δsg*-Homeomorphisms in Topological Spaces

6.1 Introduction

If A and B are subsets of Euclidean space then a homeomorphism from A to B is a bijection function $f : A \to B$ such that both $f$ and its inverse functions are continuous. When such a function exists, A and B are said to be homeomorphic to each other. A property of an object which is invariant under homeomorphism is said to be topological in character. The concept of generalized homeomorphisms and gc-homeomorphisms were introduced by Maki (1991). The class of gc-homeomorphisms is properly placed between the classes of homeomorphisms and g-homeomorphisms. In this chapter, two new classes of functions called δsg*-homeomorphisms and δsg*-C-homeomorphisms are introduced. The interrelationships of these newly introduced functions with various functions are analysed and the dependence links are depicted. Furthermore δsg*-compactness and connectedness are discussed in this chapter. The composition of two δsg*-C-homeomorphisms is a δsg*-C-homeomorphism. The collection δsg*-C(A(X, τ)) of δsg*-C-homeomorphisms is a group under the composition of functions. The δsg*-C-homeomorphism $f : (X, τ) \to (Y, σ)$ induces an isomorphism from the group δsg*-C(A(X, τ)) onto the group δsg*-C(A(X, τ)).

6.2 δsg*-Closed Functions

Definition 6.2.1 A function $f : (X, τ) \to (Y, σ)$ is called a δsg*-closed function if the image of each closed set in $(X, τ)$ is δsg*-closed in $(Y, σ)$.

Example 6.2.2 Let $X = Y = \{a, b, c\}$ with $τ = \{X, φ, \{a\}, \{b,c\}\}$ and $σ = \{Y, φ, \{a\}, \{b\}, \{a,b\}\}$. Let $f : (X, τ) \to (Y, σ)$ be the function defined by $f(a) = c$, $f(b) = a$, $f(c) = c$. Then $f$ is δsg*-closed function.

Remark 6.2.3 δsg*-closedness and δsg*-continuity are independent as shown by the following examples.
Example 6.2.4 Let $X = Y = \{a, b, c\}$ with $\tau = \{X, \phi, \{a,b\}\}$ and $\sigma = \{Y, \phi, \{a\}, \{a,b\}, \{a,c\}\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be defined by $f(a) = b$, $f(b) = c$, $f(c) = c$. Then $f$ is $\delta sg^*$-closed function but not $\delta sg^*$-continuous, since for the closed set $\{b\}$ in $(Y, \sigma)$, $f^{-1}(\{b\}) = \{a\}$ is not $\delta sg^*$-closed in $(X, \tau)$.

Example 6.2.5 Let $X = Y = \{a, b, c, d\}$ with $\tau = \{X, \phi, \{a\}, \{a,b\}, \{a,c\}, \{a,b,c\}, \{a,c,d\}\}$ and $\sigma = \{Y, \phi, \{a\}, \{a,b\}\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be defined by $f(a) = c$, $f(b) = d$, $f(c) = a$, $f(d) = b$. Then $f$ is $\delta sg^*$-continuous function but not $\delta sg^*$-closed, since for the closed set $\{d\}$ in $(X, \tau)$, $f\{d\} = \{b\}$ is not $\delta sg^*$-closed in $(Y, \sigma)$.

Proposition 6.2.6 (a) Every $\delta$-closed function is $\delta sg^*$-closed but not conversely.

(b) Every $\delta$-semi closed function is $\delta sg^*$-closed but not conversely.

Example 6.2.7 Let $X = Y = \{a, b, c\}$ with $\tau = \{X, \phi, \{a\}, \{a,b\}\}$ and $\sigma = \{Y, \phi, \{a\}, \{a,b\}, \{b\}, \{a,b\}, \{a,c\}\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be defined by $f(a) = b$, $f(b) = a$, $f(c) = c$. Then $f$ is $\delta sg^*$-closed function but not $\delta$-closed (resp. $\delta$-semi closed), since for the closed set $\{c\}$ in $(X, \tau)$, $f\{c\} = \{c\}$ is not a $\delta$-closed (resp. $\delta$-semi closed) function in $(Y, \sigma)$.

Proposition 6.2.8
(a) Every $\delta sg^*$-closed function is $gs$-closed.

(b) Every $\delta sg^*$-closed function is $\delta gs$-closed.

(c) Every $\delta sg^*$-closed function is $g\delta s$-closed.

(d) Every $\delta sg^*$-closed function is $gsp$-closed.

(e) Every $\delta sg^*$-closed function is $\pi gs$-closed.

(f) Every $\delta sg^*$-closed function is $\pi gsp$-closed.

Proof: Follows from the fact that every $\delta sg^*$-closed set is $gs$-closed, $\delta gs$-closed, $g\delta s$-closed, $gsp$-closed, $\pi gs$-closed and $\pi gsp$-closed.

Example 6.2.9 Let $X = Y = \{a, b, c\}$ with $\tau = \{X, \phi, \{a\}, \{a,b\}\}$ and $\sigma = \{Y, \phi, \{a\}\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be the identity function. Then $f$ is $gs$-closed, $\delta gs$-closed, $g\delta s$-closed, $gsp$-closed, $\pi gs$-closed and $\pi gsp$-closed functions but not a $\delta sg^*$-closed function, since for the closed set $\{c\}$ in $(X, \tau)$, $f\{c\} = \{c\}$ is not $\delta sg^*$-closed in $(Y, \sigma)$.
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**Remark 6.2.10** The following examples show that $\delta g^*$-closed function is independent from $\delta g$ -closed function and $g\delta$ -closed function.

**Example 6.2.11** Let $X = Y = \{a, b, c\}$ with $\tau = \{X, \phi, \{a\}, \{a,b\}, \{a,c\}\}$ and $\sigma = \{Y, \phi, \{a\}, \{b,c\}\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be defined by $f(a) = c$, $f(b) = a$, $f(c) = b$. Then $f$ is both $\delta g$-closed and $g\delta$-closed but not $\delta g^*$-closed, since for the closed set $\{b,c\}$ in $(X, \tau)$, $f\{b,c\} = \{a,b\}$ is a both $\delta g$-closed and $g\delta$-closed set but not $\delta g^*$-closed in $(Y, \sigma)$.

**Example 6.2.12** Let $X = Y = \{a, b, c\}$ with $\tau = \{X, \phi, \{a\}, \{b\}, \{a,b\}, \{a,c\}\}$ and $\sigma = \{Y, \phi, \{a\}, \{b\}, \{a,b\}\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be the identity function. Then $f$ is $\delta g^*$-closed but not $\delta g$-closed and $g\delta$-closed, since for the closed set $\{b\}$ in $(X, \tau)$, $f\{b\} = \{b\}$ is not $\delta g$-closed and $g\delta$-closed but it is $\delta g^*$-closed in $(Y, \sigma)$.

**Remark 6.2.13** We have the following diagram

![Diagram of closed functions](image)

**6.3 Properties of $\delta g^*$-Closed Functions**

**Theorem 6.3.1** If $f : (X, \tau) \to (Y, \sigma)$ is a $\delta g^*$-closed function and $A$ is a closed subset of $(X, \tau)$, then $f|_A : (A, \tau|_A) \to (Y, \sigma)$ is a $\delta g^*$-closed function.

