δ Semi Generalized Star Closed (δsg*-closed) Sets in Topological Spaces

2.1 Introduction

Velicko (1968) introduced δ-open sets which are stronger than open sets and proved that the collection of δ-open sets denoted by τδ formed a coarser topology on (X, τ). Norman Levine (1970) introduced the concept of generalized closed (briefly g-closed) sets. By combining the concepts of δ-closedness and g-closedness, Dontchev and Ganster (1996) proposed a class of generalized closed sets called δg-closed sets. The idea of δ-semi open sets was initiated by Park (1997). Lee (2001) studied its applications. Park (2007) introduced and studied two other concepts namely gδs-closed and δgs-closed sets using δ-semi closure and proved that the class of δgs-closed sets is weaker than the class of gδs-closed sets. Sudha (2012) introduced and investigated a stronger form of δg-closed sets namely δg*-closed sets.

In this chapter, a new class of generalized closed sets called δsg*-closed sets using δ-semiclosure (δ-scl) is introduced. Some relationships like dependency, independency with various existing closed sets are analyzed. The class of δsg*-closed sets is weaker than the class of δ-semi closed sets and is stronger than the classes of gδs-closed sets and δgs-closed sets. Further, δsg*-closure operator, δsg*-interior operator and δsg*-open sets are introduced and their properties are obtained. A new characterization of semi weakly Hausdorff spaces which are the spaces with semi-T\(_{1/2}\)-semi regularization is obtained.

2.2 δsg*-Closed Sets

In this section, a new class of generalized closed sets called δ semi generalized star closed (δsg*-closed) sets is defined and some relations between δsg*-closed sets and various existing closed sets are analyzed.

**Definition 2.2.1** A subset A of a topological space (X,τ) is called δ semi generalized star -closed (briefly δsg*-closed) set if δ-scl(A) ⊆ U whenever A ⊆ U and U is g-open in (X, τ).
The class of all $\delta g^*$-closed sets of $(X, \tau)$ is denoted by $\delta SG^*(X, \tau)$.

**Proposition 2.2.2** Every $\delta$-semi closed set is a $\delta g^*$-closed set but not conversely.

**Proof:** Let $A$ be a $\delta$-semi closed set then $A = \delta$-scl$(A)$. Let $A \subseteq U$ where $U$ is $g$-open then $\delta$-scl$(A) = A \subseteq U$. Therefore $A$ is a $\delta g^*$-closed set.

**Example 2.2.3** Let $X = \{a, b, c, d\}, \tau = \{X, \phi, \{a\}, \{c\}, \{a,b\}, \{a,c\}, \{a,b,c\}, \{a,c,d\}\}$. In this topology, $g$-open sets are open sets. Then the set $\{b,c,d\}$ is $\delta g^*$-closed but not $\delta$-semi closed.

**Lemma 2.2.4** Every $\delta$-closed set is a $\delta$-semi closed set but not conversely.

**Proof:** Let $A$ be $\delta$-open then $A = \text{int}_\delta(A)$

\[\text{cl}(A) = \text{cl}(\text{int}_\delta(A))\]

we know $A \subseteq \text{cl}(A) = \text{cl}(\text{int}_\delta(A))$

\[A \subseteq \text{cl}(\text{int}_\delta(A))\]

Therefore $A$ is $\delta$-semiopen. i.e. a $\delta$-closed set is $\delta$-semi closed.

Hence $\delta$-scl$(A) \subseteq \text{cl}_\delta(A)$.

**Example 2.2.5** Let $X = \{a, b, c\}, \tau = \{X, \phi, \{a\}, \{b\}, \{a,b\}\}$. Then the set $\{a\}$ is $\delta$-semi closed but not $\delta$-closed.

**Theorem 2.2.6** Every $\delta$-closed set is a $\delta g^*$-closed set but not conversely.

**Proof:** The proof follows from Lemma 2.2.4 and Proposition 2.2.2.

**Example 2.2.7** Let $X = \{a, b, c\}, \tau = \{X, \phi, \{a\}\}$. Then the set $\{b,c\}$ is $\delta g^*$-closed but not $\delta$-closed.

**Proposition 2.2.8** Every $\delta g^*$-closed set is a $\delta g^*$-closed set but not conversely.

**Proof:** Let $A$ be a $\delta g^*$-closed set and $U$ be any $g$-open set containing $A$ in $(X, \tau)$. Since $A$ is a $\delta g^*$-closed set, $\text{cl}_g(A) \subseteq U$. From Lemma 2.2.4, $\delta$-scl$(A) \subseteq \text{cl}_\delta(A)$ and hence $\delta$-scl$(A) \subseteq U$ and hence $A$ is a $\delta g^*$-closed set.
**Example 2.2.9** Let \( X = \{a, b, c\}, \tau = \{X, \emptyset, \{a\}, \{b\}, \{a,b\}\} \). Then the set \( \{b\} \) is \( \delta\text{sg}^* \)-closed but not \( \delta\text{g}^* \)-closed.

**Theorem 2.2.10** Every \( \delta\text{sg}^* \)-closed set is a \( \delta\text{gs} \)-closed set but not conversely.

**Proof:** Let \( A \) be a \( \delta\text{sg}^* \)-closed set and \( U \) be any open set containing \( A \) in \((X, \tau)\). Since every open set is \( g \)-open and \( A \) is \( \delta\text{sg}^* \)-closed set, \( \delta\text{scl}(A) \subseteq U \). Hence \( A \) is a \( \delta\text{gs} \)-closed set.

**Example 2.2.11** Let \( X = \{a, b, c\}, \tau = \{X, \emptyset, \{a\}, \{b,c\}\} \). Then the set \( \{a,b\} \) is \( \delta\text{gs} \)-closed but not \( \delta\text{sg}^* \)-closed.

**Remark 2.2.12** From the above results we get the following relation.

**Theorem 2.2.13** Every \( \delta\text{sg}^* \)-closed set is a \( \delta\text{gs} \)-closed set but not conversely.

**Proof:** Let \( A \) be a \( \delta\text{sg}^* \)-closed set. Let \( A \subseteq U \) where \( U \) is \( \delta \)-open. By Remark 1.1.6 every \( \delta \)-open set is \( g \)-open. Therefore \( U \) is \( g \)-open. Since \( A \) is \( \delta\text{sg}^* \)-closed set, \( \delta\text{scl}(A) \subseteq U \). Therefore \( A \) is a \( \delta\text{gs} \)-closed set.

**Example 2.2.14** Let \( X = \{a, b, c, d\}, \tau = \{X, \emptyset, \{a\}\} \). Then the set \( \{a,b,c\} \) is \( \delta\text{gs} \)-closed but not \( \delta\text{sg}^* \)-closed.

**Lemma 2.2.15** Every \( \delta \)-semi closed set is semi closed but not conversely.

**Proof:** Let \( A \) be \( \delta \)-semi closed set,
\[
\text{int}(\text{cl}_\delta(A)) \subseteq A \quad \rightarrow (1)
\]
But we know \( \text{cl}(A) \subseteq \text{cl}_\delta(A) \)
\[
\Rightarrow \text{int}(\text{cl}(A)) \subseteq \text{int}(\text{cl}_\delta(A)) \quad \rightarrow (2)
\]
\((1) \& (2) \Rightarrow \text{int}(\text{cl}(A)) \subseteq A
\]
Therefore \( A \) is semi closed.
Example 2.2.16 Let \( X = \{a, b, c\}, \tau = \{X, \phi, \{a\}, \{a,b\}\}. \) Then the set \( \{b\} \) is semi closed but not \( \delta \)-semi closed.

Theorem 2.2.17 Every \( \delta g^* \)-closed set is a \( g \)-closed set but not conversely.

**Proof:** Let \( A \) be a \( \delta g^* \)-closed set and \( U \) be any open set containing \( A \) in \((X, \tau)\). Since every open set is \( g \)-open and \( A \) is \( \delta g^* \)-closed, \( \delta \)-\( scl(A) \subseteq U \). By Lemma 2.2.15, \( scl(A) \subseteq \delta \)-\( scl(A) \) and hence \( scl(A) \subseteq U \) which implies \( A \) is a \( g \)-closed set.

Example 2.2.18 Let \( X = \{a, b, c\}, \tau = \{X, \phi, \{a\}, \{a,b\}, \{a,c\}\}. \) Then the set \( \{c\} \) is \( g \)-closed but not \( \delta g^* \)-closed.

Lemma 2.2.19 Every \( \delta \)-semi closed set is a semi pre closed set but not conversely.

**Proof:** Let \( int(A) \subseteq A \). Since \( \text{cl}_\delta \rightarrow \text{cl} \) gives \( \text{cl}(A) \subseteq \text{cl}_\delta(A) \), \( \text{cl}(int(A)) \subseteq \text{cl}_\delta(int(A)) \) \( \subseteq \text{cl}_\delta(A) \). Therefore \( \text{cl}(int(A)) \subseteq \text{cl}_\delta(A) \)

\[
\text{int}(\text{cl}(int(A))) \subseteq \text{int}(\text{cl}_\delta(A)) \quad \rightarrow (1)
\]

But \( \text{int}(\text{cl}_\delta(A)) \subseteq A \) [since \( A \) is a \( \delta \)-semi closed set] \( \rightarrow (2) \)

(1) & (2) \( \Rightarrow \) \( \text{int}(\text{cl}(int(A))) \subseteq A \)

\( \Rightarrow \) \( A \) is semi pre closed set

Therefore a \( \delta \)-semi closed set is a semi pre closed set.

