CHAPTER-4

MHD CASSON VISCOUS DISSIPATIVE FLUID FLOW PAST A VERTICALLY INCLINED PLATE IN PRESENCE OF HEAT AND MASS TRANSFER: A FINITE ELEMENT TECHNIQUE

4.1. INTRODUCTION:

Non-Newtonian fluid theory is a part of fluid mechanics based on the continuum theory that a fluid particle may be considered as continuous in a structure. One of the non-newtonian fluids is a Pseudo plastic time independent fluid whose behaviour is that Viscosity decreases with increasing velocity gradient e.g. blood, polymer solutions, etc. Casson fluid is one of the pseudoplastic fluids that means shear thinning fluids (Casson, 1959). At low shear rates the shear thinning fluid is more viscous than the Newtonian fluid, and at high shear rates it is less viscous. So, MHD flow with Casson fluid is recently well-known. Harinath Reddy et al. (2016) studied the influence of radiation absorption and chemical reaction on unstable magnetohydrodynamic natural transportation of heat and mass of Casson fluid movement an oscillating vertically standing plate lodged in permeable porous medium in the presence of constant wall temperature and concentration flow using finite difference method. Venkateswarlu and Satya Narayana (2016) studied the effects of both Soret and Dufour on MHD flow of a Casson fluid past an expanded sheet in the presence of viscous dissipation, chemical reaction and variable thermal conductivity using shooting method. Rammohan Reddy et al. (2016) found the analytical solutions of MHD convective flow of a incompressible, viscoelastic, radiative, chemically reactive, electrically conducting and rotating fluid through a permeable porous medium in the presence of thermal diffusion using perturbation technique. Bhattacharyya (2013) found the numerical solutions of steady boundary layer stagnation-point flow of Casson fluid and heat transfer towards a shrinking/stretching sheet using very efficient shooting method. Mustafa and Khan (2015) investigated magnetic field effects on Casson nanofluid over non-linearly stretching sheet. Nadeem et al. (2014) analyzed the steady stagnation point flow of a non-Newtonian fluid

In the boundary layer, the influence of thermal radiation is relevant in many engineering problems because of its applications especially in high temperature engineering processes. The effect of heat radiation is important in controlling the quality of the final product as it affects the rate cooling. Due to the above fact, some of the authors have studied the effect of thermal radiation in their works, viz., Pal et al. (2013), Mukhopadhyay et al. (2011), Akbar et al. (2013), Bhattacharyya and his co-workers (2013) and (2012), Rashidi et al. (2014), Su et al. (2012). Ahmed and Mahdy (2016) deliberated the magnetohydrodynamic flow of a laminar, isochoric, viscid, electrically conducting fluid of an hasty rotating sphere in the presence of thermal radiation, heat and mass transfer effects in the stagnation region. Prakash et al. (2014) discussed both the effects of induced magnetic field and radiation on MHD heat and mass transfer flow of viscid, isochoric, Newtonian fluid over a vertically standing porous plate using perturbation technique. Raptis et al. (2003) studied the effect of radiation in an optically thin gray gas flowing past a vertically infinite plate in the presence of applied magnetic field. Ahmed (2010) found the analytical solutions of induced magnetic field and radiation on fluid flow over a erected porous surface using Perturbation technique.

Therefore this work can be considered as extension of Prakash et al. (2014). So Novelty of this paper is discussion of numerical solutions using finite element technique of the steady natural convective Casson fluid flow over an inclined plate in the presence of transverse magnetic field. Also, the study of grid independence of finite element technique is discussed through tabular form. The behaviours of different pertinent parameters on velocity, induced magnetic field, temperature and concentration profiles are discussed in detail. Also skin friction coefficient, Nusselt number and Sherwood number are tabulated for their relevant parameters.
4.2. MATHEMATICAL FORMULATION:

In this problem, we consider the effects of magnetic and viscous dissipation on steady two-dimensional magnetohydrodynamic mixed non-Newtonian incompressible Casson and radiative fluid over a vertical plate with induced magnetic field and heat and mass transfer. The coordinate system and the physical model of the problem are shown in Figure 4.1.

![Figure 4.1. Coordinate system and physical configuration](image)

For this present research work, we have assumed the following:

i. In the fluid region, the plate is taken vertically upward along $x'$ axis and $y'$ axis is perpendicular to it.

ii. It is assumed that the wall of the plate is preserved at an unvarying temperature $T_w'$ and concentration $C_w'$ higher than the ambient temperature $T_a'$ and concentration $C_a'$ respectively.

iii. A uniform magnetic field $B_\alpha$ is applied perpendicular to the fluid flow.

iv. The magnetic Reynolds number of the flow is not taken to be small and hence the induced magnetic field is not insignificant.
The thermal diffusion (Soret) and diffusion thermos (Dufour) effects are neglected due to the concentration of the diffusing species is assumed to be very small in comparison with the other chemical species.

All the fluid properties are assumed to be constant except the effect of the pressure gradient in the body force term.

There is a first order chemical reaction between the diffusing species and the fluid.

Since the flow is assumed to be in the direction of $x'$ axis, therefore all the physical quantities are functions of space coordinates in $y'$ only.

Consider viscous dissipation in the energy equation.

