CHAPTER - I

FUNCTIONS OF BOUNDED PROXIMAL VARIATION

1. Introduction

In this chapter we introduce and study the notion of functions of bounded proximal variation, (PVB) functions, which is a concept intermediate between functions of bounded variation and functions of generalized bounded variation. We distinguish an important subclass of the (PVB) functions as the class of proximally absolutely continuous functions, (PAC) functions.

In the various existing theories of integrals the functions of bounded variation in general, and the absolutely continuous functions and generalized absolutely continuous functions in particular, play an important role. Similar role will be played by the (PVB) functions and (PAC) functions in our theory of integration.
Perhaps the most important single result of this chapter is Theorem 4.6 which generalizes the classical Banach-Zarecki theorem. We also show, among other things, that the class of (PAC) functions on any given set $E$ form a linear space of functions satisfying Lusin's condition (N) on $E$, wider than the class of generalized absolutely continuous functions. The value of the notion of (PAC) functions is aptly substantiated by an interesting counterexample (Example 5.1).

2. Notations and Conventions

Throughout this work, we shall adhere to the following notations and conventions.

$\mathbb{R} =$ The real line.

$\mathbb{R}^* =$ The extended real line.

$|E| =$ The outer Lebesgue measure of a subset $E \subset \mathbb{R}$.

$\overline{E} =$ The closure of a subset $E \subset \mathbb{R}$ in $\mathbb{R}$.

$E^0 =$ The interior of a subset $E \subset \mathbb{R}$ in $\mathbb{R}$. 
The following particular symbols

\[i, j, k, m, n\]

will always denote arbitrary positive integers. By an interval we shall always mean a bounded nondegenerate connected subset of \(\mathbb{R}\) such as: \((a, b)\) (open interval), \([a, b]\) (closed interval), \([a, b)\) and \((a, b]\) (semi-closed intervals), where \(a, b \in \mathbb{R}\) and \(a < b\).

Unless indicated otherwise, all topological statements in \(\mathbb{R}\) shall refer to the usual topology on \(\mathbb{R}\). By a set we shall mean a subset of \(\mathbb{R}\), and by a number (or, constant) we shall mean a real number, unless otherwise stated. An abstract set will be called countable if it is either finite (possibly void) or denumerable. A property defined pointwise will be said to hold nearly everywhere (n.e.) \([\text{resp. almost everywhere (a.e.)}, \text{ or for almost all points}]\) on a set \(E\), if it holds at all points of \(E\) except those of a countable subset \([\text{resp. a subset of measure zero}]\). The domain of a sequence shall always be taken to be the set of all positive integers, and a sequence will be specified by showing its range in the form \(\{x_n\}\).
For the empty set $\emptyset$, we shall tacitly follow the definitions:

$$\inf \emptyset = \infty, \quad \sup \emptyset = -\infty, \quad \sum_{x \in \emptyset} x = 0.$$

Besides the usual notation $f: E \to \mathbb{R}^*$ for a function $f$ defined on a set $E$ into $\mathbb{R}^*$, we shall also use the notations

$$f \supset E \to \mathbb{R} \quad [\text{resp. } f \cap E \to \mathbb{R}]$$

to mean that $f$ is an extended real valued function defined and finite at least for all [resp. almost all] points of the set $E$. In any case, the actual domain of the function $f$ will be tacitly assumed to be a nonempty subset of $\mathbb{R}$. Contrary to the usual practice, in statements like "$f$ is ....... on $E$" concerning a given function $f$, we shall always mean that it is the function $f$ itself, and not its restriction to $E$, which is under consideration.

Given $f: E \to \mathbb{R}^*$ and $A \subseteq \mathbb{R}$, the restriction of $f$ to $A$ will be denoted as usual by $f|A$, and $f_A$ will denote the function defined on $\mathbb{R}$ as follows:
Given \( f \ni E \to \mathbb{R}, |E| > 0 \), we denote by \( \text{ess-sup}_E f \) [resp. \( \text{ess-inf}_E f \)] the infimum [resp. supremum] of the numbers \( r \) such that

\[
\left| \left\{ x \in E \mid f(x) > r \right\} \right| = 0 \quad \text{[resp.} \quad \left| \left\{ x \in E \mid f(x) < r \right\} \right| = 0 \].
\]

The function \( f \) is said to be essentially bounded above [resp. below] on \( E \) if \( \text{ess-sup}_E f < \infty \) [resp. \( \text{ess-inf}_E f > -\infty \)].

3. Preliminaries

**DEFINITION 3.1** (Sarkhel and De [34], section 5) A finite family (possibly empty) of pairwise disjoint open-intervals with end points on a set \( E \) is called a subdivision of \( E \). A sequence \( \{E_n\} \) of sets whose union is \( E \) is called an \( E \)-form with parts \( E_n \); if, moreover, each part \( E_n \) is closed in \( E \), then the \( E \)-form is said to be closed. An expanding \( E \)-form is called an \( E \)-chain.
DEFINITION 3.2. Given a function \( f: E \rightarrow \mathbb{R} \) and a number \( r > 0 \), we define

\[
V_+(f,E;r) = \sup \sum_i (f(b_i) - f(a_i)),
\]

\[
V(f,E;r) = \sup \sum_i |f(b_i) - f(a_i)|,
\]

where the suprema are taken for all subdivisions \( \{ (a_i, b_i) \} \) of \( E \) with \( \sum_i (b_i - a_i) < r \). We also define

\[
V_+(f,E;0) = \inf_{r > 0} V_+(f,E;r),
\]

\[
V(f,E;0) = \inf_{r > 0} V(f,E;r),
\]

\[
V(f,E) = \sup_{r > 0} V(f,E;r).
\]

**NOTE.** If \( 0 \leq r_1 \leq r \) and \( E_1 \subseteq E \), then we have

\[
0 \leq V_+(f,E_1;r_1) \leq V_+(f,E;r) \leq V(f,E;r) \leq \infty,
\]

\[
0 \leq V(f,E_1;r_1) \leq V(f,E;r) \leq V(f,E) \leq \infty.
\]

