4

δsg*-Continuous Functions in Topological Spaces

4.1 Introduction

Functions are important tools for studying properties of spaces and for constructing new spaces from the existing spaces. A significant theme in general topology concerns the variously modified forms of continuity by utilizing generalized closed sets. In the year 1970, Norman Levine initiated the idea of continuous functions. Noiri (1980) introduced δ-continuity. Munshi (1982) defined super continuous functions. The generalized continuity concept was studied by Balachandran (1991). Further many authors contributed their research towards continuity.

In this chapter, a new concept of continuous functions using δsg*-closed sets is established. The dependency and independency of δsg*-continuous functions with other existing continuous functions are analyzed in this chapter. The composition of δsg*-continuities is not preserved. Some modifications on domain and codomain are done with separation axioms in order to preserve composition of δsg*-continuities. Various types of continuities namely, quasi δsg*- , perfectly δsg*- , totally δsg*- , strongly δsg*- and contra δsg*-continuities are defined. Many interrelating properties on newly defined continuities are obtained.

4.2 δsg*-Continuous Functions

In this section δsg*-continuous functions in topological spaces are introduced and some of their properties are studied.

Definition 4.2.1 A function f : (X, τ) → (Y, σ) is said to be δsg*-continuous if the inverse image of every closed set in (Y, σ) is δsg*-closed in (X, τ).

Example 4.2.2 Let X = Y = {a, b, c, d} with τ = {X, ∅, {a}, {b}, {a, b}, {a, b, c}, {a, b, d}} and σ = {Y, ∅, {c}, {a, b}, {a, b, c}}. Let f : (X, τ) → (Y, σ) be the function defined by f(a) = b, f(b) = c, f(c) = a, f(d) = d. Then f is δsg*-continuous.
**Theorem 4.2.3** A function $f : (X, \tau) \to (Y, \sigma)$ is $\delta g^*$-continuous if and only if the inverse image of every open set in $(Y, \sigma)$ is $\delta g^*$-open in $(X, \tau)$.

**Proof:** **Necessity:** Let $f : (X, \tau) \to (Y, \sigma)$ be $\delta g^*$-continuous and $U$ be an open set in $(Y, \sigma)$. Then $Y \setminus U$ is closed in $(Y, \sigma)$. Since $f$ is $\delta g^*$-continuous, $f^{-1}(Y \setminus U) = X \setminus f^{-1}(U)$ is $\delta g^*$-closed in $(X, \tau)$ and hence $f^{-1}(U)$ is $\delta g^*$-open in $(X, \tau)$.

**Sufficiency:** Assume that $f^{-1}(V)$ is $\delta g^*$-open in $(X, \tau)$ for each open set $V$ in $(Y, \sigma)$. Let $V$ be a closed set in $(Y, \sigma)$. Then $Y \setminus V$ is open in $(Y, \sigma)$. By assumption, $f^{-1}(Y \setminus V) = X \setminus f^{-1}(V)$ is $\delta g^*$-open in $(X, \tau)$ which implies that $f^{-1}(V)$ is $\delta g^*$-closed in $(X, \tau)$. Hence $f$ is $\delta g^*$-continuous.

**Proposition 4.2.4 (a)** Every super continuous function is a $\delta g^*$-continuous function.

(b) Every $\delta$-semi continuous function is a $\delta g^*$-continuous function.

(c) Every $\delta g^*$-continuous function is a $\delta g^*$-continuous function.

**Proof:** (a) Let $f : (X, \tau) \to (Y, \sigma)$ be a super continuous function. Let $V$ be any closed set in $(Y, \sigma)$. Since $f$ is a super continuous function, $f^{-1}(V)$ is $\delta$-closed in $(X, \tau)$. By Theorem 2.2.6., every $\delta$-closed set is $\delta g^*$-closed which implies $f^{-1}(V)$ is $\delta g^*$-closed in $(X, \tau)$. Therefore $f$ is $\delta g^*$-continuous.

(b) and (c) follow from Proposition 2.2.2 and Proposition 2.2.8 respectively.

**Remark 4.2.5** The converse of Proposition 4.2.4 (a), (b), (c) don’t follow as seen from the following examples.

**Example 4.2.6** Let $X = Y = \{a, b, c, d\}$ with $\tau = \{X, \phi, \{a\}, \{c\}, \{a,b\}, \{a,c\}, \{a,b,c\}, \{a,c,d\}\}$ and $\sigma = \{Y, \phi, \{a\}, \{a,b\}\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be the identity function. Then $f$ is $\delta g^*$-continuous as the inverse image of every closed set in $(Y, \sigma)$ is $\delta g^*$-closed in $(X, \tau)$. $f$ is not super continuous and $\delta$-semi continuous, since for the closed set $\{b,c,d\}$ in $(Y, \sigma)$, $f^{-1}(\{b,c,d\}) = \{b,c,d\}$ is not $\delta$-closed and $\delta$-semi closed in $(X, \tau)$.

