- = 150 \)

and \( d_1^+ = 0 = d_2^- = d_2^+ \).
The dual problem takes the form:

\[ \text{maximize } Z_2 = - (250y_1 + 500y_2 + 300y_3 + 240\alpha_4^+) \]

subject to:

\[ y_1 + 2y_2 + y_3 + 0.4\alpha_4^+ \geq 0 \]
\[ y_1 + y_2 + 0.3\alpha_4^+ \geq 0 \]
\[ y_1, y_2 \text{ are unrestricted in sign,} \]
\[ \text{and } y_3 \geq 0; |\alpha_4| \leq 1. \]

Considering \( \alpha_4 = -1/2 \) and using the same solution procedure, the problem yields \( x_1 = 250, x_2 = 0. \) It is apparent from this result that the goal at the second priority level (\( P_2 \)) is attained without sacrificing the degree of attainment of the goals at the first priority level (\( P_1 \)). This solution must also be the solution for attaining the goal at the last priority level (\( P_3 \)) and thereby not sacrificing the degree of attainment of the goals at the higher priority levels.

Hence, the optimal solution of the given problem is:

\[ (x_1^*, x_2^*) = (250, 0). \]
3.4. Conclusion

This chapter establishes that the proposed dual method produces the solution according to the hierarchy of importance of the goals in the decision-making environment. In priority based sense, this dual may be named as 'prioritized dual'. At each stage of the procedure, the dual problem appears as a LP problem, and it possesses all the properties normally associated with the duality in LP. It is to be noted that, in a priority based GP, the multipliers provide a sensitivity analysis of the degree of attainment of the goals at the current priority level with respect to the degree of attainment of the goals at the higher priority levels, whereas in LP the dual variables may be interpreted as shadow prices.

The principal interesting feature of this dual formulation is that it plays a very important role in making decisions to the problems where the number of goals is larger than the number of decision variables, and have to attain multiple, competitive and often conflicting goals with varying priorities.
It is to be noted that, in a realistic situation, it may not always be possible to attain all the goals to the extent desired with the scarce resources. But the proposed dual can provide the managerial decision under the given priority structure of the goals.
PART II

GOAL PROGRAMMING: FORMULATIONS TO SOME REAL-WORLD PROBLEMS