In the above notations and terminologies, the classical definitions of functions of bounded variation
and absolutely continuous functions, and their generalizations, assume the following convenient forms. The function $f$ is absolutely continuous, $AC$, $\text{resp. of bounded variation, } VB$, on $E$ if $V(f,E;0) = 0$ $\text{resp. } V(f,E) < \infty$ (Saks [28], pp.221,223). It is $AC$ above $\text{resp. } AC$ below] on $E$ if $V_+(f,E;0) = 0$ $\text{resp. } V_+(f,E;0) = 0$; it is $(ACG)$ above $\text{resp. } (ACG)$ below] on $E$ if it is $AC$ above $\text{resp. } AC$ below] on each part of some closed $E$-form; it is generalized absolutely continuous, $(ACG)$, on $E$ if it is both $(ACG)$ above and $(ACG)$ below on $E$ (Ridder [27]). The terms $ACG$ above, $ACG$ below and $ACG$ are defined similarly, without requiring the $E$-forms to be 'closed'. (We do not follow Saks' definition of $ACG$ ([28], p.223), which requires $f\big|_E$ to be continuous)

**DEFINITION 3.3.** A function $f: E \rightarrow \mathbb{R}$ is said to be (VBG) on $E$, if it is VB on each part of some closed $E$-form.

**DEFINITION 3.4.** ([28], p.224) A function $f: E \rightarrow \mathbb{R}$ is said to satisfy Lusin's Condition (N) on $E$, if $|f(H)| = 0$ for every $H \subset E$ with $|H| = 0$.

We prove here a lemma for future use.
LEMMA 3.1. For every closed $E$-form $\{E_n\}$, there is a closed $E$-chain $\{F_n\}$ determined by a double sequence of sets $F_{kn}$ closed in $E$, $k \leq n$, such that

1. $F_n = \bigcup_{k \leq n} F_{kn}$, $n = 1, 2, 3, \ldots$,

2. $F_{kn} \subseteq F_{km} \subseteq E_k$ whenever $m \geq n \geq k$,

3. $\text{dist}(F_{in}, F_{jn}) \geq \frac{1}{n}$ for $i \neq j$, $n = 1, 2, 3, \ldots$.

(Here dist stands for the usual metric distance.)

PROOF. First set

$E_0 = \emptyset$ and $A_k = \bigcup_{i \leq k} E_{i-1}$, $k = 1, 2, \ldots$.

Then for each ordered pair $(k, n)$, $k \leq n$, define

$$F_{kn} = \left\{ x \in E_k \setminus A_k \left| \text{dist}(A_k, x) \geq \frac{1}{n} \right. \right\}.$$

Now let the sets $F_n$ be defined as in $(c_1)$. Since the sets $E_{i-1}$ are closed in $E$, clearly the sets $A_k$, $F_{kn}$ and $F_n$ are all closed in $E$. Also, it follows at once
from the definition of the sets $F_{kn}$ that the condition (c<sub>2</sub>) is satisfied, and, hence, from the defining condition (c<sub>1</sub>) we also get that $F_n \subseteq F_{n+1}$ for all $n$.

Again, given any $x \in E$, evidently there is a unique $k$ such that $x \in E_k \setminus A_k$. Since $A_k$ is closed in $E$, we have $\text{dist} (A_k, x) > 0$. (We may have $A_k = \emptyset$, for instance if $k = 1$; but recall that $\text{dist} (\emptyset, x) = \infty$ by definition.) So $x \in F_{kn}$ for all sufficiently large $n$. This shows that $E \subseteq \bigcup_{n=1}^{\infty} F_n$. But we clearly have $\bigcup_{n=1}^{\infty} F_n \subseteq \bigcup_{k=1}^{\infty} E_k = E$. Hence $\bigcup_{n=1}^{\infty} F_n = E$. Since, further, the sets $F_n$ are closed in $E$ and $F_n \subseteq F_{n+1}$ for all $n$, it follows that $\{F_n\}$ is a closed $E$-chain.

Finally, if $x \in F_{in}$ and $y \in F_{jn}$ where $i < j$, then $x \in E_i \subseteq A_j$, and, hence, by definition of $F_{jn}$ we have $\text{dist} (x, y) \geq 1/n$. This evidently verifies the condition (c<sub>3</sub>), and the proof ends.

Some results of this chapter will involve the notions of density, approximate continuity and approximate derivative (relative to a set), for which we refer to [28]. We shall have occasion to discuss these notions in
4. (PVB) and (PAC) Functions

Throughout this section we shall deal with arbitrary but fixed functions $f, g : E \rightarrow R$. It should be noted that $af + bg$ and $fg$ are defined at least on $E$, where $a, b$ are constants.

**DEFINITION 4.1.** The infimum of the set of numbers $r$, for which there is at least one $E$-chain $\{E_n\}$ such that

$$\lim_{n \rightarrow \infty} V(f, E_n; 0) \leq r,$$

i.e., $V(f, E_n; 0) \leq r$ for all $n$, is called the **proximal variation** of $f$ on $E$, and it is denoted by $PV(f, E)$. The function $f$ is said to be of **bounded proximal variation** or **proximally variationally bounded**, (PVB), on $E$ if $PV(f, E) < \infty$. If, in particular, $PV(f, E) = 0$ then $f$ is said to be **proximally absolutely continuous**, (PAC), on $E$.

**REMARK.** The properties VB, AC et cetera and (PVB) and (PAC) are all necessarily hereditary. Our notion of
(PAC) is slightly stronger than the notion of PAC introduced recently by Sarkhel and De ([34], Definition 5.1). In fact, \( f \) is PAC on \( E \) if and only if it is (PAC) on a co-countable subset of \( E \). The conditions (PVB) and (PAC) are obviously more general than the conditions VB and AC, respectively.

Our first theorem justifies the terminology 'proximally variationally bounded'.

**Theorem 4.1.** Let \( f \) be (PVB) on \( E \). Then \( f \) is VB on each part of some closed \( E \)-chain of bounded sets; in particular, therefore, \( f \) is (VBG) on \( E \). If, further, the set \( E \) is measurable, then \( f \mid_E \) is measurable and \( f \) possesses a finite approximate derivative, \( (ap)f'(x) \), at almost all points \( x \) of \( E \).