**Example 4.2.7** Let $X = Y = \{a, b, c\}$ with $\tau = \{X, \phi, \{a\}, \{b\}, \{a,b\}\}$ and $\sigma = \{Y, \phi, \{a\}, \{b\}, \{a,b\}, \{a,c\}\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be the identity function. Then $f$ is $\delta g^*$-continuous as the inverse image of every closed set in $(Y, \sigma)$ is $\delta g^*$-closed in $(X, \tau)$. $f$ is not $\delta g^*$-continuous, since for the closed set $\{b\}$ in $(Y, \sigma)$, $f^{-1}(\{b\}) = \{b\}$ is not $\delta g^*$-closed in $(X, \tau)$.
Chapter 4

Proposition 4.2.8 Let \( f : (X, \tau) \to (Y, \sigma) \) be a \( \delta \text{sg}^* \)-continuous function and \( (X, \tau) \) be a \( \delta \text{sg}^*T_\delta \)-space. Then \( f \) is super continuous.

Proof: Let \( V \) be a closed set in \( (Y, \sigma) \). Since \( f \) is \( \delta \text{sg}^* \)-continuous, \( f^{-1}(V) \) is \( \delta \text{sg}^* \)-closed in \( (X, \tau) \). Since \( (X, \tau) \) is a \( \delta \text{sg}^*T_\delta \)-space in which every \( \delta \text{sg}^* \)-closed set is \( \delta \)-closed, \( f^{-1}(V) \) is \( \delta \)-closed in \( (X, \tau) \). Hence \( f \) is super continuous.

Proposition 4.2.9

(a) Every \( \delta \text{sg}^* \)-continuous function is a \( g\delta s \)-continuous function.

(b) Every \( \delta \text{sg}^* \)-continuous function is a \( \text{gs} \)-continuous function.

(c) Every \( \delta \text{sg}^* \)-continuous function is a \( \text{gsp} \)-continuous function.

The converse of the above Proposition need not be true which can be seen from the following examples.

Example 4.2.10 Let \( X = Y = \{a, b, c\} \) with \( \tau = \{X, \phi, \{a\}\} \) and \( \sigma = \{Y, \phi, \{a\}, \{a, b\}\} \). Let \( f : (X, \tau) \to (Y, \sigma) \) be the identity function. Then \( f \) is \( g\delta s \)-continuous, \( \text{gs} \)-continuous and \( \text{gsp} \)-continuous as the inverse image of every closed set in \( (Y, \sigma) \) but \( f \) is not \( \delta \text{sg}^* \)-continuous, since for the closed set \( \{c\} \) in \( (Y, \sigma) \), \( f^{-1}(\{c\}) = \{c\} \) is \( g\delta s \)-closed, \( \text{gs} \)-closed and \( \text{gsp} \)-closed but not \( \delta \text{sg}^* \)-closed in \( (X, \tau) \).

Proposition 4.2.11

(a) Every \( \delta \text{sg}^* \)-continuous function is a \( \delta \text{gs} \)-continuous function.

(b) Every \( \delta \text{sg}^* \)-continuous function is a \( \text{gspr} \)-continuous function.

(c) Every \( \delta \text{sg}^* \)-continuous function is a \( \pi \text{gs} \)-continuous function.

(d) Every \( \delta \text{sg}^* \)-continuous function is a \( \pi \text{gsp} \)-continuous function.

(e) Every \( \delta \text{sg}^* \)-continuous function is a \( \pi \text{gb} \)-continuous function.

The converse of the above Proposition need not be true which can be seen from the following examples.

Example 4.2.12 Let \( X = Y = \{a, b, c\} \) with \( \tau = \{X, \phi, \{a\}\} \) and \( \sigma = \{Y, \phi, \{a\}, \{a, b\}, \{a, c\}\} \). Let \( f : (X, \tau) \to (Y, \sigma) \) be the identity function. Then \( f \) is \( \delta \text{gs} \)-continuous,
A study on \( \delta g^{*} \)-Closed Sets in Topological Spaces

Chapter 4

gspr-continuous, \( \pi g s \)-continuous, \( \pi g s p \)-continuous and \( \pi g b \)-continuous as the inverse image of every closed set in \((Y, \sigma)\) is \( \delta g s \)-closed, gspr-closed, \( \pi g s \)-closed, \( \pi g s p \)-closed and \( \pi g b \)-closed but not \( \delta g^{*} \)-continuous, since for only closed set \( \{b\} \) in \((Y, \sigma)\), \( f^{-1}(\{b\}) = \{b\} \) is \( \delta g s \)-closed, gspr-closed, \( \pi g s \)-closed, \( \pi g s p \)-closed and \( \pi g b \)-closed but not \( \delta g^{*} \)-closed in \((X, \tau)\).