The rheological equation of state for the Cauchy stress tensor of Casson fluid (Dash et al.(1996)) is written as $\tau = \tau_0 + \mu \dot{\alpha}^*$

equivalently $\tau_{ij} = \begin{cases} 2 \left( \mu_B + \frac{p_y}{\sqrt{2\pi}} \right) e_{ij}, & \pi > \pi_c \\ 2 \left( \mu_B + \frac{p_y}{\sqrt{2\pi}} \right) e_{ij}, & \pi < \pi_c \end{cases}$

where $\tau$ is shear stress, $\tau_0$ is Casson yield stress, $\mu$ is dynamic viscosity, $\dot{\alpha}^*$ is shear rate, $\pi = e_{ij} e_{ij}$ and $e_{ij}$ is the $(i, j)^{th}$ component of deformation rate, $\pi$ is the product based on the non-Newtonian fluid, $\pi_c$ is a critical value of this product, $\mu_B$ is plastic dynamic viscosity of the non-Newtonian fluid, $p_y = \frac{\mu_B \sqrt{2\pi}}{\gamma}$

denote the yield stress of fluid. Some fluids require a gradually increasing shear stress to maintain a constant strain rate and are called Rheopctic, in the case of Casson fluid (Non-Newtonian) flow where $\pi > \pi_c$, $\mu = \mu_B + \frac{p_y}{\sqrt{2\pi}}$

Substituting Eq. (4.3) into Eq. (4.4), then, the kinematic viscosity can be written as $\nu = \frac{\mu}{\rho} = \frac{\mu_B}{\rho} \left( 1 + \frac{1}{\gamma} \right)$

Finally $\gamma$ is the Casson fluid parameter and as $\gamma \to \infty$, the governing equations of the Casson fluid model ($\gamma \to \infty$) given by Eqs. (4.8)-(4.11) become the governing
equations of the Newtonian fluid model ($\gamma \rightarrow \infty$). Then under usual Boussinesq’s approximation along with the assumptions considered to the flow, the fundamental governing partial differential equations that illustrate the physical situation are given by

**Equation of Conservation of Electric Charge:**

$$\nabla \cdot \bar{J} = 0 \text{ where } \bar{J} = \left( J_x, J_y, J_z \right) \quad (4.6)$$

**Gauss Law of Magnetism:**

$$\frac{\partial H'_y}{\partial y'} = 0 \Rightarrow H'_y = B_o \text{ (a constant)} \quad (4.7)$$

**Equation of Conservation of Momentum:**

$$v' \frac{\partial u'}{\partial y'} = \left( \frac{1}{\gamma} \right) \frac{\partial^2 u'}{\partial y'^2} + \beta(T' - T'_{\infty})(\cos \psi) + \beta'(C' - C'_{\infty})(\cos \psi) + \left( \frac{\mu B_o}{\rho} \right) \frac{\partial H'_x}{\partial y'} \quad (4.8)$$

**Equation of Conservation of Energy:**

$$v' \frac{\partial T'}{\partial y'} = \left( \frac{k}{\rho C_p} \right) \frac{\partial^2 T'}{\partial y'^2} \left( 1 + \frac{1}{\gamma} \right) \frac{\partial q'}{\partial y'} + v \frac{\partial q'}{\partial y'} + \left( 1 + \frac{1}{\gamma} \right) \left( \frac{\partial u'}{\partial y'} \right)^2 + \left( \frac{1}{\sigma \rho C_p} \right) \left( \frac{\partial H'_x}{\partial y'} \right)^2 \quad (4.9)$$

**Equation of Conservation of Magnetic Induction:**

$$v' \frac{\partial H'_x}{\partial y'} = \left( \frac{1}{\sigma \mu_e} \right) \frac{\partial^2 H'_x}{\partial y'^2} + B_o \left( \frac{\partial u'}{\partial y'} \right) \quad (4.10)$$

**Species Diffusion Equation:**

$$v' \frac{\partial C'}{\partial y'} = D \frac{\partial^2 C'}{\partial y'^2} - k' \left( C' - C'_{\infty} \right) \quad (4.11)$$

subject to the appropriate boundary conditions (Prakash et al. 2014)

$$u' = 0, \quad v' = -V_o, \quad T' = T'_{\infty}, \quad C' = C'_{\infty}, \quad H'_x = 0 \quad \text{ at } \quad y' = 0$$

$$u' \rightarrow U_o, \quad T' \rightarrow T'_{\infty}, \quad C' \rightarrow C'_{\infty}, \quad H'_x \rightarrow 0 \quad \text{ as } \quad y' \rightarrow \infty \quad (4.12)$$

In case of an optically thin gray gas, the expression for local radiant (Prakash et al. 2014) is given by

$$\frac{\partial q'}{\partial y'} = -4a \sigma^* \left( T'_{\infty}^4 - T'^4 \right) \quad (4.13)$$
It is assumed that the temperature differences within the flow are sufficiently small and that \( T^4 \) may be expressed as a linear function of the temperature. This is obtained by expanding \( T^4 \) in a Taylor series about \( T^4 \) and neglecting the higher order terms, thus we get (Prakash et al. 2014)

\[
T^4 \approx 4T^3 - 3T^4
\]  

(4.14)