**Proof.** Fix any real number \( e > 0 \). Since \( f \) is (PVB) on \( E \), there exist an \( E \)-chain \( \{ E_n \} \) and a corresponding sequence of numbers \( r_n > 0 \) such that

\[
(1) \quad V(f, E_n; 2r_n) < PV(f, E) + e \quad \text{for all } n.
\]
Set $F_n = E \cap E_n \cap [-n, n]$. Then, since $E_n \subseteq E_{n+1}$ for all $n$, it follows at once that $\{F_n\}$ is a closed $E$-chain of bounded sets.

Now fix an index $n$, and consider first any subdivision

\[
\left\{(a_i, b_i)\right\}
\]

of $F_n$ with \[\sum_i (b_i - a_i) < r_n.\] There is an index $m > n$ such that $E_m$ contains all the end points $a_i, b_i$. Since $a_i, b_i \in F_n$, for each $i$ we can choose points $a'_i, b'_i \in E_n$ arbitrarily close to the points $a_i, b_i$, respectively. If $a_i \in E_n$ [resp. $b_i \in E_n$], we may take $a'_i = a_i$ [resp. $b'_i = b_i$]; also, if $b_i = a_j$ for some $i, j$, then we may take $b'_i = a'_j$. We note that the points $a'_i, b'_i$ necessarily belong to $E_m$, since $m > n$ implies $E_m \supseteq E_n$.

This being so, choosing the points $a'_i, b'_i$ properly, and using modulus inequalities, we readily obtain

\[
\sum_i |f(b'_i) - f(a'_i)| \leq \sum_i |f(b'_i) - f(a'_i)| + 2V(f, E_m; r_m) \leq V(f, E_n; 2r_n) + 2V(f, E_m; 2r_m) \leq 3(PV(f, E) + \epsilon) \quad \text{by} \quad (1).
\]

Consequently we have
Now, fix a finite number of points $c_0, c_1, c_2, \ldots, c_p$ such that $-n = c_0 < c_1 < c_2 < \ldots < c_p = n$ and

$$c_k - c_{k-1} < r_n \quad \text{for} \quad k = 1, 2, \ldots, p.$$ 

The condition (2) plainly implies that $f$ is bounded on each of the sets $F_n \cap [c_{k-1}, c_k]$, and, hence, $f$ is bounded on $F_n$. Let $M_n$ denote the upper bound of $|f|$ on $F_n$. Consider now any subdivision $\{ (a_i, b_i) \}$ of $F_n$. Let $N_k$ denote the set of those indices $i$ for which $a_i, b_i \in F_n \cap [c_{k-1}, c_k], k = 1, 2, \ldots, p$; and let $N_0$ denote the set of those indices $i$ which do not belong to $\bigcup_{k=1}^p N_k$. We observe that, if $i \in N_0$ then the points $a_i$ and $b_i$ belong to two distinct intervals $[c_{j-1}, c_j]$ and $[c_{k-1}, c_k]$. Since, further, the intervals $(a_i, b_i)$ are pairwise disjoint, it follows at once that $N_0$ contains at most $p$ indices. Also, for the index sets $N_k$ we can use (2) in view of (3). Thus we have
So $f$ is VB on each of the sets $F_n$, and this completes the proof of the first part of the theorem.

When the set $E$ is measurable, then the condition (VBG) implies that the function $f|_E$ is measurable and, hence by a theorem of Denjoy - Khintchine ([28], p.222), also that $(ap)f'(x) = (ap)(f|_E)'(x)$ exists finitely at almost all points $x$ of $E$. This completes the proof of the theorem.

**Lemma 4.1.** For any two constants $a, b$ and for any $r \geq 0$, $V(af + bg, E; r) \leq |a| V(f, E; r) + |b| V(g, E; r)$.

**Proof.** Let $r > 0$. Given any subdivision $\{(a_i, b_i)\}$ of $E$ with $\sum_i (b_i - a_i) < r$, we have
\[ \sum_i \left| (af(b_i) + bg(b_i)) - (af(a_i) + bg(a_i)) \right| \]
\[ \leq |a| \sum_i |f(b_i) - f(a_i)| + |b| \sum_i |g(b_i) - g(a_i)| \]
\[ \leq |a| V(f,E;r) + |b| V(g,E;r). \]

Hence, the required inequality follows at once for \( r > 0 \), and thence for \( r = 0 \) by letting \( r \to 0^+ \).

**THEOREM 4.2.** (i) For any two constants \( a, b \), we have
\[ PV(af+bg,E) \leq |a| \cdot PV(f,E) + |b| \cdot PV(g,E). \]

(ii) If \( PV(g,E) = 0 \), then \( PV(f+g,E) = PV(f,E) \).

**PROOF.** Given any \( e > 0 \), there exist \( E \)-chains \( \{ A_n \} \) and \( \{ B_n \} \) such that
\[ V(f,A_n;0) \leq PV(f,E) + e \text{ for all } n \]
and
\[ V(g,B_n;0) \leq PV(g,E) + e \text{ for all } n. \]
Let now $E_n = A_n \cap B_n$, then it is easily seen that \{E_n\} is an $E$-chain. Further, using Lemma 4.1 and noting that $E_n$ is a subset of both $A_n$ and $B_n$, we have, for each $n$,

\[ V(af + bg, E_n; 0) \leq |a| \cdot V(f, E_n; 0) + |b| \cdot V(g, E_n; 0) \]

\[ \leq |a| \cdot V(f, A_n; 0) + |b| \cdot V(g, B_n; 0) \]

\[ \leq |a| \cdot PV(f, E) + |b| \cdot PV(g, E) + e_1, \]

where $e_1 = (|a| + |b|).e$. This evidently implies (i).

Next, by (i) we have

\[ PV(f, E) \leq PV(f + g, E) + PV(-g, E) \]

\[ \leq PV(f, E) + PV(g, E) + PV(-g, E). \]

This gives (ii), since it follows at once from the definition that $PV(g, E) = 0$ implies $PV(-g, E) = 0$.

The following immediate corollary to Theorem 4.2 implies the important fact that the class of functions
defined and (PVB) \[\text{resp. (PAC)}\] on a given set form a linear space.