**Remark 4.2.13** We have the following diagram

**Proposition 4.2.14** Let \( f : (X, \tau) \rightarrow (Y, \sigma) \) be a gs-continuous function and \((X, \tau)\) be a \( g_{TS_{g^{*}}} \)-space. Then \( f \) is \( \delta g^{*} \)-continuous.

**Proof:** Let \( V \) be a closed set in \((Y, \sigma)\). Since \( f : (X, \tau) \rightarrow (Y, \sigma) \) is gs-continuous, \( f^{-1}(V) \) is gs-closed in \((X, \tau)\). Since \((X, \tau)\) is a \( g_{TS_{g^{*}}} \)-space in which every gs-closed set is \( \delta g^{*} \)-closed, \( f^{-1}(V) \) is \( \delta g^{*} \)-closed in \((X, \tau)\). Hence \( f \) is \( \delta g^{*} \)-continuous.

**Remark 4.2.15** The following examples show that the notions of \( \delta g^{*} \)-continuity and semi (resp. sg, \( g^{*} \), \( g^{#} \), \( g_{S}^{g} \)) continuity are independent.

**Example 4.2.16** Let \( X = Y = \{a, b, c\} \) with \( \tau = \{X, \emptyset, \{a\}, \{a, b\}\} \) and \( \sigma = \{Y, \emptyset, \{a, b\}\} \). Let \( f : (X, \tau) \rightarrow (Y, \sigma) \) be define by \( f(a) = b, f(b) = c, f(c) = a \). Then \( f \) is semi (resp. sg, \( g^{*} \), \( g^{#} \), \( g_{S}^{g} \)) continuous but not \( \delta g^{*} \)-continuous, since for the only closed set \( \{c\} \) in \((Y, \sigma)\), \( f^{-1}(\{c\}) = \{b\} \) is semi (resp. sg, \( g^{*} \), \( g^{#} \), \( g_{S}^{g} \)) closed but not \( \delta g^{*} \)-closed in \((X, \tau)\).
**Example 4.2.17** Let $X = Y = \{a, b, c\}$ with $\tau = \{X, \phi, \{a\}, \{a, b\}\}$ and $\sigma = \{Y, \phi, \{a, b\}\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be define by $f(a) = c$, $f(b) = b$, $f(c) = c$. Then $f$ is $\delta_{sg}^*$-continuous but not semi (resp. $sg$, $g^s$, $g^#_s$) continuous, since for the only closed set $\{c\}$ in $Y$, $f^{-1}(\{c\}) = \{a, c\}$ is $\delta_{sg}^*$-closed but not semi (resp. $sg$, $g^s$, $g^#_s$, $g^*_s$) closed in $(X, \tau)$.

**Remark 4.2.18** We have the following diagram

**Proposition 4.2.19** Let $f : (X, \tau) \to (Y, \sigma)$ be a $\delta_{sg}^*$-continuous function. Then for every subset $A$ of $(X, \tau)$, $f(\delta_{sg}^*\text{cl}(A)) \subseteq \text{cl}(f(A))$.

**Proof:** Let $f : (X, \tau) \to (Y, \sigma)$ be a $\delta_{sg}^*$-continuous function and $A$ be any subset of $(X, \tau)$. Then $\text{cl}(f(A))$ is a closed set in $(Y, \sigma)$. Since $f$ is $\delta_{sg}^*$-continuous, we have

$$f^{-1}(\text{cl}(f(A))) \text{ is } \delta_{sg}^*\text{-closed in } (X, \tau) \quad \to (1)$$

Since $f(A) \subseteq \text{cl}(f(A))$, $A \subseteq f^{-1}(\text{cl}(f(A))) \quad \to (2)$

(1) and (2) we get, $f^{-1}(\text{cl}(f(A)))$ is a $\delta_{sg}^*$-closed set containing $A$.

By definition of $\delta_{sg}^*$-closure, we have $\delta_{sg}^*\text{cl}(A) \subseteq f^{-1}(\text{cl}(f(A)))$ which implies that $f(\delta_{sg}^*\text{cl}(A)) \subseteq \text{cl}(f(A))$.

**Corollary 4.2.20** Let $f : (X, \tau) \to (Y, \sigma)$ be a super continuous function. Then for every subset $A$ of $(X, \tau)$, $f(\delta_{sg}^*\text{cl}(A)) \subseteq \text{cl}(f(A))$.