**Proof:** Let $B \subseteq A$ be a closed set in $(A, \tau|_A)$. Since $A$ is closed in $(X, \tau)$, $B$ is closed in $(X, \tau)$. Since $f$ is a $\delta g^*$-closed function, $f(B) = (f|_A)(B)$ is $\delta g^*$-closed in $(Y, \sigma)$. Hence $f|_A$ is a $\delta g^*$-closed function.
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**Theorem 6.3.2** A function \( f: (X, \tau) \rightarrow (Y, \sigma) \) is \( \delta\text{sg}^* \)-closed if and only if for each subset \( G \) of \( (Y, \sigma) \) and for each open set \( U \) of \( (X, \tau) \) containing \( f^{-1}(G) \), there exists a \( \delta\text{sg}^* \)-open set \( B \) of \( (Y, \sigma) \) such that \( G \subseteq B \) and \( f^{-1}(B) \subseteq U \).

**Proof:** Let \( f \) be a \( \delta\text{sg}^* \)-closed function and let \( G \) be a subset of \( (Y, \sigma) \), \( U \) be an open set of \( (X, \tau) \) containing \( f^{-1}(G) \). Then \( X \setminus U \) is closed in \( (X, \tau) \). Since \( f \) is \( \delta\text{sg}^* \)-closed, \( f(X \setminus U) \) is a \( \delta\text{sg}^* \)-closed set in \( (Y, \sigma) \). Hence \( Y \setminus f(X \setminus U) \) is a \( \delta\text{sg}^* \)-open set in \( (Y, \sigma) \). Take \( B = Y \setminus f(X \setminus U) \). Then \( B \) is \( \delta\text{sg}^* \)-open in \( (Y, \sigma) \) containing \( G \) such that \( f^{-1}(B) \subseteq U \).

Conversely, Let \( F \) be a closed subset of \( (X, \tau) \). Then \( f^{-1}(Y \setminus f(F)) \subseteq X \setminus F \) and \( X \setminus F \) is open. By hypothesis, there is a \( \delta\text{sg}^* \)-open set \( B \) of \( (Y, \sigma) \) such that \( Y \setminus f(F) \subset B \) and \( f^{-1}(B) \subset X \setminus F \). Therefore \( F \subset X \setminus f^{-1}(B) \). Hence \( Y \setminus B \subset f(F) \subset f(X \setminus f^{-1}(B)) \subset Y \setminus B \), which implies \( f(F) = Y \setminus B \) and hence \( f(F) \) is \( \delta\text{sg}^* \)-closed in \( (Y, \sigma) \). Thus \( f \) is a \( \delta\text{sg}^* \)-closed function.

**Theorem 6.3.3** A bijection \( f: (X, \tau) \rightarrow (Y, \sigma) \) is a \( \delta\text{sg}^* \)-closed function if and only if \( f(U) \) is \( \delta\text{sg}^* \)-open in \( (Y, \sigma) \) for every open set \( U \) in \( (X, \tau) \).

**Proof:** Let \( f: (X, \tau) \rightarrow (Y, \sigma) \) be a \( \delta\text{sg}^* \)-closed function and \( U \) be an open set in \( (X, \tau) \). Then \( X \setminus U \) is a closed set in \( (X, \tau) \). Since \( f \) is a \( \delta\text{sg}^* \)-closed function, \( f(X \setminus U) \) is a \( \delta\text{sg}^* \)-closed set in \( (Y, \sigma) \). Since \( f \) is bijection, \( f(X \setminus U) = X \setminus f(U) \) and hence \( X \setminus f(U) \) is \( \delta\text{sg}^* \)-closed in \( (Y, \sigma) \). Hence \( f(U) \) is \( \delta\text{sg}^* \)-open in \( (Y, \sigma) \).

Conversely, Let \( U \) be a closed subset of \( (X, \tau) \). Then \( X \setminus U \) is an open set in \( (X, \tau) \). By the hypothesis, \( f(X \setminus U) \) is \( \delta\text{sg}^* \)-open in \( (Y, \sigma) \). Since \( f \) is bijective, \( f(X \setminus U) = X \setminus f(U) \) and hence \( f(U) \) is \( \delta\text{sg}^* \)-closed in \( (Y, \sigma) \). Thus \( f \) is a \( \delta\text{sg}^* \)-closed function.

**Remark 6.3.4** Bijection of \( f \) is necessary in the above theorem which can be seen in the following example.

**Example 6.3.5** Let \( X = Y = \{a, b, c\} \) with \( \tau = \{X, \phi, \{a\}\} \) and \( \sigma = \{Y, \phi, \{a\}, \{a,b\}\} \). Let \( f: (X, \tau) \rightarrow (Y, \sigma) \) be defined by \( f(a) = b, f(b) = a, f(c) = a \). Here \( f \) is not bijective. Then for the only open set \( \{a\} \) in \( (X, \tau) \), \( f(\{a\}) \) is \( \delta\text{sg}^* \)-open in \( (Y, \sigma) \) but \( f \) is not a \( \delta\text{sg}^* \)-closed function as for the closed set \( \{b,c\} \) in \( (X, \tau) \), \( f(\{b,c\}) = \{a\} \) is not \( \delta\text{sg}^* \)-closed in \( (Y, \sigma) \).
6.4 Composition of $\delta s g^*$-Closed Functions

**Remark 6.4.1** The composition of two $\delta s g^*$-closed functions is not a $\delta s g^*$-closed function as shown in the following example.

**Example 6.4.2** Let $X = Y = Z = \{a, b, c\}$ with $\tau = \{X, \phi, \{a\}, \{a, b\}\}$, $\sigma = \{Y, \phi, \{a, b\}\}$ and $\eta = \{Z, \phi, \{a\}, \{b, c\}\}$. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be defined by $f(a) = b$, $f(b) = a$, $f(c) = c$ and $g : (Y, \sigma) \rightarrow (Z, \eta)$ be defined by $g(a) = c$, $g(b) = b$, $g(c) = a$. Then $f$ and $g$ are $\delta s g^*$-closed functions but their composition $g \circ f : (X, \tau) \rightarrow (Z, \eta)$ is not $\delta s g^*$-closed function, since for the closed set $\{b, c\}$ in $(X, \tau)$, $(g \circ f)\{b, c\} = \{a, c\}$ is not $\delta s g^*$-closed in $(Z, \eta)$.

**Theorem 6.4.3** If $f : (X, \tau) \rightarrow (Y, \sigma)$ is a closed function and $g : (Y, \sigma) \rightarrow (Z, \eta)$ is a $\delta s g^*$-closed function then $g \circ f : (X, \tau) \rightarrow (Z, \eta)$ is a $\delta s g^*$-closed function.

**Proof:** Let $V$ be a closed subset of $(X, \tau)$. Since $f$ is a closed function, $f(V)$ is closed in $(Y, \sigma)$. Since $g$ is a $\delta s g^*$-closed function, $g(f(V))$ is $\delta s g^*$-closed in $(Z, \eta)$. That is, $(g \circ f)(V)$ is $\delta s g^*$-closed in $(Z, \eta)$. Hence $g \circ f$ is a $\delta s g^*$-closed function.