Hence \( spcl(A) \subseteq \delta \)-\( scl(A) \).

Example 2.2.20 Let \( X = \{a, b, c\}, \tau = \{X, \phi, \{a\}, \{a,b\}\}. \) Then the set \( \{a\} \) is semi pre closed but not \( \delta \)-semi closed.

Theorem 2.2.21 Every \( \delta g^* \)-closed set is a gspr-closed set but not conversely.

**Proof:** Let \( A \) be a \( \delta g^* \)-closed set and \( U \) be any regular open set containing \( A \) in \((X, \tau)\). Since every regular open set is \( g \)-open [by Remark 1.1.6] and \( A \) is a \( \delta g^* \)-closed set, \( \delta \)-\( scl(A) \subseteq U \). By Lemma 2.2.19, \( spcl(A) \subseteq \delta \)-\( scl(A) \) and hence \( spcl(A) \subseteq U \) and hence \( A \) is a gspr-closed set.
Example 2.2.22 Let $X = \{a, b, c\}, \tau = \{X, \emptyset, \{a\}, \{b\}, \{a,b\}, \{a,c\}\}$. Then the set $\{a,b\}$ is gspr-closed but not $\delta g^*$-closed.

Theorem 2.2.23 Every $\delta g^*$-closed set is a $\pi gsp$-closed set but not conversely.

Proof: Let $A$ be a $\delta g^*$-closed set and $U$ be any $\pi$-open set containing $A$ in $(X, \tau)$. Since every $\pi$-open set is g-open set and $A$ is a $\delta g^*$-closed set, $\delta - scl(A) \subseteq U$. By Lemma 2.2.19, $spcl(A) \subseteq \delta - scl(A)$ and hence $spcl(A) \subseteq U$. Therefore $A$ is a $\pi gsp$-closed set.

Example 2.2.24 Let $X = \{a, b, c\}, \tau = \{X, \emptyset, \{a\}, \{a,b\}\}$. Then the set $\{b\}$ is $\pi gsp$-closed but not $\delta g^*$-closed.

Theorem 2.2.25 Every $\delta g^*$-closed set is a $\pi gs$-closed set but not conversely.

Proof: Let $A$ be a $\delta g^*$-closed set and $U$ be any $\pi$-open set containing $A$ in $(X, \tau)$. Since every $\pi$-open set is g-open and $A$ is a $\delta g^*$-closed set, $\delta - scl(A) \subseteq U$. By Lemma 2.2.15, $scl(A) \subseteq \delta - scl(A)$ and so we have $scl(A) \subseteq U$. Therefore $A$ is a $\pi gs$-closed set.

Example 2.2.26 Let $X = \{a, b, c\}, \tau = \{X, \emptyset, \{a\}, \{a,b\}, \{a,c\}\}$. Then the set $\{a,c\}$ is $\pi gs$-closed but not $\delta g^*$-closed.

Lemma 2.2.27 Every $\delta$-semi closed set is a b-closed set but not conversely.

Proof: We know, $cl(A) \subseteq cl_\delta(A)$

\[ int(cl(A)) \subseteq int(cl_\delta(A)) \rightarrow (1) \]

\[ cl(int(A)) \cap int(cl(A)) \subseteq int(cl(A)) \rightarrow (2) \]

(1) & (2) \Rightarrow $int(cl(A)) \cap int(cl(A)) \subseteq int(cl_\delta(A)) \rightarrow (3) $

If $A$ is $\delta$-semi closed then $int(cl_\delta(A)) \subseteq A \rightarrow (4) $

(3) & (4) \Rightarrow $cl(int(A)) \cap int(cl(A)) \subseteq A$

Therefore $A$ is a b-closed set. Hence $bcl(A) \subseteq \delta - scl(A)$. 
**Example 2.2.28** Let \( X = \{a, b, c\}, \tau = \{X, \phi, \{a\}, \{b,c\}\}. \) Then the set \( \{b\} \) is b-closed but not \( \delta \)-semi closed.

**Corollary 2.2.29** For every subset \( A \) of \((X, \tau), \) \( \text{bcl}(A) \subseteq \delta \text{-scl}(A) \)

**Theorem 2.2.30** Every \( \delta s^* \)-closed set is a \( \pi gb \)-closed set but not conversely.

**Proof:** Let \( A \) be \( \delta s^* \)-closed set and \( U \) be any \( \pi \)-open set containing \( A \) in \((X, \tau). \) Since every \( \pi \)-open is \( g \)-open and \( A \) is \( \delta s^* \)-closed set, \( \delta \text{-scl}(A) \subseteq U. \) By Corollary 2.2.29, \( \text{bcl}(A) \subseteq \delta \text{-scl}(A) \) and so we have \( \text{bcl}(A) \subseteq U \) and hence \( A \) is a \( \pi gb \)-closed set.

**Example 2.2.31** Let \( X = \{a, b, c\}, \tau = \{X, \phi, \{a\}\}. \) Then the set \( \{a\} \) is \( \pi gb \)-closed but not \( \delta s^* \)-closed.

**Theorem 2.2.32** Every \( \delta s^* \)-closed set is a \( gsp \)-closed set but not conversely.

**Proof:** Let \( A \) be a \( \delta s^* \)-closed set and \( U \) be any open set containing \( A \) in \((X, \tau). \) Since every open set is \( g \)-open and \( A \) is \( \delta s^* \)-closed set, \( \delta \text{-scl}(A) \subseteq U. \) By Lemma 2.2.19, \( \text{spcl}(A) \subseteq \delta \text{-scl}(A) \) and so we have \( \text{spcl}(A) \subseteq U \) and hence \( A \) is a \( gsp \)-closed set.

**Example 2.2.33** Let \( X = \{a, b, c\}, \tau = \{X, \phi, \{a,b\}\}. \) Then the set \( \{b\} \) is \( gsp \)-closed but not \( \delta s^* \)-closed.

**Remark 2.2.34** The following figure gives the dependence of \( \delta s^* \)-closed set with various eleven closed sets.

In this diagram, \( A \rightarrow B \) represents \( A \) implies \( B \) but not reversible.
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Remark 2.2.35 The following counter examples show that \( \delta sg^* \)-closedness is independent from closedness.

Example 2.2.36 Let \( X = \{a, b, c, d\} \), \( \tau = \{X, \phi, \{a\}, \{c\}, \{a,b\}, \{a,c\}, \{a,b,c\}, \{a,c,d\}\} \). In this topology the subset \( \{a,b\} \) is \( \delta sg^* \)-closed but not closed.

Example 2.2.37 Let \( X = \{a, b, c\} \), \( \tau = \{X, \phi, \{a\}, \{a,b\}, \{a,c\}\} \). In this topology the subset \( \{c\} \) is closed but not \( \delta sg^* \)-closed.

Remark 2.2.38 The following counter examples show that \( \delta sg^* \)-closedness is independent from \( \alpha \)-closedness, semiclosedness and \( b \)-closedness.

Example 2.2.39 Let \( X = \{a, b, c\} \), \( \tau = \{X, \phi, \{a\}, \{a,b\}\} \). In this topology the subset \( \{b\} \) is \( \alpha \)-closed, semi closed and \( b \)-closed but not \( \delta sg^* \)-closed.

Example 2.2.40 Let \( X = \{a, b, c\} \), \( \tau = \{X, \phi, \{a\}, \{a,b\}\} \). In this topology the subset \( \{a,c\} \) is \( \delta sg^* \)-closed but not \( \alpha \)-closed, semiclosed and \( b \)-closed.

Remark 2.2.41 The following counter examples show that \( \delta sg^* \)-closedness is independent from semi-pre closedness, \( sg \)-closedness, \( g^*s \)-closedness and \( g^#s \)-closedness.

Example 2.2.42 Let \( X = \{a, b, c\} \), \( \tau = \{X, \phi, \{a\}, \{a,b\}\} \). In this topology the subset \( \{b\} \) is semi pre closed, \( sg \)-closed, \( g^*s \)-closed and \( g^#s \)-closed but not \( \delta sg^* \)-closed.

Example 2.2.43 Let \( X = \{a, b, c\} \), \( \tau = \{X, \phi, \{a\}, \{a,b\}\} \). In this topology the subset \( \{a,c\} \) is \( \delta sg^* \)-closed but not semi pre closed, \( sg \)-closed, \( g^*s \)-closed and \( g^#s \)-closed.

Remark 2.2.44 The following counter examples show that \( \delta sg^* \)-closedness is independent from \( g \)-closedness, \( ag \)-closedness, \( a\#g \)-closedness, \( g^* \)-closedness, \( \delta g \)-closedness, \( w\delta g^* \)-closedness and \( \delta \#g \)-closedness.

Example 2.2.45 Let \( X = \{a, b, c\} \), \( \tau = \{X, \phi, \{a\}\} \). In this topology the subset \( \{c\} \) is \( g \)-closed, \( ag \)-closed, \( a\#g \)-closed, \( g^* \)-closed, \( gp \)-closed, \( \delta g \)-closed, \( w\delta g^* \)-closed and \( \delta \#g \)-closed but not \( \delta sg^* \)-closed.