Using the following non-dimensional quantities (4.15)

\[
\begin{align*}
\frac{u}{U_o} = u', & \quad y = \frac{y'V_o}{v}, & \quad B = \frac{H_0}{U_o}, & \quad \theta = \frac{T' - T_o}{T'_w - T_o}, & \quad \phi = \frac{C'_w - C'_o}{C'_o - C'_w}, & \quad M = \frac{\mu B_o}{\rho V_o}, & \quad Gr = \frac{\nu g (T'_w - T_o)}{U_o V_o^2} \\
Gm = \frac{\nu g (C'_w - C'_o)}{U_o V_o^2}, & \quad Pr = \frac{\nu C_p}{\alpha}, & \quad Sc = \frac{v}{D}, & \quad Re = \frac{U_o x'}{v}, & \quad Pr_m = \sigma v u_o
\end{align*}
\]

(4.15)

and with the help of Eqs. (4.13) and (4.14), the governing Eqs. (4.8)-(4.11) reduce to

\[
\left(1 + \frac{1}{\gamma}\right) \frac{d^2 u}{dy^2} + \frac{du}{dy} + M \frac{dB}{dy} + Gr (\cos \psi) \theta + Gc (\cos \psi) \phi = 0
\]  

(4.16)

\[
\frac{d^2 \theta}{dy^2} + (Pr) \frac{d \theta}{dy} - \left(\frac{R(Pr)}{4}\right) \theta + (Ec)(Pr) \left(\frac{du}{dy}\right)^2 + \left(\frac{Ec(Pr)}{Pr_m}\right) \left(\frac{dB}{dy}\right)^2 = 0
\]  

(4.17)

\[
\frac{d^2 B}{dy^2} + (M)(Pr_m) \frac{du}{dy} + (Pr_m) \frac{dB}{dy} = 0
\]  

(4.18)

\[
\frac{d^2 \phi}{dy^2} + (Sc) \frac{d \phi}{dy} - (Kr)(Sc) \phi = 0
\]  

(4.19)

with associated initial and boundary conditions

\[
\begin{align*}
& u = 0, \quad \theta = 1, \quad \phi = 1, \quad B = 0 \quad \text{at} \quad y = 0 \\
& u \to 1, \quad \theta \to 0, \quad \phi \to 0, \quad B \to 0 \quad \text{as} \quad y \to \infty
\end{align*}
\]

(4.20)

All the symbols are defined in nomenclature, non-dimensional parameters for this boundary layer flow are given by

The Skin-friction coefficient at the plate, given by

\[
C_f = \left(1 + \frac{1}{\gamma}\right) \frac{\tau_w'}{\rho U_o V} = \left(1 + \frac{1}{\gamma}\right) \left(\frac{\partial u}{\partial y}\right)_{y=0}
\]  

(4.21)

The heat transfer coefficient (Nusselt number), given by
\[ Nu = -x' \left( \frac{\partial T'}{\partial y'} \right)_{y=0} / \left( T'_w - T'_x \right) \Rightarrow Nu \text{Re}^{-1} = - \left( \frac{\partial \theta}{\partial y} \right)_{y=0} \] (4.22)

The mass transfer coefficient (Sherwood number), given by

\[ Sh = -x' \left( \frac{\partial C'}{\partial y'} \right)_{y=0} / \left( C'_w - C'_x \right) \Rightarrow Sh \text{Re}^{-1} = - \left( \frac{\partial \phi}{\partial y} \right)_{y=0} \] (4.23)

4.3. METHOD OF SOLUTION BY FINITE ELEMENT METHOD:

The finite element method is a numerical and computer based technique of solving a variety of practical engineering problems that arise in different fields such as, in heat transfer, fluid mechanics (Srinivasa Raju et al. (2015), Bhargava and Rana (2011), Sheri and Raju (2015), Sheri and Raju (2016), Sivaiah and Raju (2013), Srinivasa Raju et al. (2016)), chemical processing (Lin and Lo (2003)), rigid body dynamics (Dettmer and Peric (2006)), solid mechanics (Hansbo and Hansbo (2003)), and many other fields. It is recognized as the most powerful numerical analysis tool in solving complex problems that arise in engineering.

Because of simplicity and accuracy this method is widely used in the design of engineering models, designing processes and also applied to many number of physical problems. Here, the governing partial differential equations are reduced to matrix form. Then, by describing the geometry of the problem the FEM is analysed with greater flexibility. The discretization of the domain is executed by highly flexible pieces or elements can be easily describe the complex regions. This method follows, by considering the piecewise continuity of the solutions. An excellent description of finite element formulations is available in Bathe (1996) and Reddy (1985). The steps involved in the finite element analysis areas follows.

**Step 1: Discretization of the domain:**

In this process, the entire domain of the problem taken into consideration is divided into non overlapping finite number of connected sub-domains called elements and the group of all these elements is known as finite element mesh, such that these sub-domains cover the entire region of the problem taken into consideration.
Step 2: Formation of the element equations:

i. The variational formulation of the differential equation is applied on the typical element

ii. An approximate solution of the element equations are generated after the invention of variational formulation over the typical element.

iii. The element matrix is constructed by using the element interpolation functions, which is also known as Stiffness matrix

STEP 3: Assembling the element equations:

By imposing the inter element continuity conditions, the algebraic equations are achieved and assembled yielding number of mathematical equations governing the whole domain known to be Global Finite Element model.

Step 4: Imposing the boundary conditions:

The Neumann and Dirichlet's boundary conditions (4.20) are imposed on the above accumulated mathematical equations.