**Proof:** Follows from the Proposition 4.2.4 (a), every super continuous function is $\delta_{sg}^*$-continuous and Proposition 4.2.19.
Proposition 4.2.21 Let \( f : (X, \tau) \rightarrow (Y, \sigma) \) be a function. If for each point \( x \in X \) and each open set \( V \) in \( (Y, \sigma) \) containing \( f(x) \), there exists a \( \delta_{sg}^{*} \)-open set \( U \) in \( (X, \tau) \) containing \( x \) such that \( f(U) \subseteq V \), then for every subset \( A \) of \( (X, \tau) \), \( f(\delta_{sg}^{*}\text{cl}(A)) \subseteq \text{cl}(f(A)) \).

Proof: Let \( A \) be any subset of \( (X, \tau) \) and \( y \in f(\delta_{sg}^{*}\text{cl}(A)) \). Therefore \( y = f(x) \) for some \( x \in \delta_{sg}^{*}\text{cl}(A) \subseteq X \). Let \( V \) be any open set in \( (Y, \sigma) \) such that \( f(x) \in V \). Then by hypothesis, there exists a \( \delta_{sg}^{*} \)-open set \( U \) in \( (X, \tau) \) containing \( x \) with \( f(U) \subseteq V \). By Theorem 2.5.16, \( U \cap A \neq \emptyset \), then \( f(U \cap A) \neq \emptyset \) which implies that \( V \cap f(A) \neq \emptyset \). Hence \( y \in \text{cl}(f(A)) \).

Composition of Functions

Remark 4.2.22 The composition of two \( \delta_{sg}^{*} \)-continuous functions need not be \( \delta_{sg}^{*} \)-continuous as seen from the following example.

Example 4.2.23 Let \( X = Y = Z = \{a, b, c\} \) with \( \tau = \{X, \emptyset, \{a\}, \{b, c\}\} \), \( \sigma = \{Y, \emptyset, \{a, b\}\} \) and \( \eta = \{Z, \emptyset, \{c\}\} \). Let \( f : (X, \tau) \rightarrow (Y, \sigma) \) be defined by \( f(a) = c \), \( f(b) = b \), \( f(c) = a \) and \( g : (Y, \sigma) \rightarrow (Z, \eta) \) be defined by \( g(a) = a \), \( g(b) = c \), \( g(c) = b \). Then the functions \( f \) and \( g \) are \( \delta_{sg}^{*} \)-continuous but their composition \( g \circ f : (X, \tau) \rightarrow (Z, \eta) \) is not \( \delta_{sg}^{*} \)-continuous, since for the only closed set \( \{a, b\} \) in \( (Z, \eta) \), \( (g \circ f)^{-1}\{a, b\} = \{a, c\} \) is not \( \delta_{sg}^{*} \)-closed in \( (X, \tau) \).

Theorem 4.2.24 If \( f : (X, \tau) \rightarrow (Y, \sigma) \) is \( \delta_{sg}^{*} \)-continuous and \( g : (Y, \sigma) \rightarrow (Z, \eta) \) is super continuous then \( g \circ f : (X, \tau) \rightarrow (Z, \eta) \) is super continuous.

Proof: Let \( V \) be a closed set in \( (Z, \eta) \). Since \( g \) is super continuous, \( g^{-1}(V) \) is \( \delta \)-closed in \( (Y, \sigma) \). As every \( \delta \)-closed set is closed, now \( g^{-1}(V) \) is closed in \( (Y, \sigma) \). Since \( f : (X, \tau) \rightarrow (Y, \sigma) \) is \( \delta_{sg}^{*} \)-continuous, \( f^{-1}(g^{-1}(V)) = (g \circ f)^{-1}(V) \) is \( \delta_{sg}^{*} \)-closed in \( (X, \tau) \). Hence \( g \circ f : (X, \tau) \rightarrow (Z, \eta) \) is \( \delta_{sg}^{*} \)-continuous.

Theorem 4.2.25 If \( f : (X, \tau) \rightarrow (Y, \sigma) \) is super continuous and \( g : (Y, \sigma) \rightarrow (Z, \eta) \) is super continuous then \( g \circ f : (X, \tau) \rightarrow (Z, \eta) \) is \( \delta_{sg}^{*} \)-continuous.

Proof: The proof follows from the fact that the product of two super continuous functions is super continuous and Proposition 4.2.4 (a).

Proposition 4.2.26 Let \( f : (X, \tau) \rightarrow (Y, \sigma) \) be any topological space and \( (Y, \sigma) \) be a \( \delta_{sg}^{*} \)\( T_{\delta} \)-space. If \( f : (X, \tau) \rightarrow (Y, \sigma) \) and \( g : (Y, \sigma) \rightarrow (Z, \eta) \) are \( \delta_{sg}^{*} \)-continuous functions, then their composition \( g \circ f : (X, \tau) \rightarrow (Z, \eta) \) is \( \delta_{sg}^{*} \)-continuous function.
Proof: Let V be a closed set in \((Z, \eta)\). Since \(g : (Y, \sigma) \to (Z, \eta)\) is \(\delta\)-continuous, \(g^{-1}(V)\) is \(\delta\)-closed in \((Y, \sigma)\). Since \((Y, \sigma)\) is a \(\delta\)-T\(\delta\) space, \(g^{-1}(V)\) is \(\delta\)-closed in \((Y, \sigma)\).