Example 2.2.46 Let \( X = \{a, b, c\} \), \( \tau = \{X, \phi, \{a\}, \{b\}, \{a,b\}\} \). In this topology the subset \( \{a\} \) is \( \delta sg^* \)-closed but not \( g \)-closed, \( ag \)-closed, \( a\#g \)-closed, \( g^* \)-closed, \( gp \)-closed, \( \delta g \)-closed, \( w\delta g^* \)-closed and \( \delta \#g \)-closed.
Remark 2.2.47 The following counter examples show that $\delta sg^*$-closedness is independent from pre closedness, $g^*$-closedness, $\hat{g}$-closedness, $g^p$-closedness and $(gs)^*$-closedness.

Example 2.2.48 Let $X = \{a, b, c\}$, $\tau = \{X, \phi, \{a\}, \{b\}, \{a,b\}\}$. In this topology the subset $\{b\}$ is $\delta sg^*$-closed but not pre closed, $g^*$-closed, $\hat{g}$-closed, $g^p$-closed and $(gs)^*$-closed.

Example 2.2.49 Let $X = \{a, b, c\}$, $\tau = \{X, \phi, \{a\}\}$. In this topology the subset $\{c\}$ is pre closed, $g^*$-closed, $\hat{g}$-closed, $g^p$-closed and $(gs)^*$-closed but not $\delta sg^*$-closed.

Remark 2.2.50 The following counter examples show that $\delta sg^*$-closedness is independent from $g\alpha$-closedness, $\alpha g^*$-closedness, $sg^*$-closedness, $g\delta$-closedness, $\delta g^\#$-closedness and $\Delta^*$-closedness.

Example 2.2.51 Let $X = \{a, b, c\}$, $\tau = \{X, \phi, \{a\}, \{b\}, \{a,b\}\}$. In this topology the subset $\{b\}$ is $\delta sg^*$-closed but not $g\alpha$-closed, $\alpha g^*$-closed, $sg^*$-closed, $g\delta$-closed, $\delta g^\#$-closed and $\Delta^*$-closed.

Example 2.2.52 Let $X = \{a, b, c\}$, $\tau = \{X, \phi, \{a\}\}$. In this topology the subset $\{c\}$ is $g\alpha$-closed, $\alpha g^*$-closed, $sg^*$-closed, $g\delta$-closed, $\delta g^\#$-closed and $\Delta^*$-closed but not $\delta sg^*$-closed.

Remark 2.2.53 The following counter examples show that $\delta sg^*$-closedness is independent from $\pi g$-closedness, $\pi g\alpha$-closedness and $\pi gp$-closedness.

Example 2.2.54 Let $X = \{a, b, c, d\}$, $\tau = \{X, \phi, \{a\}, \{c\}, \{a,b\}, \{a,c\}, \{a,b,c\}, \{a,c,d\}\}$. In this topology the subset $\{c\}$ is $\delta sg^*$-closed but not $\pi g$-closed, $\pi g\alpha$-closed and $\pi gp$-closed.

Example 2.2.55 Let $X = \{a, b, c, d\}$, $\tau = \{X, \phi, \{a\}, \{c\}, \{a,b\}, \{a,c\}, \{a,b,c\}, \{a,c,d\}\}$. In this topology the subset $\{a,c,d\}$ is $\pi g$-closed, $\pi g\alpha$-closed and $\pi gp$-closed but not $\delta sg^*$-closed.

Remark 2.2.56 The following diagram depicts the independence of $\delta sg^*$-closed set between various g-closed sets (In this diagram, $A \iff B$ represents A and B are independent).
2.3 Properties of $\delta sg^*-$Closed Sets in Topological Spaces

Theorem 2.3.1
(a) Every finite union of $\delta sg^*-$closed sets may fail to be a $\delta sg^*-$closed set.

(b) Every finite intersection of $\delta sg^*-$closed sets may fail to be a $\delta sg^*-$closed set.

The following examples support the above theorem.

Example 2.3.2 Let $X = \{a,b,c\}$, $\tau = \{X,\phi,\{a\},\{b\},\{a,b\}\}$. Consider $A = \{a\}$ and $B = \{b\}$ then $A$ and $B$ are $\delta sg^*-$closed sets but $A \cup B = \{a,b\}$ is not a $\delta sg^*-$closed set in $(X, \tau)$.

Example 2.3.3 Let $X = \{a,b,c,d\}$, $\tau = \{X,\phi,\{c\},\{a,b\},\{a,b,c\}\}$. Consider $A = \{a,b\}$ and $B = \{a,d\}$ then $A$ and $B$ are $\delta sg^*-$closed sets but $A \cap B = \{a\}$ is not a $\delta sg^*-$closed set in $(X, \tau)$.

Theorem 2.3.4 Let $A$ be a $\delta sg^*-$closed set of $(X, \tau)$. Then $\delta$-scl($A$) \ $A$ does not contain a non-empty $g$-closed set.

Proof: Suppose that $A$ is $\delta sg^*-$closed, let $F$ be a $g$-closed set contained in $\delta$-scl($A$) \ $A$. Now $X \setminus F$ is a $g$-open set in $(X, \tau)$ such that $A \subseteq X \setminus F$. Since $A$ is a $\delta sg^*-$closed set of $(X, \tau)$, $\delta$-scl($A$) \ $X \setminus F$. Thus $F \subseteq X \setminus \delta$-scl($A$). Also $F \subseteq \delta$-scl($A$) \ $A$. Therefore $F \subseteq X \setminus \delta$-scl($A$) \ $F \subseteq \delta$-scl($A$) \ $A$. Therefore $F = \phi$. Hence $F = \phi$. 

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**Proposition 2.3.5** If \( A \) is a \( g \)-open set and a \( \delta g^* \)-closed set of \((X, \tau)\) then \( A \) is a \( \delta \)-semi closed set of \((X, \tau)\).

**Proof:** Since \( A \) is \( g \)-open and \( \delta g^* \)-closed. Let \( A \subseteq A \), where \( A \) is \( g \)-open and \( \delta \text{-scl}(A) \subseteq A \) which implies \( \delta \text{-scl}(A) = A \). Hence \( A \) is \( \delta \)-semi closed.

**Corollary 2.3.6** If \( A \) is \( \delta g^* \)-closed, \( g \)-open and \( F \) is \( \delta \)-semi closed in \((X, \tau)\), then \( A \cap F \) is \( \delta \)-semi closed.

**Proof:** Since \( A \) is \( \delta g^* \)-closed and \( g \)-open, \( A \) is \( \delta \)-semi closed by Proposition 2.3.5. Since \( F \) is \( \delta \)-semi closed in \((X, \tau)\). Therefore \( A \cap F \) is \( \delta \)-semi closed in \((X, \tau)\) [By Lemma 1 (Caldas, 2003)].

**Remark 2.3.7** The intersection of a \( \delta \)-semi closed set and a \( \delta g^* \)-closed set is not \( \delta g^* \)-closed. It is seen from the following example.

**Example 2.3.8** Let \( X = \{a, b, c, d\}, \tau = \{X, \phi, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}\} \). Then \( \{a, b\} \) is \( \delta \)-semi closed and \( \{b, d\} \) is \( \delta g^* \)-closed but \( \{a, b\} \cap \{b, d\} = \{b\} \) is not a \( \delta g^* \)-closed set in \((X, \tau)\).

**Proposition 2.3.9** If \( A \) is a \( \delta g^* \)-closed set in a space \((X, \tau)\) and \( A \subseteq B \subseteq \delta \text{-scl}(A) \), then \( B \) is also a \( \delta g^* \)-closed set.

**Proof:** Let \( U \) be a \( g \)-open set of \((X, \tau)\) such that \( B \subseteq U \) then \( A \subseteq U \). Since \( A \) is a \( \delta g^* \)-closed set, \( \delta \text{-scl}(A) \subseteq U \). Also since \( B \subseteq \delta \text{-scl}(A) \), \( \delta \text{-scl}(B) \subseteq \delta \text{-scl}(\delta \text{-scl}(A)) = \delta \text{-scl}(A) \). Hence \( \delta \text{-scl}(B) \subseteq U \). Therefore \( B \) is also a \( \delta g^* \)-closed set.

**Theorem 2.3.10** Let \( A \) be a \( \delta g^* \)-closed set of \((X, \tau)\). Then \( A \) is \( \delta \)-semi closed if and only if \( \delta \text{-scl}(A) \setminus A \) is \( g \)-closed.

**Proof:** **Necessity:** Let \( A \) be a \( \delta \)-semi closed subset of \((X, \tau)\). Then \( \delta \text{-scl}(A) = A \) and so \( \delta \text{-scl}(A) \setminus A = \phi \), which is \( g \)-closed.

**Sufficiency:** Let \( \delta \text{-scl}(A) \setminus A \) be \( g \)-closed. Since \( A \) is \( \delta g^* \)-closed, by Theorem 2.3.4., \( \delta \text{-scl}(A) \setminus A \) does not contain a non-empty \( g \)-closed set which implies \( \delta \text{-scl}(A) \setminus A = \phi \). That is \( \delta \text{-scl}(A) = A \). Hence \( A \) is \( \delta \)-semi closed.
**Definition 2.3.11** Let $B \subseteq A \subseteq X$. Then $B$ is $\delta sg^*$-closed relative to $A$ if $(\delta-scl)_A(B) \subseteq U$, whenever $B \subseteq U$, $U$ is g-open in $A$.

**Theorem 2.3.12** Let $B \subseteq A \subseteq X$ and suppose that $B$ is $\delta sg^*$-closed in $(X, \tau)$, then $B$ is $\delta sg^*$-closed relative to $A$. The converse is true if $A$ is $\delta$-semi closed in $(X, \tau)$.