Step 5: Solution of assembled equations:

The mathematical equations so obtained are solved by Gauss elimination method, and the final matrix equation is solved by iterative process. For the computation, the value of \( y \) is varied from 0 to \( y_{\max} = 9 \), where \( y_{\max} \) is infinity i.e., away from the momentum, energy and concentration edge layers. The entire domain is divided into a set of 90 line segments of equal width 0.1, each element being two-noded.

Variational formulation:

The variational formulation associated with Eqs. (4.16)-(4.19) over a two-noded linear typical element \((y_v, y_{ei})\) is given as

\[
\int_{y_v}^{y_{ei}} w_i \left[ \left( 1 + \frac{1}{\gamma} \right) \frac{d^2 u}{dy^2} + \frac{du}{dy} + M \frac{dB}{dy} + Gr(\cos \psi)\Theta + Gc(\cos \psi)\Phi \right] dy = 0 \quad (4.24)
\]

\[
\int_{y_v}^{y_{ei}} w_i \left[ 4(Pr m)\frac{d^2 \Theta}{dy^2} + 4(Pr m)\frac{d\Theta}{dy} - (Pr m)(R)(Pr)\Theta + 4(Pr m)(Ec)(Pr) \left( \frac{du}{dy} \right)^2 + 4(Ec)(Pr) \left( \frac{dB}{dy} \right)^2 \right] dy = 0 \quad (4.25)
\]

\[
\int_{y_v}^{y_{ei}} w_3 \left[ \frac{d^2 B}{dy^2} + (M)(Pr m) \frac{du}{dy} + (Pr m) \frac{dB}{dy} \right] dy = 0 \quad (4.26)
\]
\[
\int_{y_e}^{y_{e+1}} w_1 \left[ \frac{d^2 \phi}{dy^2} + (Sc) \frac{d\phi}{dy} - (Kr)(Sc) \phi \right] dy = 0 \tag{4.27}
\]

Where \( w_1, w_2, w_3, w_4 \) (arbitrary) are test functions referred as the variations in \( u, \theta, B \) and \( \phi \) respectively.

After reducing the non-linearity and order of integration, we arrive at the following system of equations

\[
\int_{y_e}^{y_{e+1}} \left[ \left( 1 + \frac{1}{\gamma} \right) \frac{du}{dy} \left( \frac{dw_1}{dy} \right) - \left( \frac{dw_3}{dy} \right) \frac{du}{dy} - M(w_1) \frac{dB}{dy} - Gr(w_1)(\cos \psi) \theta - Gc(w_1)(\cos \psi) \phi \right] dy = 0 \tag{4.28}
\]

\[
\int_{y_e}^{y_{e+1}} \left[ \left( 4(Pr m) \frac{\partial \theta}{\partial y} \right) + 4(Pr m) \left( \frac{\partial w_2}{\partial y} \right) \left( \frac{\partial \theta}{\partial y} \right) - 4(Pr m) \left( Ec \right) \left( Pr \right) w_2 \left( \frac{\partial u}{\partial y} \right) \left( \frac{\partial u}{\partial y} \right) \right] dy = 0 \tag{4.29}
\]

\[
\int_{y_e}^{y_{e+1}} \left[ \left( \frac{dB}{dy} \right) \left( \frac{dw_3}{dy} \right) - M(w_3) \frac{dB}{dy} - (Pr m) \left( \frac{dB}{dy} \right) \right] dy - \left[ \left( \frac{\partial \theta}{\partial y} \right) \right]_{y_e}^{y_{e+1}} = 0 \tag{4.30}
\]

\[
\int_{y_e}^{y_{e+1}} \left[ \left( \frac{d\phi}{dy} \right) \left( \frac{dw_4}{dy} \right) - (Sc)(w_4) \frac{d\phi}{dy} + (Kr)(Sc)(w_4) \phi \right] dy - \left[ \left( \frac{\partial \phi}{\partial y} \right) \right]_{y_e}^{y_{e+1}} = 0 \tag{4.31}
\]

**Finite element formulation:**

In eqs. (4.28)-(4.31) substituting finite element approximations of the form:

\[
u = \sum_{j=1}^{2} u_j \psi_j, \quad \theta = \sum_{j=1}^{2} \theta_j \psi_j, \quad B = \sum_{j=1}^{2} B_j \psi_j, \quad \phi = \sum_{j=1}^{2} \phi_j \psi_j
\tag{4.32}
\]

With \( w_1 = w_2 = w_3 = w_4 = \psi_i \) (i = 1, 2) are the velocity(\( u_j \)), temperature(\( \theta_j \)), induced magnetic field(\( B_j \)) and concentration(\( \phi_j \)) respectively at the \( j^{th} \) node of \( e^{th} \) typical element \((y_e, y_{e+1})\) and \( \psi_i \) are the shape functions for this element \((y_e, y_{e+1})\) and are taken as:
$$\psi_1^e = \frac{y_{e1} - y}{y_{e1} - y_e} \quad \text{and} \quad \psi_2^e = \frac{y - y_e}{y_{e1} - y_e}, \quad y_e \leq y \leq y_{e1}$$  \hfill (4.33)