As every \(\delta\)-closed set is closed, now \(g^{-1}(V)\) is closed in \((Y, \sigma)\). Since \(f : (X, \tau) \to (Y, \sigma)\) is \(\delta\)-continuous, \(f^{-1}(g^{-1}(V)) = (g \circ f)^{-1}(V)\) is \(\delta\)-closed in \((X, \tau)\). Hence \(g \circ f : (X, \tau) \to (Z, \eta)\) is \(\delta\)-continuous.

### 4.3 Quasi \(\delta\)sg\(^*\)-Continuous Functions and Perfectly \(\delta\)sg\(^*\)-Continuous Functions

**Definition 4.3.1** A function \(f : (X, \tau) \to (Y, \sigma)\) is called **quasi \(\delta\)sg\(^*\)-continuous** if the inverse image of every \(\delta\)sg\(^*\)-closed set in \((Y, \sigma)\) is closed in \((X, \tau)\).

**Example 4.3.2** Let \(X = Y = \{a, b, c\}\) with \(\tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}\}\) and \(\sigma = \{Y, \emptyset, \{a, b\}\}\). Let \(f : (X, \tau) \to (Y, \sigma)\) be the identity function. Then \(f\) is quasi \(\delta\)sg\(^*\)-continuous.

**Theorem 4.3.3** A function \(f : (X, \tau) \to (Y, \sigma)\) is quasi \(\delta\)sg\(^*\)-continuous if and only if the inverse image of every \(\delta\)sg\(^*\)-open set in \((Y, \sigma)\) is open in \((X, \tau)\).

**Proof:** Follows from the definitions.

**Proposition 4.3.4** If \(f : (X, \tau) \to (Y, \sigma)\) is strongly continuous, then it is quasi \(\delta\)sg\(^*\)-continuous but not conversely.

**Proof:** Let \(U\) be a \(\delta\)sg\(^*\)-open set in \((Y, \sigma)\). Since \(f : (X, \tau) \to (Y, \sigma)\) is strongly continuous for any subset \(U\), \(f^{-1}(U)\) is a clopen set in \((X, \tau)\). Therefore \(f\) is quasi \(\delta\)sg\(^*\)-continuous.

**Example 4.3.5** Let \(X = Y = \{a, b, c\}\) with \(\tau = \{X, \emptyset, \{a\}, \{b, c\}\}\) and \(\sigma = \{Y, \emptyset, \{a\}\}\). Let \(f : (X, \tau) \to (Y, \sigma)\) be the identity function. Then the function \(f\) is quasi \(\delta\)sg\(^*\)-continuous but not strongly continuous, since for the subset \(\{b\}\) in \((Y, \sigma)\), \(f^{-1}(\{b\}) = \{b\}\) is not clopen in \((X, \tau)\).

**Proposition 4.3.6** If \(f : (X, \tau) \to (Y, \sigma)\) is totally continuous and \((Y, \sigma)\) is a \(\delta\)sg\(^*\)-T\(\delta\) space then it is quasi \(\delta\)sg\(^*\)-continuous but not conversely.

**Proof:** Let \(U\) be any \(\delta\)sg\(^*\)-open set in \((Y, \sigma)\). Since \((Y, \sigma)\) is a \(\delta\)sg\(^*\)-T\(\delta\) space, \(U\) is a \(\delta\)-open set which implies that \(U\) is open. Since \(f\) is totally continuous, \(f^{-1}(U)\) is clopen set in \((X, \tau)\). That is \(f^{-1}(U)\) is open in \((X, \tau)\). Hence \(f\) is quasi \(\delta\)sg\(^*\)-continuous.
**Example 4.3.7** Let \( X = Y = \{a, b, c\} \) with \( \tau = \{X, \phi, \{a\}, \{a,b\}\} \) and \( \sigma = \{Y, \phi, \{a,b\}\} \). Let \( f : (X, \tau) \rightarrow (Y, \sigma) \) be defined by \( f(a) = a, f(b) = a, f(c) = c \). Then the function \( f \) is quasi \( \delta_{sg^*} \)-continuous but not totally continuous, since for the open set \( \{a\} \) in \( (Y, \sigma) \), \( f^{-1}(\{a\}) = \{a,b\} \) is open but not closed in \( (X, \tau) \).

**Remark 4.3.8** The Proposition 4.3.6 does not hold good if \( (Y, \sigma) \) is not a \( \delta_{sg^*}T_{\delta} \)-space.