**Proof:** Suppose that $B$ is $\delta sg^*$-closed in $(X, \tau)$. Let $B \subseteq U$, $U$ is g-open in $A$. Since $U$ is g-open in $A$, $U = V \cap A$, where $V$ is g-open in $(X, \tau)$. Hence $B \subseteq U \subseteq V$. Since $B$ is $\delta sg^*$-closed in $(X, \tau)$, $\delta-scl(B) \subseteq V$. Hence $\delta-scl(B) \cap A \subseteq V \cap A$ which in turn implies that $(\delta-scl)_A(B) \subseteq V \cap A = U$. Therefore $B$ is $\delta sg^*$-closed relative to $A$.

Now, to prove the converse, assume that $B \subseteq A \subseteq X$ where $A$ is $\delta$-semi closed in $(X, \tau)$ and $B$ is $\delta sg^*$-closed relative to $A$. Let $B \subseteq U$, $U$ is g-open in $(X, \tau)$. Then $A \cap U$ is g-open in $A$. Since $B \subseteq A$ and $B \subseteq U$, $B \subseteq A \cap U$.

Since $B$ is $\delta sg^*$-closed relative to $A$, $(\delta-scl)_A(B) \subseteq A \cap U$ \hspace{1cm} (1)

Since $A$ is $\delta$-semi closed in $(X, \tau)$, $\delta-scl(A) = A$.

Since $\delta-scl(B) \subseteq \delta-scl(A)$. Hence $\delta-scl(B) \subseteq A$.

Therefore $\delta-scl(B) \cap A = \delta-scl(B) \Rightarrow (\delta-scl)_A(B) = \delta-scl(B)$ \hspace{1cm} (2)

(1) and (2) implies $\delta-scl(B) \subseteq A \cap U \subseteq U$.

Therefore $B$ is $\delta sg^*$-closed in $(X, \tau)$.

**Theorem 2.3.13** Let $A$ be a subset of a $T_{1/2}$-space $(X, \tau)$ then

(a) $A$ is $\delta sg^*$-closed if and only if $A$ is $g\delta s$-closed.

(b) If in addition, $(X, \tau)$ is semi regular then $A$ is $\delta sg^*$-closed if and only if $A$ is $gs$-closed.

(c) If in addition, $(X, \tau)$ is $T_b$ (resp. $T_d$) $A$ is $\delta sg^*$-closed if and only if $A$ is closed (resp. $g$-closed).

**Proof:** (a) In general $\delta sg^*$-closed is $g\delta s$-closed by Theorem 2.2.10. Conversely Let $A$ be a $g\delta s$-closed set. Let $A \subseteq U$, where $U$ is g-open. In a $T_{1/2}$-space, g-open sets coincide with open sets. Since $A$ is a $g\delta s$-closed set, $\delta-scl(A) \subseteq U$. Therefore $A$ is a $\delta sg^*$-closed set.
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(b) Every $\delta sg^*$-closed is gs-closed from Theorem 2.2.17. Conversely let $A$ be gs-closed. Let $A \subseteq U$ where $U$ is g-open. In a $T_{1/2}$-space, every g-open set is open and since $A$ is gs-closed, $scl(A) \subseteq U$. In a semi regular space, $\tau = \tau_\delta$ then $\delta scl(A) = scl(A) \subseteq U$. Hence $A$ is $\delta sg^*$-closed.

(c) The proof follows from Theorem 2.8 of (Park, 2007) and from (a).

Definition 2.3.14 A partition space (Nieminen, 1977) is a space where every open set is closed.

Remark 2.3.15 In a partition space, open sets coincide with $\delta$-open sets and the concepts of $\delta$-closure and $\delta$-semi closure coincide for any set.

Theorem 2.3.16 For a subset $A$ of a $T_{1/2}$ partition space $(X, \tau)$ the following are equivalent:

(a) $A$ is $\delta sg^*$-closed
(b) $A$ is $\delta g$-closed
(c) $A$ is $\delta g^+$-closed
(d) $A$ is $gd_\delta$-closed
(e) $A$ is $\delta g s$-closed

Proof: (b) $\iff$ (c) $\iff$ (d) $\iff$ (e) is proved in [Theorem 2.6 of (Park, 2007)]

(a) $\iff$ (b) In a $T_{1/2}$-space, g-open sets coincide with open sets and hence by Remark 2.3.15 the proof follows.

The previous observation leads to the problem of finding the spaces $(X, \tau)$ in which the gs-closed sets of $(X, \tau_\delta)$ are $g\delta s$-closed in $(X, \tau)$. While we have not been able to completely resolve this problem, we offer partial solutions. For that reason we will call the spaces with semi-$T_{1/2}$ semi-regularization semi-weakly Hausdorff. Recall that a space is called almost weakly Hausdorff (Dontchev, 1996) if its semi-regularization is $T_{1/2}$. Clearly almost weakly Hausdorff spaces are semi-weakly Hausdorff, but conversely.

Example 2.3.17 (Park, 2007) Let $X = \{a,b,c,d\}$ with $\tau = \{X,\phi,\{a\},\{b\},\{a,b\}\}$. Then $(X, \tau)$ is clearly semi-weakly Hausdorff but not almost weakly Hausdorff.

Remark 2.3.18 In a semi weakly Hausdorff space, every singleton is either $\delta$-semi open or $\delta$-semi closed.
Theorem 2.3.19 For a subset A of a semi-weakly Hausdorff space (X, τ) the following are equivalent:

(a) A is gs-closed in (X, τ),
(b) A is δ-semi closed in (X, τ),
(c) A is δsg*-closed in (X, τ).

Proof: (a) ⇒ (b) Let A ⊆ X be gs-closed subset (X, τ). Let x ∈ δ-scl(A)

Case 1: {x} is δ-semi open. Since x ∈ δ-scl(A), every δ-semi neighborhood intersects A. Here {x} is a δ-semi neighborhood and hence {x} ∩ A ≠ ∅ which implies x ∈ A.

Case 2: {x} is not δ-semi open. Then {x} is δ-semi closed. By Remark 2.3.18, X \ {x} is δ-semi open.

Assume that x ∉ A. Since A is gs-closed in (X, τ), then δ-scl(A) ⊆ X \ {x}, i.e. x ∉ δ-scl(A). This is a contradiction to our assumption x ∈ δ-scl(A). Hence x ∈ A. Thus δ-scl(A) = A or equivalently A is δ-semi closed in (X, τ).

(b) ⇒ (c) By Proposition 2.2.2.

(c) ⇒ (a) Let A ⊆ U, where U is open in (X, τ). Then U is δ-open in (X, τ). Every δ-open is g-open. Since A is δsg*-closed in (X, τ), δ-scl(A) ⊆ U. By Lemma 7.3 of (Noiri, 2004), δ-scl(A) = scl(A) in (X, τ). That is scl(A) ⊆ U. Thus A is gs-closed in (X, τ).

Theorem 2.3.20 For a space (X, τ) the following are equivalent:

(a) Every g-open set of X is a δ-semi closed set
(b) Every subset of X is a δsg*-closed set.

Proof: (a) ⇒ (b) Let A ⊆ X such that A ⊆ U, where U is g-open. By (a), U is δ-semi closed and thus δ-scl(U) = U. Hence A is δsg*-closed set.

(b) ⇒ (a) Let U be a g-open set of (X, τ) and U ⊆ U, then by (b) δ-scl(U) ⊆ U or equivalently U is δ-semi closed.

Definition 2.3.21 A topological space (X, τ) is called an $R_1$-space if every two different points with distinct closures have disjoint neighborhoods.

Remark 2.3.22 In $R_1$-spaces the concepts of closure and δ-closure coincide for compact sets [Theorem 3.6 in (Jankovic, 1980)].
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**Theorem 2.3.23** For a compact subset A of an $R_1$-topological space $(X, \tau)$ the following conditions are equivalent, when $(X, \tau)$ is also $T_{1/2}$.

(a) $A$ is a $\delta sg^*$-closed set

(b) $A$ is a $gs$-closed set

**Proof:** (a) $\implies$ (b) By Theorem 2.2.17.

(b) $\implies$ (a) Let $A \subseteq U$, where $U$ is $g$-open. In a $T_{1/2}$-space $g$-open sets coincide with open sets. By Remark 2.3.2, the rest of the proof is obvious.

**Corollary 2.3.24** In Hausdorff spaces, a finite set is $gs$-closed if and only if it is $\delta sg^*$-closed.

### 2.4 $\delta sg^*$-Open Sets

In this section we introduce the concept of $\delta sg^*$-open sets in topological spaces and study some of their properties. It is interesting to note some extensions related $\delta sg^*$-closed sets don’t hold good for $\delta sg^*$-closed sets since $\delta$-semi closed $\iff$ g-closed. The singletons are characterized through $\delta sg^*$-open sets here.

**Definition 2.4.1** A subset $A$ of a topological space $(X, \tau)$ is called $\delta sg^*$-open if its complement $X \setminus A$ is $\delta sg^*$-closed in $(X, \tau)$. The collection of all $\delta sg^*$-open sets in $(X, \tau)$ is denoted by $\delta SG^*O(X, \tau)$.

**Theorem 2.4.2** If a subset $A$ of a topological space $(X, \tau)$ is $\delta$-semi open, then it is $\delta sg^*$-open in $(X, \tau)$.

**Proof:** Let $A$ be a $\delta$-semi open set in topological space $(X, \tau)$. Then $A^c$ is $\delta$-semi closed in $(X, \tau)$. By Proposition 2.2.2, $A^c$ is $\delta sg^*$-closed in $(X, \tau)$. Hence $A$ is $\delta sg^*$-open in $(X, \tau)$.