The finite element model for $e^o$ element $(y_e, y_{e1})$ is given by

$$\begin{bmatrix} K_{11}^o & K_{12}^o & K_{13}^o & K_{14}^o & \begin{bmatrix} \psi_e^o \end{bmatrix} \\ K_{21}^o & K_{22}^o & K_{23}^o & K_{24}^o & \begin{bmatrix} \theta_e^o \end{bmatrix} \\ K_{31}^o & K_{32}^o & K_{33}^o & K_{34}^o & \begin{bmatrix} B^e \end{bmatrix} \\ K_{41}^o & K_{42}^o & K_{43}^o & K_{44}^o & \begin{bmatrix} \phi_e^o \end{bmatrix} \end{bmatrix} + \begin{bmatrix} M_{11}^o \\ M_{12}^o \\ M_{13}^o \\ M_{14}^o \end{bmatrix} = \begin{bmatrix} \psi_{e1}^o \\ \theta_{e1}^o \\ B_e \phi_e^o \end{bmatrix}$$  \hfill (4.34)

Where $\{K_m\}, \{M_m\}, \{\psi_e^o\}, \{\theta_e^o\}, \{B_e\}, \{\phi_e^o\}$ and $\{b_e^o\}$,

$(m, n = 1, 2, 3, 4)$ are the set of matrices defined as follows:

$$K_{11}^o = \left( 1 + \frac{1}{y} \right) \int_{y_e}^{y_{e1}} \left( \frac{\partial \psi_e^o}{\partial y} \right) \left( \frac{\partial \psi_e^o}{\partial y} \right) dy = \int_{y_e}^{y_{e1}} \left( \frac{\partial \psi_e^o}{\partial y} \right) dy, \quad K_{12}^o = -M \int_{y_e}^{y_{e1}} \left( \frac{\partial \psi_e^o}{\partial y} \right) dy,$$

$$K_{13}^o = -(Gr + Gc) \int_{y_e}^{y_{e1}} \left( \psi_e^o \right) \left( \psi_e^o \right) dy, \quad K_{14}^o = K_{24}^o = K_{33}^o = K_{34}^o = K_{43}^o = K_{44}^o = 0,$$

$$M_{11}^o = M_{22}^o = M_{33}^o = M_{44}^o = \int_{y_e}^{y_{e1}} \left( \psi_e^o \right) \left( \psi_e^o \right) dy,$$

$$M_{12}^o = M_{13}^o = M_{21}^o = M_{23}^o = M_{31}^o = M_{32}^o = M_{43}^o = M_{44}^o = M_{41}^o = M_{42}^o = M_{43}^o = 0,$$

$$K_{21}^o = 4(Pr m) (Pr) \int_{y_e}^{y_{e1}} \left( \psi_e^o \right) \left( \frac{\partial \psi_e^o}{\partial y} \right) dy + 4(Pr m) \int_{y_e}^{y_{e1}} \left( \frac{\partial \psi_e^o}{\partial y} \right) \left( \frac{\partial \psi_e^o}{\partial y} \right) dy,$$

$$K_{22}^o = (Pr m) (Pr) (R) \int_{y_e}^{y_{e1}} \left( \psi_e^o \right) \left( \psi_e^o \right) dy, \quad b_{4e}^o = \left[ \left( \psi_e^o \right) \left( \frac{\partial \psi_e^o}{\partial y} \right) \right]_{y_e}^{y_{e1}},$$

$$K_{23}^o = 4(Pr m) (Ec) (Pr) \int_{y_e}^{y_{e1}} \left( \psi_e^o \right) \left( \frac{\partial \psi_e^o}{\partial y} \right) \left( \frac{\partial \psi_e^o}{\partial y} \right) dy - \frac{4}{4}(Ec) (Pr) \int_{y_e}^{y_{e1}} \left( \psi_e^o \right) \left( \frac{\partial \psi_e^o}{\partial y} \right) \left( \frac{\partial \psi_e^o}{\partial y} \right) dy,$$

$$K_{24}^o = \int_{y_e}^{y_{e1}} \left( \psi_e^o \right) \left( \frac{\partial \psi_e^o}{\partial y} \right) dy - M (Pr m) \int_{y_e}^{y_{e1}} \left( \psi_e^o \right) \left( \frac{\partial \psi_e^o}{\partial y} \right) dy, \quad K_{32}^o = -(Pr m) \int_{y_e}^{y_{e1}} \left( \psi_e^o \right) \left( \frac{\partial \psi_e^o}{\partial y} \right) dy,$$

$$K_{31}^o = \int_{y_e}^{y_{e1}} \left( \psi_e^o \right) \left( \frac{\partial \psi_e^o}{\partial y} \right) dy - (Sc) \int_{y_e}^{y_{e1}} \left( \psi_e^o \right) \left( \frac{\partial \psi_e^o}{\partial y} \right) dy,$$
In one-dimensional region, linear and quadratic elements, or element of higher order can be taken. Here, the domain of the problem taken into consideration is divided into 90 intervals of equal intervals of equal width 0.1. All the four functions u, B, θ, ϕ are evaluated at each node. Hence, assembling the element equations yield a set of 364 non-linear equations. Gauss elimination iterative procedure is employed to system of equations maintaining the accuracy 0.00005. When the relative error between the consecutive values of the iterations reaches desired accuracy iterative procedure is stopped, then concluded that we arrive a convergent solution. The algorithm is executed in MATLAB running on PC. We conclude that this method has excellent convergence.