**Example 4.3.9** Let \( X = Y = \{a, b, c, d\} \) with \( \tau = \{X, \phi, \{a,b\}, \{c,d\}\} \) and \( \sigma = \{Y, \phi, \{a,b\}\} \). Here \( (Y, \sigma) \) is not a \( \delta_{sg^*}T_{\delta} \)-space. Let \( f : (X, \tau) \rightarrow (Y, \sigma) \) be the identity function. Then the function \( f \) is totally continuous but not quasi \( \delta_{sg^*} \)-continuous as the inverse image of the \( \delta_{sg^*} \)-open set \( \{a\} \) in \( (Y, \sigma) \) is not a open set in \( (X, \tau) \).

**Definition 4.3.10** A function \( f : (X, \tau) \rightarrow (Y, \sigma) \) is called **perfectly \( \delta_{sg^*} \)-continuous** if the inverse image of every \( \delta_{sg^*} \)-open set in \( (Y, \sigma) \) is clopen in \( (X, \tau) \).

**Example 4.3.11** Let \( X = Y = \{a,b,c\} \) with \( \tau = \{X, \phi, \{a\}, \{b,c\}\} \) and \( \sigma = \{Y, \phi, \{a\}, \{a,b\}, \{a,c\}\} \). Let \( f : (X, \tau) \rightarrow (Y, \sigma) \) be the identity function. Then \( f \) is perfectly \( \delta_{sg^*} \)-continuous.

**Proposition 4.3.12** A function \( f : (X, \tau) \rightarrow (Y, \sigma) \) is perfectly \( \delta_{sg^*} \)-continuous if and only if the inverse of every \( \delta_{sg^*} \)-closed set in \( (Y, \sigma) \) is clopen in \( (X, \tau) \).

**Proof:** Similar to Theorem 4.2.3.

**Proposition 4.3.13** If a function \( f : (X, \tau) \rightarrow (Y, \sigma) \) is perfectly \( \delta_{sg^*} \)-continuous, then it is quasi \( \delta_{sg^*} \)-continuous but not conversely.

**Proof:** Let \( f \) be perfectly \( \delta_{sg^*} \)-continuous. Let \( V \) be open in \( (Y, \sigma) \). Then by definition of perfectly \( \delta_{sg^*} \)-continuous, \( f^{-1}(V) \) is clopen. That is \( f^{-1}(V) \) is open. Therefore \( f \) is quasi \( \delta_{sg^*} \)-continuous.

**Example 4.3.14** Let \( X = Y = \{a,b,c\} \) with \( \tau = \{X, \phi, \{a\}, \{a,b\}\} \) and \( \sigma = \{Y, \phi, \{a\}, \{a,b\}, \{a,c\}\} \). Let \( f : (X, \tau) \rightarrow (Y, \sigma) \) be the identity function. Then \( f \) is quasi \( \delta_{sg^*} \)-continuous, but not perfectly \( \delta_{sg^*} \)-continuous, since for the \( \delta_{sg^*} \)-closed set \( \{b,c\} \) in \( (Y, \sigma) \), \( f^{-1}(\{b,c\}) = \{b,c\} \) is not clopen in \( (X, \tau) \).
**Proposition 4.3.15** If \( f : (X, \tau) \to (Y, \sigma) \) is strongly continuous, then it is perfectly \( \delta sg^* \)-continuous but not conversely.

**Proof:** Let \( U \) be a \( \delta sg^* \)-open set in \((Y, \sigma)\). Since \( f : (X, \tau) \to (Y, \sigma) \) is strongly continuous, \( f^{-1}(U) \) is a clopen set in \((X, \tau)\). Therefore \( f \) is perfectly \( \delta sg^* \)-continuous.

**Example 4.3.16** Let \( X = Y = \{a,b,c\} \) with \( \tau = \{X, \emptyset, \{a\}, \{b\}, \{a,b\}, \{a,c\}\} \) and \( \sigma = \{Y, \emptyset, \{a\}, \{b,c\}\} \). Let \( f : (X, \tau) \to (Y, \sigma) \) be defined by \( f(a) = b, f(b) = a, f(c) = c \). Then \( f \) is perfectly \( \delta sg^* \)-continuous but not strongly continuous, since for the subset \( \{b\} \) in \((Y, \sigma)\), \( f^{-1}(\{b\}) = \{a\} \) is not clopen in \((X, \tau)\).

**Remark 4.3.17** From the above observation, we have the following diagram.

None of the above implications is reversible.