**Remark 2.4.3** The converse of the above theorem need not be true as seen in the following example.

**Example 2.4.4** Let $X = \{a, b, c\}$, $\tau = \{X, \phi, \{a\}, \{a,b\}, \{a,c\}\}$. Then the subset $\{a\}$ is $\delta sg^*$-open but not $\delta$-semi open in $(X, \tau)$. 

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**Proposition 2.4.5** Every clopen set in \((X, \tau)\) is \(\delta\)-sg*-open.

**Proof:** Let \(A\) be a clopen set. Then \(\text{cl}(A) = A\) and \(\text{int}(A) = A\). Hence \(\text{int}(\text{cl}(A)) = A\). Thus \(A\) is regular open since regular open \(\rightarrow \delta\)-open \(\rightarrow \delta\)-semi open. Therefore \(A\) is \(\delta\)-semi open. By Theorem 2.4.2, we get \(A\) is \(\delta\)-sg*-open.

**Remark 2.4.6** A \(\delta\)-sg*-open set is need not be clopen as seen from the following example.

**Example 2.4.7** Let \(X = \{a, b, c\}\), \(\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}\). Then the subset \(\{a, b\}\) is \(\delta\)-sg*-open but not clopen.

The following proposition can be proved similar to Theorem 2.4.2.

**Proposition 2.4.8** Every \(\delta\)-sg*-open set is gs-open (resp. g\(\delta\)-s-open, \(\delta\)gs-open, gsp-open, gspr-open, \(\pi\)gb-open, \(\pi\)gs-open, \(\pi\)gsp-open)

**Lemma 2.4.9** For a subset \(A\) of \((X, \tau)\), \(\delta\)-scl\((X \setminus A) = X \setminus \delta\)-sint\((A)\).

**Theorem 2.4.10** A subset \(A\) of a topological space \((X, \tau)\) is \(\delta\)-sg*-open if and only if \(G \subseteq \delta\)-sint\((A)\) whenever \(A \supseteq G\) and \(G\) is g-closed.

**Proof:** Assume that \(A\) is \(\delta\)-sg*-open. Then \(X \setminus A\) is \(\delta\)-sg*-closed. Let \(G\) be a g-closed set in \((X, \tau)\) contained in \(A\). Then \(X \setminus G\) is g-open set in \((X, \tau)\) containing \(X \setminus A\). Since \(X \setminus A\) is \(\delta\)-sg*-closed, \(\delta\)-scl\((X \setminus A) \subseteq X \setminus G\), equivalently \(G \subseteq \delta\)-sint\((A)\) by Lemma 2.4.9.

Conversely assume that \(G\) is contained in \(\delta\)-sint\((A)\), whenever \(G\) is contained in \(A\) and \(G\) is g-closed in \((X, \tau)\). Let \(X \setminus A\) be contained in \(F\), where \(F\) is g-open. Then \(X \setminus F \subseteq A\). By criteria, \(X \setminus F \subseteq \delta\)-sint\((A)\). This implies \(\delta\)-scl\((X \setminus A) \subseteq F\) by Lemma 2.4.9. Thus \(X \setminus A\) is \(\delta\)-sg*-closed. Hence \(A\) is \(\delta\)-sg*-open.

**Proposition 2.4.11** If \(\delta\)-sint\((A) \subseteq B \subseteq A\) and \(A\) is \(\delta\)-sg*-open in \((X, \tau)\), then \(B\) is \(\delta\)-sg*-open in \((X, \tau)\).

**Proof:** Follows from Lemma 2.4.9 and Proposition 2.3.9.

**Theorem 2.4.12** If \(A\) and \(B\) are \(\delta\)-sg*-open sets in \((X, \tau)\), then \(A \cap B\) is \(\delta\)-sg*-open in \((X, \tau)\).
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**Proof:** Let A and B be $\delta sg^*$-open sets in $(X, \tau)$. Then $X \setminus A$ and $X \setminus B$ are $\delta sg^*$-closed sets and $(X \setminus A) \cup (X \setminus B) = X \setminus (A \cap B)$ is $\delta sg^*$-closed. Hence $A \cap B$ is $\delta sg^*$-open.

**Theorem 2.4.13** If A is $\delta sg^*$-open in $(X, \tau)$ and F is g-open such that $\delta sint(A) \cup (X \setminus A) \subseteq F$ then $F = X$.

**Proof:** Let A be a $\delta sg^*$-open set and F be g-open and $\delta sint(A) \cup (X \setminus A) \subseteq F$. This gives $X \setminus F \subseteq X \setminus (\delta sint(A) \cup (X \setminus A)) \subseteq (X \setminus \delta sint(A)) \cap A \subseteq (X \setminus \delta sint(A)) \setminus (X \setminus A) \subseteq \delta scl(X \setminus A) \setminus (X \setminus A)$. Since $X \setminus A$ is $\delta sg^*$-closed and $X \setminus F$ is g-closed by Theorem 2.3.4, it follows that $X \setminus F = \emptyset$. Therefore $F = X$.

**Note 2.4.14** In the case of $\delta g^*$-closed sets (Sudha (2014)), the converse part of Theorem 2.4.13 holds good (Theorem 2.4.7 in Ph.D thesis of Sudha). As $\delta$-semi closed $\nrightarrow$ g-closed, the converse part cannot be proved in the above theorem. This can be seen from the following example.

**Example 2.4.15** Let $X = \{a, b, c, d\}$, $\tau = \{X, \emptyset, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}\}$. Let $A = \{a, c, d\}$, $\delta sint(A) = \{c\} \cup (X \setminus A) = \{b, c\} \subseteq \{a, b, c\} = F \neq X$.

**Definition 2.4.16** The intersection of all g-open subsets of $(X, \tau)$ containing A is called the g-kernel of A and is denoted by $gker(A)$.

\[i.e., gker(A) = \bigcap \{U / U \text{ is g-open in } (X, \tau) \text{ and } A \subseteq U\}\]

The following theorem characterizes singletons through $\delta sg^*$-closed sets.

**Theorem 2.4.17** Every singleton is either g-closed or $\delta sg^*$-open in $(X, \tau)$.

**Proof:** If $\{a\}$ is g-closed, then there is nothing to prove. Suppose that $\{a\}$ is not g-closed in $(X, \tau)$, then $X \setminus \{a\}$ is not g-open and the only g-open set containing $X \setminus \{a\}$ is the space X itself. That is $X \setminus \{a\} \subseteq X$. Therefore $\delta scl(X \setminus \{a\}) \subseteq X$ and $X \setminus \{a\}$ is $\delta sg^*$-closed and hence $\{a\}$ is $\delta sg^*$-open.

**Note 2.4.18** Theorem 2.4.17 gives a decomposition for $(X, \tau)$ as $X = X_1 \cup X_2$ where $X_1 = \{x \in X / \{x\} \text{ is g-closed}\}$ and $X_2 = \{x \in X / \{x\} \text{ is } \delta sg^*-open\}$.

**Theorem 2.4.19** For a subset A of $(X, \tau)$, the following properties are equivalent

(a) A is $\delta sg^*$-closed

(b) $\delta scl(A) \subseteq gker(A)$ holds
\(\text{(c) (i) } \delta\text{-scl}(A) \cap X_1 \subseteq A\)

\(\text{(ii) } \delta\text{-scl}(A) \cap X_2 \subseteq g\text{-ker}(A)\)

\textbf{Proof: \(\text{(a)} \Rightarrow \text{(b)}\)} Let \(x \notin g\text{-ker}(A)\). Then there exists a set \(U \in \text{GO}(X, \tau)\) such that \(A \subseteq U\) and \(x \notin U\). Since \(A\) is \(\delta\text{sg}-\text{closed}\), \(\delta\text{-scl}(A)\subseteq U\) and \(x \notin \delta\text{-scl}(A)\).

\(\text{(b) } \Rightarrow \text{(a)}\) Let \(A \subseteq U\), where \(U\) is \(g\)-open. Since \(U\) is \(g\)-open containing \(A\), by definition of \(g\text{-ker}(A)\) we get, \(g\text{-ker}(A) \subseteq U\) \(\rightarrow (1)\)

By \(\text{(b)}\), \(\delta\text{-scl}(A) \subseteq g\text{-ker}(A)\) \(\rightarrow (2)\)

\((1)\) and \((2)\) implies \(\delta\text{-scl}(A) \subseteq U\). Therefore \(A\) is \(\delta\text{sg}-\text{closed}\).

\(\text{(b) } \Rightarrow \text{(c)}\)

\(\text{(i) Let } x \in \delta\text{-scl}(A) \cap X_1\) \(\rightarrow (1)\)

\(x \in \delta\text{-scl}(A)\) then by \(\text{(b)}\), \(x \in g\text{-ker}(A)\) \(\rightarrow (2)\)

\((1) \Rightarrow x \in X_1, \{x\}\) is \(g\)-closed \(\rightarrow (3)\)

If \(x \notin A\) and say \(U = X \setminus \{x\}\) is a \(g\)-open set \(\text{[by (3)]}\) and \(A \subseteq U\) \(\rightarrow (4)\)

That is \(U\) is a \(g\)-open set containing \(A\). By definition of \(g\text{-kernel}\) of \(A\), \(g\text{-ker}(A) \subseteq U\).

\((2) \Rightarrow x \in U\), which is a contradiction to \(U = X \setminus \{x\}\). Therefore \(x \in A\).