**COROLLARY 4.2.1.** If \(f \) and \(g\) are both (PVB) \[\text{resp. (PAC)}\] on \(E\), then \(af + bg\) is also (PVB) \[\text{resp. (PAC)}\] on \(E\).

**THEOREM 4.3.** If both the functions \(f\) and \(g\) are bounded and (PVB) on \(E\), then their product \(fg\) is (PVB) on \(E\).

**PROOF.** Let \(M\) be a common upper bound of \(|f|\) and \(|g|\) on \(E\). Then for any two points \(a, b \in E\) we have

\[
|f(b)g(b) - f(a)g(a)| = |g(b)(f(b)-f(a)) + f(a)(g(b)-g(a))| \\
\leq M \left( |f(b)-f(a)| + |g(b) - g(a)| \right).
\]

(1)

Now, there exist \(E\)-chains \(\{A_n\}\) and \(\{B_n\}\) such that

\[
V(f, A_n; 0) < PV(f, E) + 1 \text{ for all } n.
\]
and

\[ V(g, B_n; 0) \leq PV(g, E) + 1 \text{ for all } n. \]

Considering the E-chain \( \{ E_n \} \), \( E_n = A_n \cap B_n \), and using (1), we readily obtain that, for each \( n \),

\[ V(f, E_n; 0) \leq M_n (V(f, A_n; 0) + V(g, B_n; 0)) \]

\[ \leq M_n (PV(f, E) + PV(g, E) + 2), \]

whence the theorem follows at once.

THEOREM 4.4. For any closed E-form \( \{ E_n \} \), we have \( PV(f, E) \leq \sum_{n=1}^{\infty} PV(f, E_n) \).

PROOF. Given any \( \varepsilon > 0 \), for each \( k \) there exist an \( E_k \)-chain \( \{ E_{kn} \} \) and a corresponding sequence \( \{ r_{kn} \} \) of positive numbers such that

(1) \( V(f, E_{kn}; r_{kn}) \leq PV(f, E_k) + 2^{-k} \cdot \varepsilon \) for all \( n \).
Now, considering the closed E-chain \( \{ F_n \} \) furnished by Lemma 3.1 corresponding to the closed E-form \( \{ E_n \} \) under consideration, and setting

\[
H_n = \bigcup_{k \leq n} (F_{kn} \cap E_{kn}), \quad n = 1, 2, \ldots,
\]

it is easily seen from the conditions \((c_1)\) and \((c_2)\) of Lemma 3.1 that \( \{ H_n \} \) is an E-chain.

Let us put, for each \( n \),

\[
r_n = \min \left\{ \frac{1}{n}, r_{1n}, r_{2n}, \ldots, r_{nn} \right\}.
\]

If \( \{(a_p, b_p)\} \) is any subdivision of the set \( H_m \), \( m \) fixed with \( \sum_p (b_p - a_p) < r_m \), then, since by the condition \((c_2)\) of Lemma 3.1 we have \( \text{dist}(F_{1m}, F_{jm}) \geq \frac{1}{m} \geq r_m \) for \( i \neq j \), the end points of an interval \((a_p, b_p)\) must both belong to precisely one of the sets \( F_{km} \cap E_{km} \), where \( k \leq m \), and so taking note of the definition of the number \( r_m \) and using (1), we clearly have
\[
\sum_{p} |f(b_p) - f(a_p)| \leq \sum_{k \leq m} V(f, F_{km} \cap E_{km}; r_{km})
\]

\[
\leq \sum_{k \leq m} V(f, E_{km}; r_{km}).
\]

\[
\leq \sum_{k \leq m} (PV(f, E_{k}) + 2^{-k}e)
\]

\[
\leq \sum_{n=1}^{\infty} PV(f, E_n) + e.
\]

Hence it follows that

\[
V(f, H_m; r_m) \leq \sum_{n=1}^{\infty} PV(f, E_n) + e \text{ for all } m.
\]

Consequently we have

\[
PV(f, E) \leq \sum_{n=1}^{\infty} PV(f, E_n) + e,
\]

and the proof ends by noting that \(e > 0\) is arbitrary.

This theorem leads to the following important corollaries, of which the first two are immediate.

**COROLLARY 4.4.1.** If there exists a closed E-form \(\{E_n\}\) such that

\[
\sum_{n=1}^{\infty} PV(f, E_n) < \infty,
\]

then \(f\) is necessarily (PVB) on \(E\).
COROLLARY 4.4.2. If $f$ is (PAC) on each part of some closed $E$-form, then $f$ is necessarily (PAC) on $E$.

REMARK. Following the principle of generalization of AC to (ACG), one might be tempted to generalize (PAC) to (PACG). But from Corollary 4.4.2 it is clear that such an endeavor will lead us to nothing new.

COROLLARY 4.4.3. If $f$ and $g$ are both (PAC) on $E$, then their product $fg$ is (PAC) on $E$.

PROOF. By Theorem 4.1, we can find closed $E$-chains $\{A_n\}$ and $\{B_n\}$ such that $f$ is VB on each part $A_n$ and $g$ is VB on each part $B_n$. Considering the closed $E$-chain $\{E_n\}$, where $E_n = A_n \cap B_n$, we see that $f$ and $g$ are both VB on each part $E_n$. Let $M_n$ denote a common upper bound of $|f|$ and $|g|$ on $E_n$. Then for any two points $a, b \in E_n$, we have

$$|f(b)g(b) - f(a)g(a)| = |g(b)(f(b) - f(a)) + f(a)(g(b) - g(a))|$$

$$\leq M_n(|f(b) - f(a)| + |g(b) - g(a)|).$$
Consequently, since \( f \) and \( g \) are both necessarily (PAC) on each part \( E_n \), it follows at once that \( fg \) is (PAC) on each part \( E_n \). Hence, by Corollary 4.4.2, \( fg \) is (PAC) on \( E \).