**Proposition 4.3.18** Let \((X, \tau)\) be a discrete topological space, \((Y, \sigma)\) be any topological space and \( f : (X, \tau) \to (Y, \sigma) \) be a function. Then the following are equivalent:

(a) \( f \) is perfectly \( \delta sg^* \)-continuous

(b) \( f \) is quasi \( \delta sg^* \)-continuous

**Proof:** (a) \(\Rightarrow\) (b) Follows from the Proposition 4.3.13.

(b) \(\Rightarrow\) (a) Let \( U \) be any \( \delta sg^* \)-open set in \((Y, \sigma)\). By hypothesis, \( f^{-1}(U) \) is open in \((X, \tau)\). Since \((X, \tau)\) is a discrete space, \( f^{-1}(U) \) is also closed in \((X, \tau)\). That is \( f^{-1}(U) \) is clopen in \((X, \tau)\). Hence \( f \) is perfectly \( \delta sg^* \)-continuous.

**Proposition 4.3.19** Let \( f : (X, \tau) \to (Y, \sigma) \) and \( g : (Y, \sigma) \to (Z, \eta) \) be any two functions. Then their composition \( g \circ f : (X, \tau) \to (Z, \eta) \) is
(a) $\delta_{sg^*}$-continuous if $g$ is strongly continuous and $f$ is $\delta_{sg^*}$-continuous

(b) Perfectly $\delta_{sg^*}$-continuous, if $g$ is perfectly $\delta_{sg^*}$-continuous and $f$ is continuous.

**Proof:** (a) Let $g : (Y, \sigma) \rightarrow (Z, \eta)$ be strongly continuous and $f : (X, \tau) \rightarrow (Y, \sigma)$ be $\delta_{sg^*}$-continuous. Let $V$ be any closed set in $(Z, \eta)$. Since $g$ is strongly continuous, $g^{-1}(V)$ is both open and closed in $(Y, \sigma)$. Since $f$ is $\delta_{sg^*}$-continuous, $f^{-1}(g^{-1}(V)) = (g \circ f)^{-1}(V)$ is $\delta_{sg^*}$-closed in $(X, \tau)$. Hence $g \circ f : (X, \tau) \rightarrow (Z, \eta)$ is $\delta_{sg^*}$-continuous.

(b) Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be continuous, $g : (Y, \sigma) \rightarrow (Z, \eta)$ be perfectly continuous. Let $V$ be any $\delta_{sg^*}$-closed in $(Z, \eta)$. Since $g$ is perfectly $\delta_{sg^*}$-continuous, $g^{-1}(V)$ is clopen in $(Y, \sigma)$. Since $f$ is continuous, $f^{-1}(g^{-1}(V)) = (g \circ f)^{-1}(V)$ is clopen in $(X, \tau)$. Hence $g \circ f : (X, \tau) \rightarrow (Z, \eta)$ is perfectly $\delta_{sg^*}$-continuous.

**Remark 4.3.20** Any totally continuous function need not be a perfectly $\delta_{sg^*}$-continuous function as seen from the following example.

**Example 4.3.21** Let $X = Y = \{a,b,c\}$ with $\tau = \{X, \phi, \{a\}, \{b,c\}\}$ and $\sigma = \{Y, \phi, \{a,b\}\}$. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be defined by $f(a) = c$, $f(b) = a$, $f(c) = b$. Then $f$ is totally continuous but not perfectly $\delta_{sg^*}$-continuous, since for the $\delta_{sg^*}$-open set $\{a\}$ in $(Y, \sigma)$, $f^{-1}(\{a\}) = \{b\}$ is not a clopen set in $(X, \tau)$.

### 4.4 Totally $\delta_{sg^*}$-Continuous, Strongly $\delta_{sg^*}$-Continuous and Contra $\delta_{sg^*}$-Continuous Functions

**Definition 4.4.1** A function $f : (X, \tau) \rightarrow (Y, \sigma)$ is called **totally $\delta_{sg^*}$-continuous** if the inverse image of every closed subset of $(Y, \sigma)$ is a $\delta_{sg^*}$-clopen set in $(X, \tau)$.

**Example 4.4.2** Let $X = Y = \{a,b,c\}$ with $\tau = \{X, \phi, \{a\}, \{b,c\}\}$ and $\sigma = \{Y, \phi, \{a\}\}$. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be the identity function. Then $f$ is totally $\delta_{sg^*}$-continuous.

**Proposition 4.4.3** A function $f : (X, \tau) \rightarrow (Y, \sigma)$ is totally $\delta_{sg^*}$-continuous if and only if the inverse image of every open subset of $(Y, \sigma)$ is a $\delta_{sg^*}$-clopen set in $(X, \tau)$.

**Proof:** Follows from the definitions.
**Definition 4.4.4** A function \( f : (X, \tau) \rightarrow (Y, \sigma) \) is called **strongly \( \delta sg^* \)-continuous** if the inverse image of every subset of \((Y, \sigma)\) is a \( \delta sg^* \)-clopen set in \((X, \tau)\).

**Proposition 4.4.5** A strongly \( \delta sg^* \)-continuous function \( f : (X, \tau) \rightarrow (Y, \sigma) \) is totally \( \delta sg^* \)-continuous but not conversely.