\(\text{(ii) Always } \delta\text{-scl}(A) \cap X_2 \subseteq \delta\text{-scl}(A)\) \(\rightarrow (1)\)

\(\text{(b) } \Rightarrow \delta\text{-scl}(A) \subseteq g\text{-ker}(A)\) \(\rightarrow (2)\)

\((1) \& (2) \Rightarrow \delta\text{-scl}(A) \cap X_2 \subseteq g\text{-ker}(A)\)

\(\text{(c) } \Rightarrow \text{(b)}\) \(\delta\text{-scl}(A) = \delta\text{-scl}(A) \cap X\)

\[= \delta\text{-scl}(A) \cap [X_1 \cup X_2]\]

\[= [\delta\text{-scl}(A) \cap X_1] \cup [\delta\text{-scl}(A) \cap X_2]\] [using \((c)\) (i) and (ii)]

\[\subseteq A \cup g\text{-ker}(A).\]

Therefore \(\delta\text{-scl}(A) \subseteq g\text{-ker}(A)\).
Corollary 2.4.20 Let \( P = \{ A \subseteq X / \delta\text{-scl}(A) \cap X_2 \subseteq g\text{-ker}(A) \} \). Then

(a) If \( \bigcap_{i \in I} A_i \in P \) and \( A_i \) is \( \delta\text{sg}^* \)-closed in \((X, \tau)\) for each \( i \), then \( \bigcap_{i \in I} A_i \) is \( \delta\text{sg}^* \)-closed.

(b) If \( P = P(X) \) and \( A_i \) is \( \delta\text{sg}^* \)-closed in \((X, \tau)\) for each \( i \in I \), then \( \bigcap_{i \in I} A_i \) is \( \delta\text{sg}^* \)-closed in \((X, \tau)\).

(c) If \( \delta\text{-scl}(A_i) \cap X_2 \subseteq g\text{-ker}(A_i) \) and \( A_i \) is \( \delta\text{sg}^* \)-closed in \((X, \tau)\) for each \( i \in I \), then \( \bigcap_{i \in I} A_i \) is \( \delta\text{sg}^* \)-closed in \((X, \tau)\).

Proof: (a) By Theorem 2.4.19, \( \delta\text{-scl}(A_i) \cap X_1 \subseteq A_i \) for each \( i \in I \). Then we have \( \delta\text{-scl}\left(\bigcap_{i \in I} A_i \right) \cap X_1 \subseteq \bigcap_{i \in I} A_i \) using assumption and Theorem 2.4.19 (c) \( \bigcap_{i \in I} A_i \) is \( \delta\text{sg}^* \)-closed.

(b) follow from (a)

(c) Let \( \delta\text{-scl}(A_i) \cap X_2 \subseteq g\text{-ker}(A_i) \). Since \( \delta\text{-scl}(\bigcap_{i \in I} A_i) = \bigcap_{i \in I} \delta\text{-scl}(A_i) \), we have \( \delta\text{-scl}(\bigcap_{i \in I} A_i) \cap X_2 = [\bigcap_{i \in I} \delta\text{-scl}(A_i) \cap X_2] \subseteq g\text{-ker}(A_i) = g\text{-ker}(\bigcap_{i \in I} A_i) \). Using assumption and Theorem 2.4.19(c), \( \bigcap_{i \in I} A_i \) is \( \delta\text{sg}^* \)-closed.

Theorem 2.4.21 If a subset \( A \) is \( \delta\text{sg}^* \)-closed in \((X, \tau)\), then \( \delta\text{-scl}(A) \setminus A \) is \( \delta\text{sg}^* \)-open.

Proof: Suppose that \( A \) is \( \delta\text{sg}^* \)-closed in \((X, \tau)\). Let \( F \subseteq \delta\text{-scl}(A) \setminus A \) and \( F \) be \( g \)-closed. Since \( A \) is \( \delta\text{sg}^* \)-closed, \( \delta\text{-scl}(A) \setminus A \) does not contain non-empty \( g \)-closed set (by Theorem 2.3.4) hence \( F = \emptyset \). Thus \( F \subseteq \delta\text{-sint}[\delta\text{-scl}(A) \setminus A] \). Hence \( \delta\text{-scl}(A) \setminus A \) is \( \delta\text{sg}^* \)-open.

2.5 \( \delta\text{sg}^* \) - Closure Operator

In this section, the notion of \( \delta\text{sg}^* \)-closure of a set is introduced and some of its properties are studied. The newly defined \( \delta\text{sg}^* \)-closure operator doesn’t satisfy the axioms of Kuratowski closure operator.

Definition 2.5.1 The \( \delta \) semi generalized star -closure of \( A \) (briefly \( \delta\text{sg}^* \text{cl}(A) \)) of a topological space \((X, \tau)\) is defined as follows.

\[
\delta\text{sg}^* \text{cl}(A) = \cap \{ F \subseteq X : A \subseteq F \text{ and } F \in \delta\text{SG}^* \mathcal{C}(X, \tau) \}
\]
where $\delta^*_SG^*C(X, \tau)$ is set of all $\delta^*_sg$-closed subsets of $(X, \tau)$.

**Remark 2.5.2** For a subset $A$ of $(X, \tau)$, $A \subseteq gs-cl(A) \subseteq \delta^*_sg cl(A) \subseteq \delta-scl(A) \subseteq cl_\delta(A)$.

**Proposition 2.5.3** Let $A$ be any subset of $(X, \tau)$. If $A$ is $\delta^*_sg$-closed in $(X, \tau)$ then $\delta^*_sg cl(A) = A$.

**Proof:** Let $A$ be $\delta^*_sg$-closed in $(X, \tau)$. By definition, $\delta^*_sg cl(A) = \cap \{F \subseteq X : A \subseteq F \text{ and } F \text{ is } \delta^*_sg \text{-closed in } (X, \tau)\}$. Since $A$ is $\delta^*_sg$-closed, $F$ in the above intersection is $A$ and hence $\delta^*_sg cl(A) = A$.

**Remark 2.5.4** Let $A \subseteq X$. Then $\delta^*_sg cl(A)$ need not be a $\delta^*_sg$-closed set.

**Example 2.5.5** Let $X = \{a,b,c,d\}$, $\tau = \{X, \phi, \{c\}, \{a,b\}, \{a,b,c\}\}$

$$\delta^*_sg cl\{a\} = \cap \{\text{all } \delta^*_sg \text{-closed sets containing } \{a\}\}$$

$$= \{a,b\} \cap \{a,d\} \cap \{a,b,d\} \cap \{a,c,d\} \cap X$$

$$= \{a\} \neq \delta^*_sg \text{-closed set}$$

**Theorem 2.5.6** For any two subsets $A$ and $B$ of $(X, \tau)$. Then the following statements are true:

(a) $\delta^*_sg cl(X) = X$ and $\delta^*_sg cl(\phi) = \phi$

(b) $A \subseteq \delta^*_sg cl(A)$

(c) If $B$ is any $\delta^*_sg$-closed set containing $A$, then $\delta^*_sg cl(A) \subseteq B$

(d) If $A \subseteq B$, then $\delta^*_sg cl(A) \subseteq \delta^*_sg cl(B)$

(e) $\delta^*_sg cl(\delta^*_sg cl(A)) = \delta^*_sg cl(A)$

(f) $\delta^*_sg cl(A) \cup \delta^*_sg cl(B) \subseteq \delta^*_sg cl(A \cup B)$. The reverse inclusion is not satisfied shown in Example 2.5.7.

(g) $\delta^*_sg cl(A \cap B) = \delta^*_sg cl(A) \cap \delta^*_sg cl(B)$

**Proof:** (a) Follows from Definition 2.5.1.

(b) By the definition of $\delta^*_sg$-closure of $A$, it is obvious that $A \subseteq \delta^*_sg cl(A)$. 

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(c) Let B be any δsg*-closed set containing A. Since δsg*cl(A) is the intersection of all δsg*-closed sets containing A, δsg*cl(A) is contained in every δsg*-closed set containing A. Hence δsg*cl(A) ⊆ B.

(d) Follows from Definition 2.5.1.

(e) By (c), δsg*cl(A) ⊆ B. Let A be any subset of X. By the definition of δsg*-closure, δsg*cl(A) = ∩{F ⊆ X / A ⊆ F and F ∈ δSG* cl(X, τ)}. If A ⊆ F ∈ δSG* cl(X, τ), then δsg* cl(A) ⊆ F. Since F is δsg*-closed set containing δsg* cl(A). By (c), δsg* cl(δsg* cl(A)) ⊆ F. Hence δsg* cl(δsg* cl(A)) ⊆ ∩{F ⊆ X : A ⊆ F and F ∈ δSG* cl(X, τ)} = δsg* cl(A). That is δsg* cl(δsg* cl(A)) = δsg* cl(A).

(f) Since A ⊆ A∪B and B ⊆ A∪B, by (b), δsg* cl(A) ⊆ δsg* cl(A∪B) and δsg* cl(B) ⊆ δsg* cl(A∪B). Hence δsg* cl(A)∪ δsg* cl(B) ⊆ δsg* cl(A∪B).