4.4. STUDY OF GRID INDEPENDENCE:

To study the grid independency/dependency, for the different mesh sizes 0.0001 and 0.001, the numerical values for u, B, θ and ϕ are evaluated displayed in table 4.1. From the table, no significant difference between the values of u, B, θ, ϕ corresponding to the grid size 0.01, 0.001 and 0.0001 were observed, we conclude that the results of u, B, θ, ϕ are independent of grid size.

Table-4.1. The numerical values of u, B, θ and ϕ for variation of mesh sizes

<table>
<thead>
<tr>
<th>Mesh (Grid) Size = 0.0001</th>
<th>u</th>
<th>B</th>
<th>θ</th>
<th>ϕ</th>
</tr>
</thead>
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<tr>
<td>0.0000000</td>
<td>0.0000000</td>
<td>1.0000000</td>
<td>1.0000000</td>
<td></td>
</tr>
<tr>
<td>8.4460812</td>
<td>- 0.2041255</td>
<td>0.5721116</td>
<td>0.4081449</td>
<td></td>
</tr>
<tr>
<td>5.9442046</td>
<td>- 0.1491618</td>
<td>0.2910177</td>
<td>0.1737355</td>
<td></td>
</tr>
<tr>
<td>3.5963027</td>
<td>- 0.0830346</td>
<td>0.1394854</td>
<td>0.0760272</td>
<td></td>
</tr>
<tr>
<td>2.2666039</td>
<td>- 0.0416294</td>
<td>0.0645083</td>
<td>0.0337616</td>
<td></td>
</tr>
<tr>
<td>1.5977422</td>
<td>- 0.0197623</td>
<td>0.0290949</td>
<td>0.0150684</td>
<td></td>
</tr>
<tr>
<td>1.2759373</td>
<td>- 0.0090444</td>
<td>0.0128627</td>
<td>0.0067062</td>
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</tr>
<tr>
<td>1.1238416</td>
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<td>0.0055456</td>
<td>0.0029605</td>
<td></td>
</tr>
<tr>
<td>1.0519732</td>
<td>- 0.0016569</td>
<td>0.0022434</td>
<td>0.0012739</td>
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</tbody>
</table>

<table>
<thead>
<tr>
<th>Mesh (Grid) Size = 0.001</th>
<th>u</th>
<th>B</th>
<th>θ</th>
<th>ϕ</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0000000</td>
<td>0.0000000</td>
<td>1.0000000</td>
<td>1.0000000</td>
<td></td>
</tr>
<tr>
<td>8.4462466</td>
<td>- 0.2040831</td>
<td>0.5722164</td>
<td>0.4081766</td>
<td></td>
</tr>
</tbody>
</table>
### Table-4.2: Comparison between the present Velocity, Induced magnetic field and Temperature results with the results of Prakash et al. (2014) for $Prm << M$.

<table>
<thead>
<tr>
<th>y</th>
<th>Results of Prakash et al. (2014)</th>
<th>Present numerical results</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$u$</td>
<td>$B$</td>
</tr>
<tr>
<td></td>
<td>$M = 0.5$</td>
<td>$M = 0.5$</td>
</tr>
<tr>
<td>0</td>
<td>0.00000</td>
<td>0.00000</td>
</tr>
<tr>
<td>2</td>
<td>3.4380</td>
<td>-0.2078</td>
</tr>
<tr>
<td>4</td>
<td>1.8516</td>
<td>-0.3093</td>
</tr>
<tr>
<td>6</td>
<td>1.2335</td>
<td>-0.2961</td>
</tr>
<tr>
<td>8</td>
<td>1.0591</td>
<td>-0.2537</td>
</tr>
<tr>
<td>10</td>
<td>1.0146</td>
<td>-0.2105</td>
</tr>
</tbody>
</table>

For program code validation, we compared the present numerical results with analytical results of Prakash et al. (2014), Raptis et al. (2003) and Ahmed (2010) in tables 4.2, 4.3 and 4.4 respectively which are available in literature. From these tables, we observed that the results are in good agreement with their study.
Table 4.3: Comparison between the present velocity results with the velocity results of Prakash et al. (2014), Raptis et al. (2003) and Ahmed (2010) for $Prm = 0.1, M = 0.25, Gr = 5.0, \gamma = 1.0, \psi = 30$ and $Gc = 5.0$.

<table>
<thead>
<tr>
<th></th>
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<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.00000</td>
<td>0.0000</td>
<td>0.00000</td>
<td>0.00000000</td>
</tr>
<tr>
<td>2</td>
<td>3.39344</td>
<td>3.2584</td>
<td>3.35839</td>
<td>3.40218974</td>
</tr>
<tr>
<td>4</td>
<td>1.88766</td>
<td>1.8523</td>
<td>1.85232</td>
<td>1.88624982</td>
</tr>
<tr>
<td>6</td>
<td>1.24757</td>
<td>1.2406</td>
<td>1.24063</td>
<td>1.25430981</td>
</tr>
<tr>
<td>8</td>
<td>1.06255</td>
<td>1.0613</td>
<td>1.06134</td>
<td>1.07639412</td>
</tr>
<tr>
<td>10</td>
<td>1.01511</td>
<td>1.0149</td>
<td>1.01492</td>
<td>1.01569247</td>
</tr>
</tbody>
</table>

Table 4.4: Comparison between the present Induced magnetic field results with the results of Prakash et al. (2014), Raptis et al. (2003) and Ahmed (2010) for $Prm = 0.1, M = 0.25, Gr = 5.0, \gamma = 1.0, \psi = 30$ and $Gc = 5.0$.