**Remark 6.4.4** If $f : (X, \tau) \rightarrow (Y, \sigma)$ is a $\delta s g^*$-closed function and $g : (Y, \sigma) \rightarrow (Z, \eta)$ is a closed function then their composition need not be a $\delta s g^*$-closed function as seen from the following example.

**Example 6.4.5** Let $X = \{a, b, c\} = Y$ with $\tau = \{\phi, X, \{a\}, \{a, b\}\}$, $\sigma = \{\phi, Y, \{a, b\}\}$ and $\eta = \{\phi, Z, \{a\}, \{b, c\}\}$. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a function such that $f(a) = b$, $f(b) = a$, $f(c) = c$. Let $g : (Y, \sigma) \rightarrow (Z, \eta)$ be a function such that $g(a) = c$, $g(b) = b$, $g(c) = a$. Then $f$ is a $\delta s g^*$-closed function and $g$ is a closed function. But their composition function $(g \circ f) : (X, \tau) \rightarrow (Z, \eta)$ is not a $\delta s g^*$-closed function since for the closed set $\{b, c\}$ in $(X, \tau)$, $(g \circ f)\{b, c\} = \{a, c\}$ is not $\delta s g^*$-closed in $(Z, \eta)$.

**Proposition 6.4.6** Let $f : (X, \tau) \rightarrow (Y, \sigma)$ and $g : (Y, \sigma) \rightarrow (Z, \eta)$ be $\delta s g^*$-closed functions and $(Y, \sigma)$ be a $\delta s g^* T_\delta$-space. Then their composition $g \circ f : (X, \tau) \rightarrow (Z, \eta)$ is a $\delta s g^*$-closed function.

**Proof:** Let $A$ be a closed set in $(X, \tau)$. Since $f$ is $\delta s g^*$-closed, $f(A)$ is $\delta s g^*$-closed in $(Y, \sigma)$. Since $(Y, \sigma)$ is a $\delta s g^* T_\delta$-space, $f(A)$ is $\delta$-closed. Hence $f(A)$ is closed in $(Y, \sigma)$. Since $g$ is
\( \delta \mathsf{sg}^* \)-closed, \( g(f(A)) = (g \circ f)(A) \) is \( \delta \mathsf{sg}^* \)-closed in \( (Z, \eta) \). Hence \( g \circ f \) is a \( \delta \mathsf{sg}^* \)-closed function.

**Theorem 6.4.7** Let \( f : (X, \tau) \to (Y, \sigma) \) and \( g : (Y, \sigma) \to (Z, \eta) \) be any two functions:

(a) If \( g \circ f : (X, \tau) \to (Z, \eta) \) is \( \delta \mathsf{sg}^* \)-closed and \( g \) is \( \delta \mathsf{sg}^* \)-irresolute injective then \( f \) is a \( \delta \mathsf{sg}^* \)-closed function.

(b) If \( g \circ f : (X, \tau) \to (Z, \eta) \) is \( \delta \mathsf{sg}^* \)-irresolute and \( g \) is \( \delta \mathsf{sg}^* \)-closed injective then \( f \) is a \( \delta \mathsf{sg}^* \)-continuous function.

**Proof:** (a) Let \( U \) be a closed set in \( (X, \tau) \). Since \( g \circ f \) is \( \delta \mathsf{sg}^* \)-closed, \( g \circ f(U) \) is \( \delta \mathsf{sg}^* \)-closed in \( (Z, \eta) \). Therefore \( g(f(U)) \) is \( \delta \mathsf{sg}^* \)-closed in \( (Z, \eta) \). Since \( g \) is \( \delta \mathsf{sg}^* \)-irresolute, \( g^{-1}(g(f(U))) \) is \( \delta \mathsf{sg}^* \)-closed in \( (Y, \sigma) \). Since \( g \) is injective, \( g^{-1}(g(f(U))) = f(U) \) is \( \delta \mathsf{sg}^* \)-closed in \( (Y, \sigma) \). Hence \( f \) is a \( \delta \mathsf{sg}^* \)-closed function.

(b) Let \( V \) be a closed set in \( (Y, \sigma) \). Since \( g \) is \( \delta \mathsf{sg}^* \)-closed, \( g(V) \) is \( \delta \mathsf{sg}^* \)-closed in \( (Z, \eta) \). Since \( g \circ f \) is \( \delta \mathsf{sg}^* \)-irresolute, \( (g \circ f)^{-1}(g(V)) \) is \( \delta \mathsf{sg}^* \)-closed in \( (X, \tau) \). Therefore \( f^{-1}(g^{-1}(g(V))) \) is \( \delta \mathsf{sg}^* \)-closed in \( (X, \tau) \). Since \( g \) is injective, \( g^{-1}(g(f(U))) = V \) and hence \( f^{-1}(V) \) is \( \delta \mathsf{sg}^* \)-closed in \( (X, \tau) \). Therefore \( f \) is a \( \delta \mathsf{sg}^* \)-continuous function.

**Theorem 6.4.8** Let \( f : (X, \tau) \to (Y, \sigma) \) and \( g : (Y, \sigma) \to (Z, \eta) \) be any two functions such that their composition \( g \circ f : (X, \tau) \to (Z, \eta) \) is a \( \delta \mathsf{sg}^* \)-closed function. If \( f \) is continuous then \( g \) is a \( \delta \mathsf{sg}^* \)-closed function.

**Proof:** Let \( V \) be a closed set in \( (Y, \sigma) \). Since \( f \) is continuous, \( f^{-1}(V) \) is closed in \( (X, \tau) \). Since \( g \circ f \) is \( \delta \mathsf{sg}^* \)-closed, \( (g \circ f)^{-1}(g(V)) \) is \( \delta \mathsf{sg}^* \)-closed in \( (X, \tau) \). Therefore \( f^{-1}(g^{-1}(g(V))) = f^{-1}(V) \) is \( \delta \mathsf{sg}^* \)-closed in \( (X, \tau) \). Hence \( g \) is a \( \delta \mathsf{sg}^* \)-closed function.

**Proposition 6.4.9** Let \( f : (X, \tau) \to (Y, \sigma) \) and \( g : (Y, \sigma) \to (Z, \eta) \) be two functions such that their composition \( g \circ f : (X, \tau) \to (Z, \eta) \) is a \( \delta \mathsf{sg}^* \)-closed function. Then the following statements are true.

(a) If \( f \) is a surjective continuous function, then \( g \) is a \( \delta \mathsf{sg}^* \)-closed function.

(b) If \( f \) is a surjective \( g \)-continuous function and \( (X, \tau) \) is a \( T_{1/2} \) -space, then \( g \) is a \( \delta \mathsf{sg}^* \)-closed function.

(c) If \( f \) is a quasi \( \delta \mathsf{sg}^* \)-continuous function and injective, then \( f \) is a closed function.
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Proof: (a) Let A be any closed set in (Y, σ). Since f is continuous, f⁻¹(A) is closed in (X, τ). Since g ∘ f is a δsg*-closed function, (g ∘ f)(f⁻¹(A)) is δsg*-closed in (Z, η). Since f is surjective, (g ∘ f)(f⁻¹(A)) = g(A). Hence g(A) is δsg*-closed in (Z, η). Therefore g : (Y, σ) → (Z, η) is a δsg*-closed function.