**COROLLARY 4.4.4.** Let \( I = [a,c] \) and \( J = [c,b] \). Then

\[
PV(f,E\cap(I\cup J)) = PV(f,E \cap I) + PV(f,E \cap J).
\]

**PROOF.** Given any \( e > 0 \), there exist an \( E \cap (I \cup J) \)-chain \( \{ E_n \} \) and a corresponding sequence of positive numbers \( r_n \) such that

\[
(1) \quad V(f,E_n;2r_n) \leq PV(f,E \cap (I \cup J)) + e \quad \text{for all } n.
\]

Evidently \( \{ E_n \cap I \} \) is an \( E \cap I \)-chain and \( \{ E_n \cap J \} \) is an \( E \cap J \)-chain. Also, for each \( n \), we clearly have

\[
V(f,E_n \cap I; r_n) + V(f,E_n \cap J; r_n) \leq V(f,E_n;2r_n).
\]

So, letting \( n \to \infty \) and using (1), it follows that

\[
PV(f,E \cap I) + PV(f,E \cap J) \leq PV(f,E \cap (I \cup J)) + e,
\]
which by Theorem 4.4 completes the proof, since \( e > 0 \) is arbitrary and since \( E \cap I \) and \( E \cap J \) together with a sequence of empty sets determine a closed \( E \cap (I \cup J) \)-form.

**COROLLARY 4.4.5.** Let \( \operatorname{PV}(f, E \cap [a, b]) < \infty \). Then \( \operatorname{PV}(f, E \cap [a, x]) \to 0 \) as \( x \to a^+ \), and \( \operatorname{PV}(f, E \cap [x, b]) \to 0 \) as \( x \to b^- \).

**PROOF.** Select a strictly decreasing sequence \( \{c_n\} \) in \((a, b)\) converging to \( a \), and set

\[
E_1 = E \cap \{a\} \quad \text{and} \quad E_n = E \cap [c_n, c_{n-1}] \quad \text{for } n \geq 2.
\]

Now, Corollary 4.4.4 implies that

\[
\sum_{n=m}^{k} \operatorname{PV}(f, E_n) = \operatorname{PV}(f, E \cap [c_k, c_{m-1}]) \quad \text{for } 2 \leq m < k.
\]

Since \( E \cap [c_k, c_{m-1}] \subseteq E \cap [a, c_{m-1}] \), it follows that

\[
\sum_{n=m}^{\infty} \operatorname{PV}(f, E_n) \leq \operatorname{PV}(f, E \cap [a, c_{m-1}]) \quad \text{for all } m \geq 2.
\]

Therefore, since obviously \( \operatorname{PV}(f, E_1) = 0 \), it follows from Theorem 4.4 that
In particular, we have \( \sum_{n=1}^{\infty} PV(f, E_n) = \sum_{n=1}^{\infty} PV(f, E \cap [a, c_m]) \) for all \( m \geq 2 \). Therefore, given any \( \epsilon > 0 \), there is an \( m > 2 \) for which the left-hand side of (1) is less than \( \epsilon \). Then for all \( x \in (a, c_m) \) we have, by (1), \( PV(f, E \cap [a, x]) \leq PV(f, E \cap [a, c_{m-1}]) < \epsilon \). Hence \( PV(f; E \cap [a, x]) \rightarrow 0 \) as \( x \rightarrow a^+ \).

Proof of the other part is analogous.

**THEOREM 4.5.** If \( |E| = 0 \), then \( |f(E)| \leq PV(f, E) \).

**PROOF.** Given any \( \epsilon > 0 \), there exist an \( E \)-chain \( \{ E_n \} \) and a corresponding sequence of positive numbers \( r_n \) such that

\[
(1) \quad V(f; E_n ; r_n) \leq PV(f, E) + \epsilon \quad \text{for all } n.
\]

Now, fix an index \( n \). Since \( E_n \subseteq E \) and \( |E| = 0 \), we have \( |E_n| = 0 \). So the set \( E_n \) can be covered by a countable family \( \{ I_k \} \) of pairwise disjoint open intervals
such that $\sum_{k} |I_k| < r_n$. Since $|f(E_n \cap I_k)|$ cannot exceed the oscillation of $f$ on $E_n \cap I_k$, simple computations show that

$$|f(E_n)| \leq \sum |f(E_n \cap I_k)| \leq V(f, E_n; r_n).$$

Hence it follows from (1) that

$$|f(E_n)| \leq PV(f, E) + e \text{ for all } n.$$

Consequently, since the sequence $\{f(E_n)\}$ is clearly expanding, and since the outer Lebesgue measure is regular, we have (cf. [28], (6.1), p.51)

$$|f(E)| = \left| \bigcup_{\eta=1}^{\infty} f(E_n) \right| = \lim_{\eta \to \infty} |f(E_n)| \leq PV(f, E) + e,$$

and the proof ends by noting that $e > 0$ is arbitrary.

The following corollary is immediate.

COROLLARY 4.5.1. Let $f$ be (PAC) on $E$, then $f$ satisfies Lusin's condition (N) on $E$. 
The classical Banach-Zarecki theorem ([25], p. 250; [28], Theorem 6.7, p. 227) states that, if the set E is closed and bounded and if \( f \mid E \) is continuous, then \( f \) is AC on E if and only if \( f \) satisfies Lusin's condition (N) on E and is VB on E. Our next theorem gives a similar characterization of (PAC) functions.

**THEOREM 4.6 (Generalized Banach-Zarecki theorem).**

Let E be an \( \mathcal{F}_b \) -set. Then \( f \) is (PAC) on E if and only if \( f \) satisfies Lusin's condition (N) on E and is (VBG) on E.

**PROOF.** The 'only if' part follows from Corollary 4.5.1 and Theorem 4.1. We prove below the 'if' part in the particular case when the set E is closed and bounded, and \( f \) is VB on E and satisfies the condition (N) on E. The general case will then follow at once from Corollary 4.4.2.