**Proof:** Follows from the definitions.

**Example 4.4.6** Let \( X = Y = \{a,b,c\} \) with \( \tau = \{X, \phi, \{a\}, \{b,c\}\} \) and \( \sigma = \{Y, \phi, \{a\}\} \). Let \( f : (X, \tau) \rightarrow (Y, \sigma) \) be the identity function. Then \( f \) is totally \( \delta sg^* \)-continuous but not strongly totally \( \delta sg^* \)-continuous, since for the subset \( \{c\} \) in \((Y, \sigma)\), \( f^\dagger(\{c\}) = \{c\} \) is not clopen in \((X, \tau)\).

**Proposition 4.4.7** Let \( f : (X, \tau) \rightarrow (Y, \sigma) \) be a totally \( \delta sg^* \)-continuous function, where \((Y, \sigma)\) is a discrete topological space. Then \( f \) is a strongly \( \delta sg^* \)-continuous function.

**Proof:** Proof follows from the fact that every subset is clopen in a discrete space.

**Definition 4.4.8** A function \( f : (X, \tau) \rightarrow (Y, \sigma) \) is called **contra \( \delta sg^* \)-continuous** if the inverse image of every closed set of \((Y, \sigma)\) is \( \delta sg^* \)-open in \((X, \tau)\).

**Proposition 4.4.9** Every totally \( \delta sg^* \)-continuous function is contra \( \delta sg^* \)-continuous but not conversely.

**Proof:** Follows from the Definitions 4.4.1 and 4.4.8.

**Example 4.4.10** Let \( X = Y = \{a,b,c,d\} \) with \( \tau = \{X, \phi, \{a\}, \{a,b\}\} \) and \( \sigma = \{Y, \phi, \{a,b\}\} \). Let \( f : (X, \tau) \rightarrow (Y, \sigma) \) be defined by \( f(a) = c, f(b) = d, f(c) = a, f(d) = b \). Then \( f \) is contra \( \delta sg^* \)-continuous but not totally \( \delta sg^* \)-continuous, since for the closed subset \( \{c,d\} \) in \((Y, \sigma)\), \( f^\dagger(\{c,d\}) = \{a,b\} \) is \( \delta sg^* \)-open in \((X, \tau)\) but not \( \delta sg^* \)-closed in \((X, \tau)\).

**Corollary 4.4.11** Every strongly \( \delta sg^* \)-continuous function is contra \( \delta sg^* \)-continuous.

**Proof:** Follows from the Definitions 4.4.4 and 4.4.8.

**Example 4.4.12** Example 4.4.10 will serve the purpose.
Remark 4.4.13 From the above observation, we have the following diagram.

![Diagram showing relationships between types of continuity](image)

Remark 4.4.14 The composition of two contra $\delta^{sg^*}$-continuous functions need not be a contra $\delta^{sg^*}$-continuous function as seen from the following example.

Example 4.4.15 Let $X = Y = Z = \{a,b,c\}$ with $\tau = \{X, \phi, \{a\}\}$, $\sigma = \{Y, \phi, \{a,b\}\}$ and $\eta = \{Z, \phi, \{a\}, \{a,b\}, \{a,c\}\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be defined by $f(a) = c$, $f(b) = a$, $f(c) = b$ and $g : (Y, \sigma) \to (Z, \eta)$ be defined by $g(a) = c$, $g(b) = b$, $g(c) = a$. Then the functions $f$ and $g$ are contra $\delta^{sg^*}$-continuous but their composition $g \circ f : (X, \tau) \to (Z, \eta)$ is not contra $\delta^{sg^*}$-continuous, since for the closed set $\{c\}$ in $(Z, \eta)$, $(g \circ f)^{-1}\{c\} = \{b\}$ is not $\delta^{sg^*}$-open in $(X, \tau)$.

Proposition 4.4.16 If $f : (X, \tau) \to (Y, \sigma)$ is contra $\delta^{sg^*}$-continuous function and $g : (Y, \sigma) \to (Z, \eta)$ is a continuous function, then their composition $(g \circ f) : (X, \tau) \to (Z, \eta)$ is a contra $\delta^{sg^*}$-continuous function.

Proof: Let $U$ be any closed in $(Z, \eta)$. Since $g$ is continuous, $g^{-1}(U)$ is closed in $(Y, \sigma)$. Since $f$ is a contra $\delta^{sg^*}$-continuous, $f^{-1}(g^{-1}(U))$ is $\delta^{sg^*}$-open in $(X, \tau)$. Therefore $(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U))$ is $\delta^{sg^*}$-open in $(X, \tau)$. Hence $g \circ f : (X, \tau) \to (Z, \eta)$ is a contra $\delta^{sg^*}$-continuous function.