(b) Let V be any closed set in (Y, σ). Since f is g-continuous, f⁻¹(V) is g-closed in (X, τ). Since (X, τ) is T₁/₂-space, f⁻¹(V) is closed in (X, τ). Since g ∘ f is δsg*-closed and f is surjective, (g ∘ f)(f⁻¹(V)) = g(V) is δsg*-closed in (Z, η). Therefore g : (Y, σ) → (Z, η) is a δsg*-closed function.

(c) Let V be any closed set in (X, τ). Since g ∘ f is a δsg*-closed function, (g ∘ f)⁻¹(V) is δsg*-closed in (Z, η). Since g is quasi δsɡ*-continuous and injective g⁻¹(g ∘ f)(V) = f(V) is closed in (Y, σ). Therefore f : (X, τ) → (Y, σ) is a closed function.

6.5 δsg*-Open Functions

Definition 6.5.1 A function f : (X, τ) → (Y, σ) is called a δsg*-open function if the image of each open set in (X, τ) is a δsg*-open set in (Y, σ).

Proposition 6.5.2 For any bijective function f : (X, τ) → (Y, σ) the following statements are equivalent.

(a) f⁻¹ : (Y, σ) → (X, τ) is a δsg*-continuous function.

(b) f is a δsg*-open function.

(c) f is a δsg*-closed function.

Proof: (a) ⇒ (b) Let U be an open set in (X, τ). Since f⁻¹ is a δsg*-continuous function, (f⁻¹)⁻¹(U) is δsg*-open in (Y, σ). As f is bijective, (f⁻¹)⁻¹(U) = f(U) which is δsg*-open in (Y, σ). Hence f is a δsg*-open function.

(b) ⇒ (c) Let V be a closed set in (X, τ). Then X \ V is open in (X, τ). Since f is δsg*-open, f(X \ V) is δsg*-open in (Y, σ). That is f(X \ V) = Y \ f(V) which is δsg*-open in (Y, σ). This implies that f(V) is δsg*-closed in (Y, σ). Hence f is a δsg*-closed function.
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(c) $\Rightarrow$ (a) Let V be a closed set in $(X, \tau)$. Since f is a $\delta_{sg}^*$-closed function, $f(V)$ is $\delta_{sg}^*$-closed in $(Y, \sigma)$. But $f(V) = (f^{-1})^{-1}(V)$ is $\delta_{sg}^*$-closed in $(Y, \sigma)$. Hence $f^{-1}$ is a $\delta_{sg}^*$-continuous function.

**Proposition 6.5.3** Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be any function and $g : (Y, \sigma) \rightarrow (Z, \eta)$ be injective and a $\delta_{sg}^*$-irresolute function. If their composition function $g \circ f : (X, \tau) \rightarrow (Z, \eta)$ is $\delta_{sg}^*$-open then $f$ is $\delta_{sg}^*$-open in $(Y, \sigma)$.

**Proof:** Let V be any open set in $(X, \tau)$. Since $(g \circ f)$ is $\delta_{sg}^*$-open, $(g \circ f)(V)$ is $\delta_{sg}^*$-open in $(Z, \eta)$. Since $g$ is $\delta_{sg}^*$-irresolute and injective, $g^{-1}[(g \circ f)(V)] = f(V)$ is $\delta_{sg}^*$-open in $(Y, \sigma)$.

Hence $f$ is a $\delta_{sg}^*$-open function.

## 6.6 $\delta_{sg}^*$-Homeomorphisms

**Definition 6.6.1** A bijective function $f : (X, \tau) \rightarrow (Y, \sigma)$ is called a $\delta_{sg}^*$-homeomorphism if f is both $\delta_{sg}^*$-continuous and a $\delta_{sg}^*$-open function.

**Proposition 6.6.2** Every $\delta_{sg}^*$-homeomorphism is a gs-homeomorphism but not conversely.

**Proof:** Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a $\delta_{sg}^*$-homeomorphism. Then $f$ is bijective, $\delta_{sg}^*$-continuous and $\delta_{sg}^*$-open function. Let V be a closed set in $(Y, \sigma)$. Then $f^{-1}(V)$ is $\delta_{sg}^*$-closed in $(X, \tau)$. By Theorem 2.2.17, every $\delta_{sg}^*$-closed set is gs-closed, $f^{-1}(V)$ is gs-closed in $(X, \tau)$. This implies that $f$ is gs-continuous. Let U be an open set in $(X, \tau)$. Then $f(U)$ is $\delta_{sg}^*$-open in $(Y, \sigma)$. Since every $\delta_{sg}^*$-open set is gs-open, $f(U)$ is gs-open in $(Y, \sigma)$. Hence $f$ is a gs-open function. Therefore $f$ is a gs-homeomorphism.

**Example 6.6.3** Let $X = Y = \{a, b, c\}$ with $\tau = \{X, \phi, \{a\}, \{a, b\}, \{a, c\}\}$ and $\sigma = \{Y, \phi, \{a\}, \{a, b\}\}$. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be the identity function. Then $f$ is gs-homeomorphism but not a $\delta_{sg}^*$-homeomorphism, since for the closed set $\{c\}$ in $Y$, $f^{-1}\{c\} = \{c\}$ is not $\delta_{sg}^*$-closed in $(X, \tau)$.

**Proposition 6.6.4**

(a) Every $\delta_{sg}^*$-homeomorphism is a $\delta_{gs}$-homeomorphism.

(b) Every $\delta_{sg}^*$-homeomorphism is a $g\delta_{gs}$-homeomorphism.

(c) Every $\delta_{sg}^*$-homeomorphism is a gsp-homeomorphism.
(d) Every $\delta g^*$-homeomorphism is a $\pi g s$-homeomorphism.
(e) Every $\delta g^*$-homeomorphism is a $\pi g s p$-homeomorphism.

**Proof:** Follows from the fact that every $\delta g^*$-continuous function (resp. open function) is $\delta g s^*$, $g s^*$, $g s p^*$, $\pi g s^*$, $\pi g s p^*$-continuous function (resp. open function). The reverse relations don’t hold good which can be seen from the following examples.

**Example 6.6.5** Let $X = Y = \{a, b, c\}$ with $\tau = \{X, \phi, \{a\}, \{a,b\}\}$ and $\sigma = \{Y, \phi, \{a\}, \{b\}, \{a,b\}\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be the function defined by $f(a) = b$, $f(b) = c$, $f(c) = a$. Then $f$ is a $\delta g s$-homeomorphism but not a $\delta g^*$-homeomorphism, since for the closed set $\{b,c\}$ in $(Y, \sigma)$, $f^{-1}\{b,c\} = \{a,b\}$ is not $\delta g^*$-closed in $(X, \tau)$. This implies $f$ is not a $\delta g^*$-homeomorphism.

**Example 6.6.6** Let $X = Y = \{a, b, c\}$ with $\tau = \{X, \phi, \{a\}\}$ and $\sigma = \{Y, \phi, \{a\}, \{b\}, \{a,b\}\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be the identity function. Then $f$ is a $g s^*$-homeomorphism but not a $\delta g^*$-homeomorphism, since for the closed set $\{a,c\}$ in $(Y, \sigma)$, $f^{-1}\{a,c\} = \{a,c\}$ is not $\delta g^*$-closed in $(X, \tau)$.