We first observe that, since \( f \) is VB on E, \( f \mid E \) is necessarily continuous n.e. on E. Let \( d_1, d_2, \ldots, d_n, \ldots \) denote the distinct points of discontinuity, if any, of \( f \mid E \). Consider now any \( e > 0 \). Since \( f \) is VB on E, it is
necessarily (PVB) on $E$. So Corollary 4.4.5 in conjunction with Corollary 4.4.4 implies that, for any point $x$, $\text{PV}(f, E \cap [x-h, x+h]) \to 0$ as $h \to 0^+$. In particular, therefore, for each point $d_n$ we can find a closed interval $I_n$ with $d_n \in I_n^0$ such that

$$(1) \quad \text{PV}(f, E \cap I_n) \leq 2^{-n}.e.$$ 

Now, clearly the set $F = E \setminus \bigcup_n I_n^0$ is closed and bounded and $f|F$ is continuous and VB and satisfies the condition (N) on $F$. So by Banach-Zarecki theorem $f$ is AC on $F$, and hence $\text{PV}(f, F) = 0$. Then by Theorem 4.4 we get

$$\text{PV}(f, E) \leq \text{PV}(f, F) + \sum_n \text{PV}(f, E \cap I_n)$$

$$< 0 + \sum_n 2^{-n}.e \quad \text{by (1)}$$

$$\leq e.$$ 

Letting $e \to 0^+$, we get $\text{PV}(f, E) = 0$. Thus $f$ is (PAC) on $E$, and this completes the proof of the theorem.
COROLLARY 4.6.1. Let \( \{ p_n \} \) and \( \{ q_n \} \) denote two sequences in \( \mathbb{R} \), such that \( \sum_{n=1}^{\infty} |p_n| < \infty \) and \( q_i \neq q_j \) for \( i \neq j \), and let \( F : \mathbb{R} \rightarrow \mathbb{R} \) be defined by (using self-evident notations)

\[
F(x) = \sum_{q_n < x} p_n \quad \text{for all } x \in \mathbb{R},
\]
or

\[
F(x) = \sum_{q_n \leq x} p_n \quad \text{for all } x \in \mathbb{R}.
\]

Then \( F \) is \( (\text{PAC}) \) on \( \mathbb{R} \) and \( F' = 0 \) a.e. on \( \mathbb{R} \).

**PROOF.** There is no distinction between the proofs in the two cases of the definition of the function \( F \). First suppose that \( p_n \geq 0 \) for all \( n \). Then \( F \) is clearly nondecreasing and \( \text{VB} \) on \( \mathbb{R} \). Also, if \( \sum_{n=1}^{\infty} p_n = p \), then

\[
F(\mathbb{R}) = [0,p] \setminus \bigcup_{n=1}^{\infty} (F(q_n^-), F(q_n^+))
\]

and

\[
\sum_{n=1}^{\infty} (F(q_n^+) - F(q_n^-)) = \sum_{n=1}^{\infty} p_n = p,
\]

and hence

\[
|F(\mathbb{R})| = 0.
\]
Hence, by Theorem 4.6, $F$ is (PAC) on $R$. Also, since $F$ is VB on $R$, $F'$ exists finitely a.e. on $R$ and $F'$ is Lebesgue integrable on $R$. But, since $F$ is nondecreasing on $R$, by a known result (Varberg [39], Theorem 12) the Lebesgue integral of $F'$ on $R$ cannot exceed $|F(R)| (=0)$, which implies that $F' = 0$ a.e. on $R$ (Natanson [25], Theorem 6, p.135). This completes the proof in the particular case when $p_n \geq 0$ for all $n$.

Since by Corollary 4.2.1 the difference of two (PAC) functions is again (PAC), the general case follows by noting that $p_n = \frac{1}{2} (|p_n| + p_n) - \frac{1}{2} (|p_n| - p_n)$.

**Theorem 4.7.** If $f$ is (ACG) on $E$, then $f$ is necessarily (PAC) on $E$. The converse is true if the set $E$ is closed and $f|E$ is continuous.

**Proof.** First suppose that $f$ is (ACG) on $E$. Then there is a closed $E$-form on each part of which $f$ is AC. Therefore, since AC implies (PAC), it follows at once from Corollary 4.4.2 that $f$ is (PAC) on $E$, which proves the first part.
To prove the converse part, we note that if $f$ is (PAC) on $E$, then $f$ satisfies the condition (N) on $E$ by Corollary 4.5.1, and by Theorem 4.1 $f$ is VB on each part of some closed $E$-chain $\{E_n\}$ of bounded sets. Therefore, if it is given further that the set $E$ is closed and $f|_E$ is continuous, then we see that, for each $n$, the set $E_n$ is closed and bounded, $f$ satisfies the condition (N) on $E_n$, $f$ is VB on $E_n$, and $f|_{E_n}$ is continuous. Hence by Banach-Zarecki theorem $f$ is AC on each part of the closed $E$-form $\{E_n\}$. Thus $f$ is (ACG) on $E$, and this completes the proof of the theorem.

We shall close this section with a simple but interesting application of the generalized Banach-Zarecki theorem to the composition of two (PAC) functions. The result is analogous to a result of G.M. Fichtenholz (see [25], Theorem 5, p.252), which states that the composition of two absolutely continuous functions defined on closed intervals is absolutely continuous if and only if it is of bounded variation.
THEOREM 4.8. Let the set $E$ be closed and let $f$ be $(PAC)$ on $E$. Also let $F \circ f : E \to \mathbb{R}$ be $(PAC)$ on $f(E)$. Then the composite function $F \circ f$ is $(PAC)$ on $E$ if and only if it is $(V BG)$ on $E$.

PROOF. We first observe that the domain of $F \circ f$ contains the set $E$. Now, the 'only if' part follows from Theorem 4.1. To prove the 'if' part, we first note that by Corollary 4.5.1 the functions $f$ and $F$ satisfy the condition $(N)$ on the sets $E$ and $f(E)$, respectively. Then, for any subset $H \subseteq E$ with $|H| = 0$, we have first $|f(H)| = 0$ and then $|F(f(H))| = 0$. Thus $F \circ f$ satisfies the condition $(N)$ on $E$. Therefore, if it is given further that $F \circ f$ is $(V BG)$ on the closed set $E$, then it follows from Theorem 4.6 that $F \circ f$ is $(PAC)$ on $E$, which completes the proof.

5. Counterexamples

In Theorem 4.7 we have seen that if $f$ is $(ACG)$ on $E$ then $f$ is necessarily $(PAC)$ on $E$, and that the converse is true if the set $E$ is closed and $f|E$ is continuous. We will now show by example how badly this converse part fails
without the additional hypotheses of closedness and continuity, showing thereby also that the notion of (PAC) is substantially more general than the notion of (ACG).