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.00000</td>
<td>0.0000</td>
<td>0.00000</td>
<td>0.00000000</td>
</tr>
<tr>
<td>2</td>
<td>-0.09437</td>
<td>-0.0875</td>
<td>-0.08750</td>
<td>-0.08842195</td>
</tr>
<tr>
<td>4</td>
<td>-0.14786</td>
<td>-0.1388</td>
<td>-0.13880</td>
<td>-0.13995418</td>
</tr>
<tr>
<td>6</td>
<td>-0.14358</td>
<td>-0.1354</td>
<td>-0.13538</td>
<td>-0.13695178</td>
</tr>
<tr>
<td>8</td>
<td>-0.12354</td>
<td>-0.1167</td>
<td>-0.11669</td>
<td>-0.12615484</td>
</tr>
<tr>
<td>10</td>
<td>-0.10263</td>
<td>-0.0970</td>
<td>-0.09699</td>
<td>-0.01245862</td>
</tr>
</tbody>
</table>

4.6. RESULTS AND DISCUSSIONS:

Numerical solutions of non-linear coupled partial differential equations (4.16)-(4.19) under boundary conditions (4.20) are obtained through finite element technique for fluid velocity, temperature, concentration and induced magnetic field. The behaviour of the fluid (i.e., fluid velocity, temperature, induced magnetic field and concentration) was analysed with various physical parameters through the Figs. (4.2)-(4.20). In this study the boundary condition for $y \to \infty$ is replaced by $y_{max} = 9$ is a sufficiently large value of $y$, where the velocity, induced magnetic field, temperature and concentration profiles can be approached to the relevant free stream velocity.
The variation of numerical values of thermal Grashof number (Gr) on velocity profiles at the boundary layer is as shown in the Figure 4.2. It is observed that an increase in Gr leads to increase in the fluid velocity due to enhancement in buoyancy force. Here, the positive values of Gr correspond to cooling of the plate. In addition, it is observed that the velocity increases sharply near the wall as Gr increases and then decays to the free stream value.

Figure 4.2. Influence of Gr on velocity profiles

Figure 4.3 shows the effect of solutal Grashof number (Gc) on the fluid velocity at the boundary layer. It is observed that the velocity increases sharply near the wall as Gc increases and then decays to the free stream value. As expected, the fluid velocity
increases and the peak value becomes more distinctive due to increase in the species buoyancy force represented by $Gc$.

The effect of the Hartmann number (or) Magnetic field parameter is shown in Figure 4.4. It is observed that the velocity of the fluid decreases with the increase of the magnetic field parameter values. The decrease in the velocity as the Magnetic field parameter increases is because the presence of a magnetic field in an electrically conducting fluid induces a force called the Lorentz force, which acts against the flow if the magnetic field is applied in the normal direction, as in the present study. This resistive force slows down the fluid velocity component.
Figs. 4.5 and 4.6 show the behaviour of fluid velocity and fluid temperature respectively with an influence of Prandtl number. Fluid velocity and temperature decreases as increasing of Prandtl number. Since Prandtl number is the ratio of momentum diffusivity to thermal conductivity. For small Prandtl number, thermal conductivity is high and momentum diffusivity is low, because of that fluid viscosity is low. Therefore reduce the heat transfer for small Prandtl number.
Figure 4.7 and 4.8 shows that, influence of Schmidt number (Sc) on velocity profile and temperature profile. It is observed that, Schmidt number (Sc) is in the concentration and it is coupled in the momentum equation. Increasing of Schmidt number (Sc) the momentum boundary layer thickness is increasing as well as fluid velocity and fluid concentration is decreasing in the entire boundary of the region.
The effect of the viscous dissipation parameter i.e., the Eckert number $Ec$ on the velocity and temperature are shown in Figs. 4.9 and 4.10 respectively. The Eckert number $Ec$ expresses the relationship between the kinetic energy in the flow and the enthalpy. It embodies the conversion of kinetic energy into internal energy by work done against the viscous fluid stresses. Greater viscous dissipative heat causes a rise in the temperature, as well as the velocity and cross flow velocity. This behaviour is evident from Figs. 4.9 and 4.10.

Figure 4.10. Influence of $Ec$ on temperature profiles

Figure 4.11. Influence of $Prm$ on velocity profiles
Figs. 4.11 and 4.12 show the influence of magnetic Prandtl number $Prm$ on the velocity of fluid and temperature of fluid. Both fluid velocity and temperature decrease as increasing of magnetic Prandtl number $Prm$.

Figure 4.12. Influence of $Prm$ on temperature profiles

Figure 4.13. Influence of $R$ on velocity profiles
The effects of the thermal radiation parameter \( R \) on the velocity and temperature profiles in the boundary layer are illustrated in Figs. 4.13 and 4.14 respectively. Increasing the thermal radiation parameter produces significant increase in the thermal condition of the fluid and its thermal boundary layer. This increase in the fluid temperature induces more flow in the boundary layer causing the velocity of the fluid there to increase. As expected, the presence of the chemical reaction significantly affects the concentration profiles as well as the velocity profiles.