**Example 6.6.7** Let $X = Y = \{a, b, c, d\}$ with $\tau = \{X, \phi, \{a\}, \{a,b\}\}$ and $\sigma = \{Y, \phi, \{a\}, \{b\}, \{a,b\}, \{a,b,c\}, \{a,b,d\}\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be the identity function. Then $f$ is a $g s p$-homeomorphism but not a $\delta g^*$-homeomorphism, since for the closed set $\{d\}$ in $(Y, \sigma)$, $f^{-1}\{d\} = \{d\}$ is not $\delta g^*$-closed in $(X, \tau)$.

**Example 6.6.8** Let $X = Y = \{a, b, c, d\}$ with $\tau = \{X, \phi, \{a,b,c\}\}$ and $\sigma = \{Y, \phi, \{a\}, \{c\}, \{a,b\}, \{a,c\}, \{a,b,c\}, \{a,c,d\}\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be the function defined by $f(a) = b$, $f(b) = a$, $f(c) = c$ $f(d) = d$. Then $f$ is a $\pi g s$-homeomorphism but not a $\delta g^*$-homeomorphism, since for the closed set $\{b\}$ in $(Y, \sigma)$, $f^{-1}\{b\} = \{a\}$ is not $\delta g^*$-closed in $(X, \tau)$.

**Example 6.6.9** Let $X = Y = \{a, b, c, d\}$ with $\tau = \{X, \phi, \{a\}\}$ and $\sigma = \{Y, \phi, \{a,b\}\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be the identity function. Then $f$ is a $\pi g s p$-homeomorphism but not a $\delta g^*$-homeomorphism, since for the closed set $\{c,d\}$ in $(Y, \sigma)$, $f^{-1}\{c,d\} = \{c,d\}$ is not $\delta g^*$-closed in $(X, \tau)$.

**Remark 6.6.10**

(a) Every $\delta$-homeomorphism is a $\delta g^*$-homeomorphism but not conversely.
Every $\delta$-semi homeomorphism is a $\delta sg^*$-homeomorphism but not conversely.

**Example 6.6.11** Let $X = Y = \{a, b, c\}$ with $\tau = \{X, \phi, \{a\}\}$ and $\sigma = \{Y, \phi, \{a\}\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be the identity function. Then $f$ is a $\delta sg^*$-homeomorphism but not $\delta$-homeomorphism and not $\delta$-semi homeomorphism as $f$ is not a $\delta$-open function and not a $\delta$-semi open function, since for the open set $\{a\}$ in $(X, \tau)$, $f\{a\} = \{a\}$ is not $\delta$-open and not $\delta$-semi open in $(Y, \sigma)$.

**Remark 6.6.12** A homeomorphism and a $\delta sg^*$-homeomorphism are independent of each other as shown in the following examples.

**Example 6.6.13** Let $X = Y = \{a, b, c, d\}$ with $\tau = \{X, \phi, \{a\}, \{a,b\}\}$ and $\sigma = \{Y, \phi, \{a,b\}\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be the identity function. Then $f$ is $\delta sg^*$-open and $\delta sg^*$-continuous. Hence $f$ is a $\delta sg^*$-homeomorphism but not a homeomorphism, since for the image of open set $\{a\}$ in $(X, \tau)$, $f\{a\} = \{a\}$ is not open in $(Y, \sigma)$. Hence $f$ is not an open function. Therefore $f$ is not a homeomorphism.

**Example 6.6.14** Let $X = Y = \{a, b, c\}$ with $\tau = \{X, \phi, \{a\}, \{a,b\}, \{a,c\}\}$ and $\sigma = \{Y, \phi, \{a\}, \{a,b\}, \{a,c\}\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be the identity function. Then $f$ is open and continuous. Hence $f$ is a homeomorphism but not a $\delta sg^*$-homeomorphism, since for the closed set $\{c\}$ in $(Y, \sigma)$, $f^{-1}\{c\} = \{c\}$ is not $\delta sg^*$-closed in $(X, \tau)$. Hence $f$ is not a $\delta sg^*$-homeomorphism.

**Remark 6.6.15** A $g$-homeomorphism and a $\delta sg^*$-homeomorphism are independent of each other as shown in the following examples.

**Example 6.6.16** Let $X = Y = \{a, b, c\}$ with $\tau = \{X, \phi, \{a\}, \{b\}, \{a,b\}, \{a,c\}\}$ and $\sigma = \{Y, \phi, \{a\}, \{b\}, \{a,b\}\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be the identity function. Then $f$ is $\delta sg^*$-open and $\delta sg^*$-continuous. Hence $f$ is a $\delta sg^*$-homeomorphism but not a $g$-homeomorphism, since for the image of open set $\{a,c\}$ in $(X, \tau)$, $f\{a,c\} = \{a,c\}$ is not $g$-open in $(Y, \sigma)$. Hence $f$ is not a $g$-open function. Therefore $f$ is not a $g$-homeomorphism.

**Example 6.6.17** Let $X = Y = \{a, b, c\}$ with $\tau = \{X, \phi, \{a\}\}$ and $\sigma = \{Y, \phi, \{a\}, \{b\}, \{a,b\}\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be the identity function. Then $f$ is a $g$-homeomorphism but
not a $\delta_{sg^*}$-homeomorphism, since for the open set $\{b\}$ in $(Y, \sigma)$, $f^{-1}\{b\} = \{b\}$ is not $\delta_{sg^*}$-open in $(X, \tau)$. Hence $f$ is not a $\delta_{sg^*}$-homeomorphism.

**Remark 6.6.18** A $sg$-homeomorphism and a $\delta_{sg^*}$-homeomorphism are independent of each other as shown in the following examples.

**Example 6.6.19** Let $X = Y = \{a, b, c\}$ with $\tau = \{X, \phi, \{b\}, \{a,b\}\}$ and $\sigma = \{Y, \phi, \{a\}, \{a,b\}\}$. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be the identity function. Then $f$ is $\delta_{sg^*}$-open and $\delta_{sg^*}$-continuous. Hence $f$ is a $\delta_{sg^*}$-homeomorphism but not a $sg$-homeomorphism, since for the open set $\{b\}$ in $(X, \tau)$, $f(\{b\}) = \{b\}$ is not $sg$-open in $(Y, \sigma)$. Hence $f$ is not a $sg$-homeomorphism.

**Example 6.6.20** Let $X = Y = \{a, b, c\}$ with $\tau = \{X, \phi, \{a\}\}$ and $\sigma = \{Y, \phi, \{a\}, \{a,b\}\}$. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be the identity function. Then $f$ is a $sg$-homeomorphism but not a $\delta_{sg^*}$-homeomorphism, since for the open set $\{a,b\}$ in $(Y, \sigma)$, $f^{-1}\{a,b\} = \{a,b\}$ is not $\delta_{sg^*}$-open in $(X, \tau)$. Hence $f$ is not a $\delta_{sg^*}$-homeomorphism.