We will also show by example how badly the classical Banach–Zarecki theorem fails without the hypotheses of continuity and closedness.

We first prepare two lemmas.

**LEMMA 5.1.** Let $G$ be an open set having right-hand density 1 at a point $c$. Then there exist two sequences $\left\{ [a_n, b_n] \right\}$ and $\left\{ [p_n, q_n] \right\}$ of closed intervals such that:

(i) $\bigcup_{n=1}^{\infty} [a_n, b_n] \subseteq (c, \infty) \cap G$,

(ii) $c < b_{n+1} < a_n < p_n < q_n < b_n$ for all $n$,

(iii) $\left\{ c \right\} \cup \bigcup_{n=1}^{\infty} [a_n, b_n]$ is closed,

(iv) $\bigcup_{n=1}^{\infty} (p_n, q_n)$ has right-hand density 1 at $c$. 
PROOF. Fix a strictly decreasing sequence \( \{c_n\} \) in
\( G \) converging to \( c \). Since the set \( G \) is open, for each \( n \)
we can easily determine a finite family \( \{[a_{ni}, b_{ni}]\}_n \) of
pairwise disjoint closed intervals contained in
\( G \cap (c_{n+1}, c_n) \) such that
\[
\sum_{i} (a_{ni}, b_{ni}) > |G \cap (c_{n+1}, c_n)| - \frac{1}{n} |(c_{n+2}, c_{n+1})|.
\]
Now, the components of the set \( \bigcup \bigcup [a_{ni}, b_{ni}] \) can
evidently be arranged in a sequence \( \{[a_n, b_n]\} \), such that
\( c < b_{n+1} < a_n \) for all \( n \). For each \( n \), then choose an
interval \( [p_n, q_n] \subset (a_n, b_n) \) such that
\[
| (a_n, b_n) \setminus [p_n, q_n] | < \frac{1}{n} |(a_{n+1}, b_{n+1}) |.
\]
It can now be easily verified that these sequences
\( \{[a_n, b_n]\} \) and \( \{[p_n, q_n]\} \) fulfill all the required
conditions (i) through (iv).

LEMMA 5.2. Given any nonempty perfect set \( P \subset R \),
there exist denumerable dense subsets \( S_0 \) and \( S_1 \) of \( P \) such
that \( S_0 \cap S_1 = \emptyset \), and such that each point of \( S_0 \cup S_1 \) is a limit point of \( L = (S_0 \cup S_1) \) on both sides.

**PROOF.** We first recall the well known facts that the space \( R \) (with the usual metric) is separable, that every subspace of a separable metric space is separable, and that every neighborhood of every point of the perfect set \( P \) contains a continuum of the points of \( P \).

Now, let \( P_0 \) denote the set of the points of \( P \) which are limit points of \( P \) on both sides. Then \( P \setminus P_0 \) is at most denumerable and \( P_0 \) has the power of the continuum. Fix a denumerable dense subset \( S_0 \) of \( P_0 \). Then \( P_0 \setminus S_0 \) has the power of the continuum. Now fix a denumerable dense subset \( S_1 \) of \( P_0 \setminus S_0 \). Then, since \( S_0 \cup S_1 \) is denumerable, evidently \( S_0 \) and \( S_1 \) fulfil all the required conditions.

**EXAMPLE 5.1.** Let \( P \) denote a perfect set of measure 0 contained in a given closed interval \( I = [a, b] \), with \( a, b \in P \). By Lemma 5.2, we can find disjoint denumerable
dense subsets $S_0$ and $S_1$ of $P$ such that each point of $S = S_0 \cup S_1$ is a limit point of $P \setminus S$ on both sides.

Let $\{c_{2n}\}$ and $\{c_{2n-1}\}$ denote enumerations (with distinct terms) of $S_0$ and $S_1$, respectively. Then define

$$v(x) = \sum_{c_n < x} (-2)^{-n} \quad \text{for all } x \in R.$$

The function $v$ is well defined since the series $\sum_{n=1}^{\infty} (-2)^{-n}$ is absolutely convergent. Now, by successive applications of Lemma 5.1, we shall associate with each point $c_k$ two sequences $\left\{\left[ a_{kn}, b_{kn} \right] \right\}$, $\left\{\left[ p_{kn}, q_{kn} \right] \right\}$ of closed intervals such that:

(i) $|v(x) - v(c_k)| < 2 \cdot 2^{-k}$ for all $x \in (c_k, b_{k1})$,

(ii) $\bigcup_{\eta=1}^{\infty} \left[ a_{kn}, b_{kn} \right] \subset (c_k, b) \setminus P$,

(iii) $c_k < b_{km} < a_{kn} < p_{kn} < q_{kn} < b_{kn}$ for all $n, m(> n)$,

(iv) $\bigcup_{\eta=1}^{\infty} (p_{kn}, q_{kn})$ has right-hand density 1 at $c_k$,

(v) $A_i \cap A_j = \emptyset$ if $i \neq j$, where $A_k = \left\{ c_k \right\} \bigcup_{\eta=1}^{\infty} \left[ a_{kn}, b_{kn} \right]$. 


To this end, we first observe that, given any point $c_k$, for every $x > c_k$ we have

$$|v(x) - v(c_k)| = \left| \sum_{c_k < c_n < x} (-2)^{-n} \right| \leq \sum_{c_k < c_n < x} 2^{-n} + 2^{-k}.$$ 

Since the first term on the right tends to 0 as $x \rightarrow c_k^+$, it follows that $|v(x) - v(c_k)| < 2.2^{-k}$ for all $x$ in some right neighborhood of $c_k$.