Figure 4.14. Influence of \( R \) on temperature profiles

Figure 4.15. Influence of \( Kr \) on velocity profiles
Figure 4.16 Influence of $Kr$ on concentration profiles

from Figs. 4.15 and 4.16. It reveals that increase in chemical reaction parameter results fall in the velocity as well as concentration of the fluid, the peak indicates the velocity of the fluid is very high nearer to the surface of the plate. It is evident that the increase in the chemical reaction significantly alters the concentration boundary layer thickness but does not alter the momentum boundary layers.

Tables 4.5 and 4.6 represent the variations of $Ec$, $R$, $\gamma$ and $Prm$, $Kr$, $\psi$ on skin-friction coefficient respectively. The skin-friction coefficient is increasing with increasing values of $Ec$ and decreasing with increasing values of $R$ and $\gamma$.

Table-4.5: Skin-friction values for variation of $Ec$, $R$ and $\gamma$

<table>
<thead>
<tr>
<th>$Ec$</th>
<th>$R$</th>
<th>$\gamma$</th>
<th>$C_f$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.001</td>
<td>0.1</td>
<td>0.5</td>
<td>3.26154832</td>
</tr>
<tr>
<td>1.000</td>
<td>0.1</td>
<td>0.5</td>
<td>3.28015548</td>
</tr>
<tr>
<td>0.001</td>
<td>0.5</td>
<td>0.5</td>
<td>3.27336541</td>
</tr>
<tr>
<td>0.001</td>
<td>0.1</td>
<td>1.0</td>
<td>3.20667124</td>
</tr>
</tbody>
</table>

Table-4.6: Skin-friction values for variation of $Prm$, $Kr$ and $\psi$

<table>
<thead>
<tr>
<th>$Prm$</th>
<th>$Kr$</th>
<th>$\psi$</th>
<th>$C_f$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>1.0</td>
<td>$30^\circ$</td>
<td>3.26154832</td>
</tr>
<tr>
<td>0.6</td>
<td>1.0</td>
<td>$30^\circ$</td>
<td>3.19224862</td>
</tr>
<tr>
<td>0.2</td>
<td>2.0</td>
<td>$30^\circ$</td>
<td>3.21448012</td>
</tr>
<tr>
<td>0.2</td>
<td>1.0</td>
<td>$45^\circ$</td>
<td>3.22466215</td>
</tr>
</tbody>
</table>
Figure 4.17 Influence of $\gamma$ on velocity profiles

Figure 4.17 shows the fluid velocity with an influence of Casson fluid parameter. With an increasing Casson parameter, the fluid velocity decreases in the entire boundary region. Since fluid viscosity will be high with increase of Casson parameter, momentum boundary layer thickness decreases.

Figure 4.18 Influence of $\psi$ on velocity profiles

An influence of angle of inclination of the plate on the velocity profile is shown in the Figure 4.18. It is observed that fluid velocity decreases with increasing inclination parameter ($\psi$).
The influences of $R$, $Pr$ and $Ec$ on Nusselt number are discussed in table 4.7 with the help of numerical values. The rate of heat transfer coefficient is increasing with increasing values of $Ec$ and the reverse effect is observed with increasing values of $R$ and $Pr$.

Table 4.7: Nusselt number values for variation of $Ec$, $R$ and $Pr$

<table>
<thead>
<tr>
<th>$Ec$</th>
<th>$R$</th>
<th>$Pr$</th>
<th>$Nu$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.001</td>
<td>0.1</td>
<td>0.71</td>
<td>0.62334821</td>
</tr>
<tr>
<td>1.000</td>
<td>0.1</td>
<td>0.71</td>
<td>0.63955478</td>
</tr>
<tr>
<td>0.001</td>
<td>0.5</td>
<td>0.71</td>
<td>0.64388721</td>
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<tr>
<td>0.001</td>
<td>0.1</td>
<td>7.00</td>
<td>0.57661548</td>
</tr>
</tbody>
</table>

The response of induced magnetic field to Hartmann number is shown in the Figure 4.19 for weak buoyancy and airflow. Clearly, as Hartmann number increases, the induced magnetic field reduces.
Figure 4.20 Influence of Prm on induced magnetic field

The effect of magnetic Prandtl number Prm on the induced magnetic field is presented in the Figure 4.20. In this figure magnetic Prandtl number Prm is set as less than unity, which implies that the magnetic diffusion rate exceeds the viscous diffusion rate. As such Prm increases, momentum diffusivity will increase. Therefore, Prm increases from 0.2 to 1.5, the induced magnetic field is found to increase absolutely in the boundary layer $0 \leq y \leq 6$, but this trend is opposite for the region $6 \leq y \leq 9$. Greater flux reversal arises in the boundary layer region $y \in [0, 6]$ and for $Prm = 0.2$ (magnetic diffusion rate exceeds the viscous diffusion rate); but this trend is reversed for the region $y \in [6, 9]$.

Table 4.8: Sherwood number values for variation of Sc and Kr

<table>
<thead>
<tr>
<th>Sc</th>
<th>Kr</th>
<th>Sh</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>1.0</td>
<td>0.59348752</td>
</tr>
<tr>
<td>0.6</td>
<td>1.0</td>
<td>0.56177823</td>
</tr>
<tr>
<td>0.2</td>
<td>2.0</td>
<td>0.55395842</td>
</tr>
</tbody>
</table>

The combined influence of Sc and Kron Sherwood number is discussed in table 4.8. From this table, we observed that the rate of mass transfer coefficient is decreasing with increasing values of Sc and Kr.