**Proposition 6.6.21** Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a bijective function and $\delta_{sg^*}$-continuous function. Then the following statements are equivalent.

(a) $f$ is a $\delta_{sg^*}$-open function.

(b) $f$ is a $\delta_{sg^*}$-homeomorphism.

(c) $f$ is a $\delta_{sg^*}$-closed function.

**Proof:** (a) $\Rightarrow$ (b) Let $f$ be a $\delta_{sg^*}$-open function. By hypothesis, $f$ is bijective and $\delta_{sg^*}$-continuous. Hence $f$ is a $\delta_{sg^*}$-homeomorphism.

(b) $\Rightarrow$ (c) Let $f$ be a $\delta_{sg^*}$-homeomorphism. Then $f$ is $\delta_{sg^*}$-open, by Proposition 6.5.2, $f$ is a $\delta_{sg^*}$-closed function.

(c) $\Rightarrow$ (a) It is obtained from the Proposition 6.5.2.
**Remark 6.6.22** The above discussion is portrayed in the following diagram.

![Diagram of homeomorphisms](image)

**Definition 6.6.23** A bijective function $f : (X, \tau) \rightarrow (Y, \sigma)$ is said to be a $\delta_{sg}^*$-homeomorphism if both $f$ and $f^{-1}$ are $\delta_{sg}^*$-irresolute.

The family of all $\delta_{sg}^*$-homeomorphisms of a topological space $(X, \tau)$ onto itself is denoted by $\delta_{sg}^*(X, \tau)$.

**Remark 6.6.24** A $\delta_{sg}^*$-homeomorphism and a $\delta_{sg}^*$-homeomorphism are independent notions as shown in the following example.

**Example 6.6.25** Let $X = Y = \{a, b, c\}$ with $\tau = \{X, \phi, \{a\}, \{a,b\}, \{a,c\}\}$ and $\sigma = \{Y, \phi, \{a\}\}$. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be the identity function. Then $f$ is a $\delta_{sg}^*$-homeomorphism since $\delta_{sg}^*\tau = \delta_{sg}^*\sigma = \{X, \phi, \{a\}\}$ but it is not a $\delta_{sg}^*$-homeomorphism, since for the open set $\{a,b\}$ in $(X, \tau)$, $f\{a,b\} = \{a,b\}$ is not $\delta_{sg}^*$-open in $(Y, \sigma)$.

**Remark 6.6.26** The composition of two $\delta_{sg}^*$-homeomorphisms is not $\delta_{sg}^*$-homeomorphism, since composition of two $\delta_{sg}^*$-continuous functions is not $\delta_{sg}^*$-continuous.

**Theorem 6.6.27** The composition of two $\delta_{sg}^*$-homeomorphisms is a $\delta_{sg}^*$-homeomorphism.

**Proof:** By Proposition 5.2.21, Composition of two $\delta_{sg}^*$-irresolute functions is $\delta_{sg}^*$-irresolute, the proof follows.
Chapter 6

**Theorem 6.6.28** Every $δsg^*$-homeomorphism from a $δsg^*Tδ$-space into another $δsg^*Tδ$-space is a homeomorphism.

**Proof:** Let $f : (X, τ) → (Y, σ)$ be a $δsg^*$-homeomorphism. Then $f$ is bijective, $δsg^*$-open and $δsg^*$-continuous functions. Let $U$ be an open set in $(X, τ)$. Since $f$ is $δsg^*$-open and since $(Y, σ)$ is $δsg^*Tδ$-space, $f(U)$ is $δ$-open which implies that $f(U)$ is an open set in $(Y, σ)$. This implies that $f$ is an open function. Let $V$ be a closed set in $(Y, σ)$. Since $f$ is $δsg^*$-continuous and since $(X, τ)$ is $δsg^*Tδ$-space, $f^{-1}(V)$ is $δ$-closed in $(X, τ)$, which implies that $f^{-1}(V)$ is closed in $(X, τ)$. Therefore $f$ is continuous. Hence $f$ is a homeomorphism.

**Theorem 6.6.29** Let $(Y, σ)$ be $δsg^*Tδ$-space. If $f : (X, τ) → (Y, σ)$ and $g : (Y, σ) → (Z, η)$ are $δsg^*$-homeomorphisms then $g ◦ f$ is a $δsg^*$-homeomorphism.

**Proof:** Let $f : (X, τ) → (Y, σ)$ and $g : (Y, σ) → (Z, η)$ be $δsg^*$-homeomorphisms. Let $U$ be an open set in $(X, τ)$. Since $f$ is a $δsg^*$-open function, $f(U)$ is a $δsg^*$-open set in $(Y, σ)$. Since $(Y, σ)$ is $δsg^*Tδ$-space, $f(U)$ is $δ$-open in $(Y, σ)$. Since every $δ$-open set is open, $f(U)$ is open in $(Y, σ)$. Also since $g$ is a $δsg^*$-open function, $g(f(U))$ is $δsg^*$-open in $(Z, η)$. Hence $g ◦ f$ is a $δsg^*$-open function.

Let $V$ be a closed set in $(Z, η)$. Since $g$ is $δsg^*$-continuous and $(Y, σ)$ is $δsg^*Tδ$-space, $g^{-1}(V)$ is closed implies that $g^{-1}(V)$ is closed in $(Y, σ)$. Since $f$ is $δsg^*$-continuous, $f^{-1}(g^{-1}(V)) = (g ◦ f)^{-1}(V)$ is a $δsg^*$-closed set in $(X, τ)$. That is $g ◦ f$ is $δsg^*$-continuous. Hence $g ◦ f$ is a $δsg^*$-homeomorphism.

**Theorem 6.6.30** Every $δsg^*$-homeomorphism from a $δsg^*Tδ$-space into another $δsg^*Tδ$-space is a $δsg^*C$-homeomorphism.

**Proof:** Let $f : (X, τ) → (Y, σ)$ be a $δsg^*$-homeomorphism. Let $V$ be a $δsg^*$-closed set in $(Y, σ)$. Then $V$ is closed in $(Y, σ)$. Since $f$ is $δsg^*$-continuous, $f^{-1}(V)$ is $δsg^*$-closed in $(X, τ)$. Hence $f$ is a $δsg^*$-irresolute function. Let $U$ be a $δsg^*$-open set in $(X, τ)$. Then $U$ is open in $(X, τ)$. Since $f$ is $δsg^*$-open, $f(U)$ is a $δsg^*$-open set in $(Y, σ)$. That is $(f^{-1})^{-1}(U)$ is $δsg^*$-open in $(Y, σ)$ and hence $f^{-1}$ is $δsg^*$-irresolute. Thus $f$ is a $δsg^*C$-homeomorphism.
Chapter 6

Theorem 6.6.31 The set $\delta^*CH(X, \tau)$ is a group under the composition of functions.