Now, since $P$ is closed and $|P| = 0$, we see that the set $G_1 = (c_1, b) \setminus P$ is open and it has right-hand density 1 at $c_1$. So, by Lemma 5.1 and by what has been seen above, we can find two sequences $\{[a_{1n}, b_{1n}]\}$ and $\{[p_{1n}, q_{1n}]\}$ of closed intervals contained in $G_1$ such that the conditions (i) through (iv) are satisfied for $k = 1$. In general, when we have already dealt with the points $c_1, c_2, \ldots, c_{k-1}$, $k \geq 2$, we note that the sets $A_i, i < k$, as defined in condition (v) are closed, and hence there is a neighborhood of the point $c_k$ which does not intersect the sets $A_i, i < k$. Since, further, $P$ is closed and $|P| = 0$, we see that the set...
For each $k$, we now define the function $F_k$ on $\mathbb{R}$ by

$$F_k(x) = \begin{cases} 
(v(c_k) - v(x)) \cdot \sin^2 \frac{\pi}{2} \left( \frac{x-a_{kn}}{b_{kn} - a_{kn}} \right) & \text{if } a_{kn} \leq x \leq p_{kn}, \\
(v(c_k) - v(x)) \cdot \sin^2 \frac{\pi}{2} \left( \frac{b_{kn} - x}{b_{kn} - q_{kn}} \right) & \text{if } q_{kn} \leq x \leq b_{kn}, \\
(v(c_k) - v(x)) & \text{if } x \in \bigcup_{n=1}^{\infty} (p_{kn}, q_{kn}), \\
0 & \text{Otherwise}. 
\end{cases}$$
Then we consider the function $F$ defined on $\mathbb{R}$ by

$$F(x) = F_0(x) + v(x), \text{ where } F_0(x) = \sum_{n=1}^{\infty} F_n(x).$$

By (i), the series $\sum_{n=1}^{\infty} F_n(x)$ is uniformly and absolutely convergent on $\mathbb{R}$. Then it is clear that the function $F$ is continuous everywhere, except at the points $c_k$ on the right. But, we have

$$F(x) = v(c_k) = F(c_k) \text{ for all } x \in \bigcup_{n=1}^{\infty} (p_{kn}, q_{kn}),$$

which by (iv) implies that $F$ is approximately continuous at $c_k$ on the right.

Again, the condition (v) implies that, for each $k$,

$$F(x) = F_k(x) + v(x) \text{ for all } x \in \bigcup_{n=1}^{\infty} [a_{kn}, b_{kn}].$$

Therefore $F$ is AC on each of the intervals $[a_{kn}, b_{kn}]$, since $v$ is constant and $F_k$ has bounded derivative on each such interval. Also, $F(x) = 0$ for all $x \leq a$, $F(x) = F(b)$ for all $x > b$, and $F(x) = v(x)$ for all $x \in \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{\infty} (a_{kn}, b_{kn})$. 
and by Corollary 4.6.1 \( v \) is \((\text{PAC})\) on \( R \). Hence Corollary 4.4.2 plainly implies that \( F \) is \((\text{PAC})\) on \( R \).

Consider, on the other hand, any \((\text{P}\setminus\text{S})\)-form \( \{ P_n \} \). Since evidently \( \text{P}\setminus\text{S} \) is a \( \mathcal{G}^\delta \) -set, there exist by Baire category theorem (Saks\([28]\), (9.2), p.54) an open interval \( J \) and an index \( n \), such that \( J \cap \text{P}\setminus\text{S} \neq \emptyset \) and \( P_n \) is dense in \( J \cap \text{P}\setminus\text{S} \). Then, by denseness of the sets \( S_0 \) and \( S_1 \), we can find points \( c_m, c_k \in J \) with \( m \) odd and \( k \) even. We recall that \( c_m \) and \( c_k \) are both limit points of \( \text{P}\setminus\text{S} \) on both sides. Therefore, by denseness of \( P_n, c_m \) and \( c_k \) are both limit points of \( P_n \) on both sides. Since, further, \( F(x) = v(x) \) for all \( x \in P_n \), and since \( v(c_m^+) - v(c_m^-) = -2^{-m} \) and \( v(c_k^+) - v(c_k^-) = 2^{-k} \), it follows at once that \( F \) is neither AC below nor AC above on the set \( P_n \). This shows that \( F \) is neither ACG below nor ACG above on \( \text{P}\setminus\text{S} \). So, \( F \) is neither ACG below nor ACG above on any superset of \( \text{P}\setminus\text{S} \).

Summing up the above results, we have the following properties of the function \( F \):
(1) $F$ is approximately continuous on $\mathbb{R}$ (in particular, $F(x) = 0$ for all $x \leq a$, $F(x) = F(b)$ for all $x > b$, and $F$ is continuous everywhere, except at the denumerable set of points $c_k$ on the right),

(2) $F$ is (PAC) on $\mathbb{R}$, and

(3) $F$ is neither ACG below nor ACG above on $I$, or on any superset of $P \setminus S$, not to speak of being (ACG).

Noting that $I$ is a closed interval, and again that $F|(I \setminus S)$ is continuous, we see how miserably the converse part of Theorem 4.7 fails if one omits any one of the two additional hypotheses therein.

EXAMPLE 5.2. By Lemma 5.2, we can find two disjoint denumerable dense subsets $S_0$ and $S_1$ of $\mathbb{R}$, such that each point of $S = S_0 \cup S_1$ is a limit point of $R \setminus S$ on both sides. Let $\{c_{2n}\}$ and $\{c_{2n-1}\}$ denote enumerations (with distinct terms) of $S_0$ and $S_1$, respectively.

Let us consider the function $F$ defined by
The function $F$ is well defined since the series
$$
\sum_{n=1}^{\infty} (-2)^{-n}
$$
is absolutely convergent.

Clearly $F$ is VB on the entire real line $\mathbb{R}$. Also, by Corollary 4.6.1, $F$ is (PAC) on $\mathbb{R}$; in particular, therefore, $F$ satisfies Lusin's condition (N) on $\mathbb{R}$, by Corollary 4.5.1. On the other hand, given any closed interval $I$, arguing as in the preceding example it is readily seen that $F$ is neither ACG below nor ACG above on $I$, or on any superset of $I \setminus S$, not to speak of being AC. Noting that $I$ is a closed interval, and that $F \big| (I \setminus S)$ is continuous, we see how miserably the classical Banach-Zarecki theorem fails if one omits either the hypothesis of continuity or the hypothesis of closedness.