(g) Since A∩B ⊆ A and A∩B ⊆ B, by (b), δsg* cl(A∩B) ⊆ δsg* cl(A) and δsg* cl(A∩B) ⊆ δsg* cl(B). Hence δsg* cl(A∩B) ⊆ δsg* cl(A)∩ δsg* cl(B). Conversely, δsg* cl(A)∩ δsg* cl(B) = [∩{F ⊆ X : A ⊆ F and F ∈ δSG* cl(X, τ)}]∩ [∩{F ⊆ X : B ⊆ F and F ∈ δSG* cl(X, τ)}] ⊆ ∩{F ⊆ X : A∩B ⊆ F and F ∈ δSG* cl(X, τ)} = δsg* cl(A∩B).

Example 2.5.7 Let X = {a,b,c,d}, τ = {X,ϕ ,{c},{a,b},{a,b,c}}. Let A = {a,b}, B = {c} and A∪B = {a,b,c} then δsg* cl{a,b} = {a,b}, δsg* cl{c} = {c} and δsg* cl{a,b,c} = X. Hence δsg* cl(A∪B) = X ⊈ δsg* cl(A)∪ δsg* cl(B).

Note 2.5.8 δsg*-closure operator is not a Kuratowski closure operator since it is not satisfying

(i) δsg* cl(A)∪ δsg* cl(B) = δsg* cl(A∪B).

Remark 2.5.9 Denote the set of δ-semi open sets as τδs.

Definition 2.5.10 Let U be any subset of (X, τ). Using δsg*-closure operator, a new class of sets denoted by δsg* τ# is defined as follows.

δsg* τ# = {U : δsg* cl(X \ U) = X \ U}

Proposition 2.5.11 For any topology τ, we have τδs ⊆ τδs ⊆ δsg* τ#.
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Proof: Obvious from Remark 2.5.2 and Definition 2.5.10.

Lemma 2.5.12 For any $A \subseteq X$, $A \subseteq \deltasg^*\text{cl}(A) \subseteq \text{cl}_\delta(A)$.

Proof: It follows from Theorem 2.2.6.

Theorem 2.5.13 Every $\deltasg^*$-closed set is $\delta$-semi closed in $(X, \tau)$ if and only if $\deltasg^*\tau'' = \tau_\delta$.

Proof: Necessity: Let every $\deltasg^*$-closed set is $\delta$-semi closed in $(X, \tau)$. We know that every $\deltasg^*$-open set is a $\delta$-semi open set $\Rightarrow \deltasg^*\tau'' \subseteq \tau_\delta$. In Proposition 2.5.11, $\tau_\delta \subseteq \deltasg^*\tau''$. Therefore $\deltasg^*\tau'' = \tau_\delta$.

Sufficiency: Let $\deltasg^*\tau'' = \tau_\delta$. Let $A$ be a $\deltasg^*$-closed set, $X \setminus A$ is $\deltasg^*$-open. Then $\deltasg^*\text{cl}(A) = A$. Therefore $X \setminus \deltasg^*\text{cl}(A) = X \setminus A \in \deltasg^*\tau''$. Since $\deltasg^*\tau'' = \tau_\delta$, then $X \setminus A \in \tau_\delta$. Hence $A$ is $\delta$-semi closed.

Corollary 2.5.14 Every $\deltasg^*$-closed set is $\delta$-closed in $(X, \tau)$ if and only if $\deltasg^*\tau'' = \tau_\delta$.

Corollary 2.5.15 Every $\deltasg^*$-closed set is closed in a semi-regular space if and only if $\deltasg^*\tau'' = \tau_\delta$.

Theorem 2.5.16 For a point $x \in X$, $x \in \deltasg^*\text{cl}(A)$ if and only if $U \cap A \neq \emptyset$ for every $\deltasg^*$-open set $U$ in $(X, \tau)$ containing $x$.

Proof: Necessity: Let $x \in \deltasg^*\text{cl}(A)$. Suppose that there exists a $\deltasg^*$-open set $U$ in $(X, \tau)$ containing $x$ such that $U \cap A = \emptyset$. Hence $X \setminus U$ is $\deltasg^*$-closed in $(X, \tau)$ containing $A$, which implies that $\deltasg^*\text{cl}(A) \subseteq X \setminus U$. Hence $x \notin \deltasg^*\text{cl}(A)$, which is a contradiction. Hence $U \cap A \neq \emptyset$.

Sufficiency: Let us assume that $U \cap A \neq \emptyset$ for every $\deltasg^*$-open set $U$ in $(X, \tau)$ containing $x$. Suppose that $x \notin \deltasg^*\text{cl}(A)$. By definition of $\deltasg^*$-closure, there exists a $\deltasg^*$-open set in $(X, \tau)$ containing $A$ such that $x \notin U$. Hence $X \setminus U$ is $\deltasg^*$-open in $(X, \tau)$ containing $x$. Since $A \subseteq U$, we have $(X \setminus U) \cap A = \emptyset$, which is a contradiction. Hence $x \in \deltasg^*\text{cl}(A)$.

Theorem 2.5.17 Let $A$ be any subset of $(X, \tau)$. Then

(a) $(\deltasg^*\text{int}(A))^c = \deltasg^*\text{cl}(A^c)$

(b) $\deltasg^*\text{int}(A) = (\deltasg^*\text{cl}(A^c))^c$
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(c) δsg* cl(A) = (δsg* int(A c)) c

Proof: Let x ∈ (δsg* int(A)) c. Then x ∉ δsg* int(A). That is every δsg*-open set U containing x is such that U ∉ A. That is every δsg*-open set U containing x is such that U ∩ A c ≠ φ. By Theorem 2.5.16, x ∈ δsg* cl(A c) and therefore (δsg* int(A)) c ⊆ δsg* cl(A c).

Conversely, Let x ∈ δsg* cl(A c). Then by Theorem 2.5.16, every δsg*-open set U containing x is such that U ∩ A c ≠ φ. x ∉ δsg* int(A). That is every δsg*-open set U containing x is such that U ∉ A. By definition of δsg*-interior of A, x ∉ (δsg* int(A)) c and δsg* cl(A c) ⊆ (δsg* int(A)) c. Hence (δsg* int(A)) c = δsg* cl(A c).

(b) Follows by taking complements in (a).

(c) Follows by replacing A by A c in (a).

2.6 δsg* - Interior Operator

In this section, the notion of δsg*-interior of a set is introduced and some of its properties are studied.

Definition 2.6.1 Let (X, τ) be a topological space and let x ∈ X. A subset N of X is said to be δsg*-neighbourhood of x if there exists a δsg*-open set G such that x ∈ G ⊆ N.

Definition 2.6.2 Let A be a subset of X. A point x ∈ A is said to be δsg*-interior point of A if A is a δsg*-neighbourhood of x. The set of all δsg*-interior points of A is called the δsg*-interior of A and is denoted by δsg* int(A).

Theorem 2.6.3 If A be a subset of X, then δsg* int(A) = ∪ {G ⊆ X : G ⊆ A and G ∈ δSG* O(X, τ)}.

Proof: Let A be a subset of X.

x ∈ δsg* int(A) ⇔ x is a δsg*-interior point of A.

⇔ A is a δsg*-neighbourhood of x.

⇔ there exists a δsg*-open set G such that x ∈ G ⊆ A.

⇔ x ∈ ∪ {G ⊆ X : G ⊆ A and G ∈ δSG* O(X, τ)}
Hence $\delta_{sg}^* \text{int}(A) = \bigcup \{G \subseteq X : G \subseteq A \text{ and } G \in \delta_{SG}^* \text{O}(X, \tau)\}$

**Theorem 2.6.4** If $A$ is a subset of $X$, then $\delta_{-sint}(A) \subseteq \delta_{sg}^* \text{int}(A)$.

**Proof:** Let $A$ be a subset of $X$.

$$x \in \delta_{-sint}(A) \implies x \in \bigcup \{G \subseteq X : G \text{ is } \delta\text{-semi open}, G \subseteq A\}.$$  

$\implies$ there exists a $\delta$-semi open set $G$ such that $x \in G \subseteq A$.

$\implies$ there exists a $\delta_{sg}^*$-open set $G$ such that $x \in G \subseteq A$, as every $\delta$-semi open set is a $\delta_{sg}^*$-open set in $X$.

$\implies x \in \bigcup \{G \subseteq X : G \text{ is } \delta_{sg}^*\text{-open}, G \subseteq A\}$

$\implies x \in \delta_{sg}^* \text{int}(A)$

Thus $x \in \delta_{-sint}(A) \implies x \in \delta_{sg}^* \text{int}(A)$. Hence $\delta_{-sint}(A) \subseteq \delta_{sg}^* \text{int}(A)$.

**Example 2.6.5** Let $X = \{a, b, c, d\}$, $\tau = \{X, \phi, \{a\}, \{a, b\}\}$. Let $A = \{a\}$ then $\delta_{-sint}(A) = \phi$ and $\delta_{sg}^* \text{int}(A) = \{a\}$. Hence $\delta_{-sint}(A) \not= \delta_{sg}^* \text{int}(A)$.

**Theorem 2.6.6** If $A$ is a subset of $X$, then $\delta_{sg}^* \text{int}(A) \subseteq \text{gs-int}(A)$, where $\text{gs-int}(A)$ is given by $\text{gs-int}(A) = \bigcup \{G \subseteq X : G \subseteq A \text{ and } G \in \text{GSO}(X, \tau)\}$.

**Proof:** Let $A$ be a subset of $X$.

$$x \in \delta_{sg}^* \text{int}(A) \implies x \in \bigcup \{G \subseteq X : G \text{ is } \delta_{sg}^*\text{-open}, G \subseteq A\}.$$  

$\implies$ there exists a $\delta_{sg}^*$-open set $G$ such that $x \in G \subseteq A$.