**Proof:** Let us define a binary operation $\ast : \delta^*CH(X, \tau) \times \delta^*CH(X, \tau) \rightarrow \delta^*CH(X, \tau)$ by $(f \ast g) = (g \circ f)$ for every $f, g \in \delta^*CH(X, \tau)$ and $\circ$ is the usual operation of composition of functions. Then by the Theorem 6.6.27, $(g \circ f) \in \delta^*CH(X, \tau)$. We know that the composition of functions is associative and the identity function $I : (X, \tau) \rightarrow (X, \tau)$ belongs to $\delta^*CH(X, \tau)$ serves as the identity element. If $f \in \delta^*CH(X, \tau)$ then $f^{-1} \in \delta^*CH(X, \tau)$ such that $(f \circ f^{-1}) = f^{-1} \circ f = I$. So the inverse exists for each element of $\delta^*CH(X, \tau)$. Therefore $\delta^*CH(X, \tau)$ is a group under the operation of composition of functions.

Theorem 6.6.32 Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a $\delta^*C$-homeomorphism. Then $f$ induces an isomorphism from the group $\delta^*CH(X, \tau)$ onto the group $\delta^*CH(Y, \sigma)$.

**Proof:** Using the function $f$, let us define a function $\theta_f : \delta^*CH(X, \tau) \rightarrow \delta^*CH(Y, \sigma)$ by $\theta_f(\mathcal{A}) = f \circ \mathcal{A} \circ f^{-1}$ for every $\mathcal{A} \in \delta^*CH(X, \tau)$. Then $\theta_f$ is a bijective function. Further for every $\mathcal{A}_1, \mathcal{A}_2 \in \delta^*CH(X, \tau)$, $\theta_f(\mathcal{A}_1, \mathcal{A}_2) = f \circ (\mathcal{A}_1 \circ \mathcal{A}_2) \circ f^{-1} = (f \circ \mathcal{A}_1 \circ f^{-1}) \circ (f \circ \mathcal{A}_2 \circ f^{-1}) = \theta_f(\mathcal{A}_1) \circ \theta_f(\mathcal{A}_2)$. Therefore $\theta_f$ is a homomorphism. Hence $\theta_f$ is an isomorphism induced by $f$.

Theorem 6.6.33 The $\delta^*C$-homeomorphism is an equivalence relation in the collection of all topological spaces.

**Proof:** The reflexivity and symmetric relations are immediate and the transitivity follows from the Theorem 6.6.27.

### 6.7 $\delta^*$ - Compactness and $\delta^*$ - Connectedness

In this section, the concept of $\delta^*$-compact spaces and $\delta^*$-connected spaces are introduced and characterizations of $\delta^*$-connected spaces are discussed.

**Definition 6.7.1** A collection $\mathcal{A}$ of subsets of a space $(X, \tau)$ is said to cover $X$ or to be a covering of $X$ if the union of the elements of $\mathcal{A}$ is equal to $X$. It is called a $\delta^*$-open covering of $X$ if its elements are $\delta^*$-open subsets of $(X, \tau)$.

**Definition 6.7.2** A nonempty collection $\{A_i, i \in \Lambda\}$, an index set of $\delta^*$-open sets in a topological space $(X, \tau)$ is called a $\delta^*$-open cover of a subset $B$ of $(X, \tau)$ if $B \subseteq \bigcup \{A_i, i \in \Lambda\}$.
Definition 6.7.3 A topological space \((X, \tau)\) is \(\delta sg^*\)-compact if every \(\delta sg^*\)-open cover of \((X, \tau)\) has a finite subcover.

Definition 6.7.4 A subset \(B\) of a topological space \((X, \tau)\) is called \(\delta sg^*\)-compact relative to \(X\) if for every collection \(\{A_i, i \in \Lambda\}\) of \(\delta sg^*\)-open subsets of \((X, \tau)\) such that \(B \subseteq \bigcup \{A_i, i \in \Lambda\}\) there exists a finite collection \(\Lambda_0\) of \(\Lambda\) such that \(B \subseteq \bigcup \{A_i, i \in \Lambda_0\}\).

Theorem 6.7.5 A \(\delta sg^*\)-closed subset \(A\) of a \(\delta sg^*\)-compact space \((X, \tau)\) is \(\delta sg^*\)-compact relative to \((X, \tau)\).

Proof: Let \(A\) be a \(\delta sg^*\)-closed subset of a \(\delta sg^*\)-compact space \((X, \tau)\). Then \(X \setminus A\) is \(\delta sg^*\)-open in \((X, \tau)\). Let \(\Omega\) be a \(\delta sg^*\)-open cover of \(A\) in \((X, \tau)\). Then \(\Omega \cup \{X \setminus A\}\) is a \(\delta sg^*\)-open cover of \((X, \tau)\). Since \((X, \tau)\) is \(\delta sg^*\)-compact, it has a finite subcover of \(\Omega\) say, \(\{P_1, P_2, P_3, \ldots P_n\} = \Omega_1\). If \(X \setminus A \notin \Omega_1\) then \(\Omega_1\) is a finite subcover of \(A\). If \(X \setminus A \in \Omega_1\) then \(\Omega_1 \setminus \{X \setminus A\}\) is a finite subcover of \(A\). Hence \(A\) is \(\delta sg^*\)-compact relative to \((X, \tau)\).

Theorem 6.7.6 Let \(f : (X, \tau) \rightarrow (Y, \sigma)\) be a surjective \(\delta sg^*\)-continuous function. If \((X, \tau)\) is \(\delta sg^*\)-compact then \((Y, \sigma)\) compact.

Proof: Let \(\{V_i : i \in \Lambda\}\) be an open cover of \((Y, \sigma)\). Since \(f\) is \(\delta sg^*\)-continuous, \(\{f^{-1}(V_i) : i \in \Lambda\}\) is a \(\delta sg^*\)-open cover of \((X, \tau)\). Since \((X, \tau)\) is \(\delta sg^*\)-compact, it has a finite subcover of \((X, \tau)\), say \(\{f^{-1}(V_1), f^{-1}(V_2), \ldots f^{-1}(V_n)\}\). Since \(f\) is surjective, \(\{V_1, V_2, \ldots V_n\}\) is a finite open cover of \((Y, \sigma)\). Hence \((Y, \sigma)\) is compact.

Theorem 6.7.7 If \(f : (X, \tau) \rightarrow (Y, \sigma)\) is a surjective quasi \(\delta sg^*\)-continuous function where \((X, \tau)\) is a compact space, then \((Y, \sigma)\) is \(\delta sg^*\)-compact.

Proof: Let \(\{V_i : i \in \Lambda\}\) be a \(\delta sg^*\)-open cover of \((Y, \sigma)\). Since \(f\) is quasi \(\delta sg^*\)-continuous, \(\{f^{-1}(V_i) : i \in \Lambda\}\) is an open cover of \((X, \tau)\). Since \((X, \tau)\) is compact, it has a finite subcover say, \(\{f^{-1}(V_1), f^{-1}(V_2), \ldots f^{-1}(V_n)\}\). Since \(f\) is surjective, \(\{V_1, V_2, \ldots V_n\}\) is a finite subcover of \((Y, \sigma)\) and therefore \((Y, \sigma)\) is \(\delta sg^*\)-compact.

Corollary 6.7.8 If \(f : (X, \tau) \rightarrow (Y, \sigma)\) is a surjective perfectly \(\delta sg^*\)-continuous function where \((X, \tau)\) is a compact space, then \((Y, \sigma)\) is \(\delta sg^*\)-compact.

Proof: Since every perfectly \(\delta sg^*\)-continuous function is quasi \(\delta sg^*\)-continuous function, the result follows.
Theorem 6.7.9 If a function \( f : (X, \tau) \rightarrow (Y, \sigma) \) is \( \delta sg^* \)-irresolute and a subset \( B \) of \( (X, \tau) \) is \( \delta sg^* \)-compact relative to \( (X, \tau) \), then the image \( f(B) \) is \( \delta sg^* \)-compact relative to \( (Y, \sigma) \).

**Proof:** Let \( \{A_i : i \in \Lambda\} \) be any collection of \( \delta sg^* \)-open subsets of \( (Y, \sigma) \) such that \( f(B) \subset \bigcup \{A_i : i \in \Lambda\} \). Then \( B \subset \bigcup \{f^{-1}(A_i) : i \in \Lambda\} \), \( \{f^{-1}(A_i) : i \in \Lambda\} \subseteq \delta SG^*O(X, \tau) \). Since by hypothesis \( B \) is \( \delta sg^* \)-compact relative to \( (X, \tau) \) there exists a finite collection \( \Lambda_o \) of \( \Lambda \) such that \( B \subset \bigcup \{f^{-1}(A_i) : i \in \Lambda_o\} \). Therefore \( f(B) \subset \bigcup \{A_i : i \in \Lambda_o\} \), which shows that \( f(B) \) is \( \delta sg^* \)-compact relative to \( (Y, \sigma) \).

**Definition 6.7.10** A topological space \( (X, \tau) \) is called \( \delta sg^* \)-connected if \( X \) cannot be expressed as a union of two disjoint nonempty \( \delta sg^* \)-open sets.

**Theorem 6.7.11** For a topological space \( (X, \tau) \) the following are equivalent:

(a) \( (X, \tau) \) is \( \delta sg^* \)-connected.

(b) \( X \) and \( \phi \) are the only subsets of \( (X, \tau) \) which are both \( \delta sg^* \)-open and \( \delta sg^* \)-closed.

(c) Each \( \delta sg^* \)-continuous function of \( (X, \tau) \) into a discrete space \( (Y, \sigma) \) with at least two points is a constant function.

**Proof:** (a) \( \Rightarrow \) (b) Let \( U \) be a \( \delta sg^* \)-open and \( \delta sg^* \)-closed subset of \( (X, \tau) \). Then \( X \setminus U \) is both \( \delta sg^* \)-open and \( \delta sg^* \)-closed. Since \( (X, \tau) \) is the disjoint union of the \( \delta sg^* \)-open sets \( U \) and \( X \setminus U \), one of these must be empty, that is \( U = \phi \) or \( U = X \).

(b) \( \Rightarrow \) (a) Suppose \( X = A \cup B \) where \( A \) and \( B \) are two non-empty disjoint \( \delta sg^* \)-open subsets of \( (X, \tau) \). Since \( A = X \setminus B \), \( A \) is \( \delta sg^* \)-closed. By assumption \( A = \phi \) or \( X \), which is a contradiction. Hence \( (X, \tau) \) is \( \delta sg^* \)-connected.

(b) \( \Rightarrow \) (c) Let \( f : (X, \tau) \rightarrow (Y, \sigma) \) be a \( \delta sg^* \)-continuous function, where \( (Y, \sigma) \) is a discrete space with at least two points. Since \( (Y, \sigma) \) is a discrete space, for each \( y \in Y \), \( \{y\} \) is both open and closed. Since \( f \) is a \( \delta sg^* \)-continuous function, \( f^{-1}(\{y\}) \) is \( \delta sg^* \)-closed and \( \delta sg^* \)-open and \( X = \bigcup \{f^{-1}(\{y\}) : y \in Y\} \). By assumption \( f^{-1}(\{y\}) = \phi \) or \( X \) for each \( y \in Y \). If \( f^{-1}(\{y\}) = \phi \) for all \( y \in Y \), then \( f \) will not be a function. If \( f^{-1}(\{y\}) = X \) for a single \( y \in Y \), then there cannot exist one more \( y_1 \in Y \) such that \( f^{-1}(\{y_1\}) = X \). Hence there exist only one \( y \in Y \)
such that $f^{-1}(\{y\}) = X$ and $f^{-1}(\{y\}) = \phi$ where $y_1 \in Y$ and $y_1 \neq y$. This shows that $f$ is a constant function.

**(c) \Rightarrow (b)** Let $U$ be both $\delta sg^*$-open and $\delta sg^*$-closed in $(X, \tau)$. Suppose $U = \phi$, Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a $\delta sg^*$-continuous function defined by $f(U) = \{y\}$ and $f(X \setminus U) = \{w\}$ for some distinct points $y$ and $w$ in $(Y, \sigma)$. By assumption $f$ is constant. Therefore $y = w$ which implies $U = X$.

**Theorem 6.7.12** If $f : (X, \tau) \rightarrow (Y, \sigma)$ is a $\delta sg^*$-continuous surjection and $(X, \tau)$ is $\delta sg^*$-connected then $(Y, \sigma)$ is connected.

**Proof:** Suppose that $(Y, \sigma)$ is not connected. Let $Y = A \cup B$ where $A$ and $B$ are disjoint non-empty open sets in $(Y, \sigma)$. Since $f$ is $\delta sg^*$-continuous and onto, $X = f^{-1}(A) \cup f^{-1}(B)$ where $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint non-empty $\delta sg^*$-open sets in $(X, \tau)$. This contradicts the fact that $(X, \tau)$ is $\delta sg^*$-connected. Hence $(Y, \sigma)$ is connected.

**Theorem 6.7.13** If $f : (X, \tau) \rightarrow (Y, \sigma)$ is $\delta sg^*$-irresolute surjection and $(X, \tau)$ is $\delta sg^*$-connected then $(Y, \sigma)$ is $\delta sg^*$-connected.

**Proof:** Suppose that $(Y, \sigma)$ is not $\delta sg^*$-connected. Let $Y = A \cup B$ where $A$ and $B$ are disjoint non-empty $\delta sg^*$-open sets in $(Y, \sigma)$. Since $f$ is $\delta sg^*$-irresolute and onto, $X = f^{-1}(A) \cup f^{-1}(B)$ where $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint non-empty $\delta sg^*$-open sets in $(X, \tau)$. This contradicts the fact that $(X, \tau)$ is $\delta sg^*$-connected. Hence $(Y, \sigma)$ is $\delta sg^*$-connected.

**Theorem 6.7.14** If $(X, \tau)$ is both $\delta sg^*T_\delta$-space and connected then $(X, \tau)$ is $\delta sg^*$-connected.

**Proof:** Suppose that $(X, \tau)$ is connected. Then $X$ cannot be expressed as disjoint union of two non-empty proper open subsets of $(X, \tau)$. Suppose $(X, \tau)$ is not $\delta sg^*$-connected, then $X = A \cup B$ where $A$ and $B$ are two disjoint non-empty $\delta sg^*$-open sets. Since $(X, \tau)$ is a $\delta sg^*T_\delta$-space, $A$ and $B$ are open in $(X, \tau)$. Hence $(X, \tau)$ is not connected which is a contradiction. Therefore $(X, \tau)$ is connected.