$\implies$ there exists a $\text{gs}$-open set $G$ such that $x \in G \subseteq A$, as every $\delta_{sg}^*$-open set is a $\text{gs}$-open set in $X$.

$\implies x \in \bigcup \{G \subseteq X : G \text{ is } \text{gs}\text{-open}, G \subseteq A\}$

$\implies x \in \text{gs-int}(A)$

Thus $x \in \delta_{sg}^* \text{int}(A) \implies x \in \text{gs-int}(A)$. Hence $\delta_{sg}^* \text{int}(A) \subseteq \text{gs-int}(A)$.

**Example 2.6.7** Let $X = \{a, b, c, d\}$, $\tau = \{X, \phi, \{a, b\}\}$. Let $A = \{b, c\}$ then $\text{gs-int}(A) = \{b, c\}$ and $\delta_{sg}^* \text{int}(A) = \{b\}$. Hence $\text{gs-int}(A) \not= \delta_{sg}^* \text{int}(A)$. 
Remark 2.6.8 For a subset A of (X, τ), \( \text{int}_\delta(A) \subseteq \delta\text{-sint}(A) \subseteq \delta^*\text{int}(A) \subseteq \text{gs-int}(A) \subseteq A \).

Theorem 2.6.9 For any two subsets A and B of (X, τ), the following statements are true:

(a) \( \delta^*\text{int}(X) = X \) and \( \delta^*\text{int}(\emptyset) = \emptyset \)

(b) \( \delta^*\text{int}(A) \subseteq A \)

(c) If B is any \( \delta^* \)-open set contained in A, then B \( \subseteq \delta^*\text{int}(A) \)

(d) If A \( \subseteq B \), then \( \delta^*\text{int}(A) \subseteq \delta^*\text{int}(B) \)

Proof: (a) Since X and \( \emptyset \) are \( \delta^* \)-open sets,

\[
\delta^*\text{int}(X) = \cup \{G : G \subseteq X \text{ and } G \in \delta^*\text{O}(X, \tau)\}
\]

\[
= X \cup \{\text{all } \delta^*\text{-open sets}\}
\]

\[
= X
\]

Similarly since \( \emptyset \) is the only \( \delta^* \)-open set contained in \( \emptyset \), \( \delta^*\text{int}(\emptyset) = \emptyset \).

(b) Let x is a \( \delta^* \)-interior point of A.

Let \( x \in \delta^*\text{int}(A) \Rightarrow x \text{ is a } \delta^* \text{-interior point of A} \Rightarrow A \text{ is a } \delta^* \text{-neighbourhood of } x \Rightarrow x \in A \)

Thus, \( x \in \delta^*\text{int}(A) \Rightarrow x \in A \). Hence \( \delta^*\text{int}(A) \subseteq A \).

(c) Let B be any \( \delta^* \)-open set such that B \( \subseteq A \). Let \( x \in B \). Since B is a \( \delta^* \)-open set contained in A, \( x \) is a \( \delta^* \)-interior point of A. That is \( x \in \delta^*\text{int}(A) \). Hence B \( \subseteq \delta^*\text{int}(A) \).

(d) Let A and B be subsets of X such that A \( \subseteq B \). Let \( x \in \delta^*\text{int}(A) \). Then \( x \) is a \( \delta^* \)-interior point of A and so A is a \( \delta^* \)-neighbourhood of \( x \). Since \( B \supseteq A \), B is also \( \delta^* \)-neighbourhood of \( x \) then \( x \in \delta^*\text{int}(B) \). Hence \( \delta^*\text{int}(A) \subseteq \delta^*\text{int}(B) \).

Proposition 2.6.10 Let A be any subset of (X, τ). If A is \( \delta^* \)-open in (X, τ) then \( \delta^*\text{int}(A) = A \).
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Proof: Let A be $\delta^*\text{sg}$-open in $(X, \tau)$. We know that $\delta^*\text{sg}\text{int}(A) \subseteq A$. Also A is a $\delta^*\text{sg}$-open set contained in A. From above Theorem 2.6.9 (c), $A \subseteq \delta^*\text{sg}\text{int}(A)$. Hence $\delta^*\text{sg}\text{int}(A) = A$.

Corollary 2.6.11 $\delta^*\text{sg}\text{int}(\delta^*\text{sg}\text{int}(A)) = \delta^*\text{sg}\text{int}(A)$

Proof: By (b) and (d) of Theorem 2.6.9, $\delta^*\text{sg}\text{int}(\delta^*\text{sg}\text{int}(A)) \subseteq \delta^*\text{sg}\text{int}(A)$.

The reverse inclusion follows from Proposition 2.6.10.

Example 2.6.12 Converse of Proposition 2.6.10 need not be true. Let $X = \{a, b, c, d\}$, $\tau = \{X, \phi, \{c\}, \{a,b\}, \{a,b,c\}\}$

$\delta^*\text{sg}\text{int}\{b,c,d\} = \bigcup \{\text{all } \delta^*\text{sg}\text{-open sets contained in } \{b,c,d\}\}$

$= \{b\} \cup \{c\} \cup \{b,c\} \cup \{c,d\} \cup \phi$

$= \{b,c,d\}$

But $\{b,c,d\}$ is not a $\delta^*\text{sg}$-open set in $(X, \tau)$.

Theorem 2.6.13 If A and B are subsets of X, then $\delta^*\text{sg}\text{int}(A) \cup \delta^*\text{sg}\text{int}(B) \subseteq \delta^*\text{sg}\text{int}(A \cup B)$.

Proof: Since $A \subseteq A \cup B$ and $B \subseteq A \cup B$, by Theorem 2.6.9 (d), $\delta^*\text{sg}\text{int}(A) \subseteq \delta^*\text{sg}\text{int}(A \cup B)$ and $\delta^*\text{sg}\text{int}(B) \subseteq \delta^*\text{sg}\text{int}(A \cup B)$. Hence $\delta^*\text{sg}\text{int}(A) \cup \delta^*\text{sg}\text{int}(B) \subseteq \delta^*\text{sg}\text{int}(A \cup B)$.

Example 2.6.14 Let $X = \{a, b, c, d\}$, $\tau = \{X, \phi, \{c\}, \{a,b\}, \{a,b,c\}\}$. Let $A = \{a\}$, $B = \{b,d\}$ and $A \cup B = \{a,b,d\}$ then $\delta^*\text{sg}\text{int}\{a\} = \{a\}$, $\delta^*\text{sg}\text{int}\{b,d\} = \{b\}$ and $\delta^*\text{sg}\text{int}\{a,b,d\} = \{a,b,d\}$. Hence $\delta^*\text{sg}\text{int}(A \cup B) = \{a,b,d\} \not\subseteq \delta^*\text{sg}\text{int}\{a\} \cup \delta^*\text{sg}\text{int}\{b\} = \{a,b\}$.

Note 2.6.15 $\delta^*\text{sg}$-interior operator is not a Kuratowski interior operator since it is not satisfying

(i) $\delta^*\text{sg}\text{int}(A) \cup \delta^*\text{sg}\text{int}(B) = \delta^*\text{sg}\text{int}(A \cup B)$.

Remark 2.6.16 As union of $\delta^*\text{sg}$-open sets is not a $\delta^*\text{sg}$-open set, $\delta^*\text{sg}\text{int}(A)$ need not be the largest $\delta^*\text{sg}$-open set contained in A as seen in the following example.

Example 2.6.17 Let $X = \{a, b, c, d\}$, $\tau = \{X, \phi, \{c\}, \{a,b\}, \{a,b,c\}\}$

$\delta^*\text{sg}\text{int}\{a,c,d\} = \bigcup \{\text{all } \delta^*\text{sg}\text{-open sets contained in } \{a,c,d\}\}$
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\[ = \{a\} \cup \{c\} \cup \{a,c\} \cup \{c,d\} \]

\[= \{a,c,d\} \]

But \(\{a,c\}\) is the largest open set contained in \(\{a,c,d\}\).

**Theorem 2.6.18** If \(A\) and \(B\) are subsets of \(X\), then \(\delta_{sg}^* \text{int}(A \cap B) \subseteq \delta_{sg}^* \text{int}(A) \cap \delta_{sg}^* \text{int}(B)\).

**Proof:** Since \(A \cap B \subseteq A\) and \(A \cap B \subseteq B\), by Theorem 2.6.9 (d), \(\delta_{sg}^* \text{int}(A \cap B) \subseteq \delta_{sg}^* \text{int}(A)\) and \(\delta_{sg}^* \text{int}(A \cap B) \subseteq \delta_{sg}^* \text{int}(B)\). Hence \(\delta_{sg}^* \text{int}(A \cap B) \subseteq \delta_{sg}^* \text{int}(A) \cap \delta_{sg}^* \text{int}(B)\).

**Example 2.6.19** Let \(X = \{a, b, c, d\}\), \(\tau = \{X, \phi, \{a\}, \{b\}, \{a,b\}, \{a,b,c\}, \{a,b,d\}\}\). Let \(\delta_{sg}^* \text{int}\{a,c,d\} = \{a,c,d\}\), \(\delta_{sg}^* \text{int}\{b,c,d\} = \{b,c,d\}\) but \(\delta_{sg}^* \text{int}(A \cap B) = \delta_{sg}^* \text{int}\{c,d\} = \phi\). Hence \(\delta_{sg}^* \text{int}(A) \cap \delta_{sg}^* \text{int}(B) \not\subseteq \delta_{sg}^* \text{int}(A \cap B)\).