CHAPTER III
CHAPTER III

1. ON THE TORSIONAL BODY FORCE WITHIN VISCO-ELASTIC HALF-SPACE.*

INTRODUCTION:
The problem of torsional time-dependent body forces applied to the surface of an elastic half-space was studied by REISSNER (58).
In a recent paper EASON (17) has considered the problem of a homogeneous isotropic elastic half-space inside which torsional body forces act in a plane parallel to free-surface. In the present paper, solution of the problem of decaying torsional body forces acting within a viscoelastic half space has been attempted. The solution has been obtained using the multiple integral transformers, and has been applied for a particular type of body force located over a plane at a constant depth from the plane face. Numerical calculations showing variation of displacement for different radial distances have been presented.

STATEMENT AND SOLUTION OF THE PROBLEM:
As reference system we take cylindrical coordinates \((r, \theta, z)\), the origin being any point of the boundary of the half space and the \(z\)-axis pointing normally into the medium. The material considered here is homogeneous...

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isotropic and viscoelas-tic of the linear type.

The stress-relations in viscoelastic medium may be taken as

\[
\left[ 1 + a_i \frac{\partial}{\partial t} \right] \tau_{i,j} = 2 k_i \left[ 1 + b_i \frac{\partial}{\partial t} \right] \varepsilon_{i,j},
\]

(1)

where \(a_i, b_i, k_i\) are material constants,

\(\tau_{i,j}, \varepsilon_{i,j}, (i,j = 1, 2, 3)\) are stress and strain tensors respectively.

For torsional disturbance the only non-trivial stress-strain relations are

\[
(1 + a_1 \frac{\partial}{\partial t}) \tau_{\theta} = 2 k_1 (1 + b_1 \frac{\partial}{\partial t}) \varepsilon_{\theta},
\]

(2)

\[
(1 + a_1 \frac{\partial}{\partial t}) \tau_{\xi} = 2 k_1 (1 + b_1 \frac{\partial}{\partial t}) \varepsilon_{\xi},
\]

(3)

where \(\tau_{\theta}, \tau_{\xi}\) are the shear stresses in the solid.

There are no longitudinal or lateral displacements, and the motion is symmetrical about the \(z\)-axis, so \(u_r = u_z = 0\) and \(u_\theta\) must be independent of \(\theta\).

Let \(u_\theta (r, z, t) = \mathbf{v} (r, z, t)\) (say)

(4)
The equation of motion are

\[
\frac{\partial \tau_{rr}}{\partial r} + \frac{1}{r} \frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\partial \tau_{r\theta}}{\partial z} + \frac{\tau_{rr} - \tau_{\theta\theta}}{r} + \rho F_r = \rho \frac{\partial^2 u_r}{\partial t^2},
\]

(5)

\[
\frac{\partial \tau_{r\theta}}{\partial r} + \frac{1}{r} \frac{\partial \tau_{\theta\theta}}{\partial \theta} + \frac{\partial \tau_{\theta\theta}}{\partial z} + \frac{\tau_{r\theta}}{r} + \rho F_\theta = \rho \frac{\partial^2 u_\theta}{\partial t^2},
\]

(6)

\[
\frac{\partial \tau_{r\theta}}{\partial r} + \frac{1}{r} \frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\partial \tau_{r\theta}}{\partial z} + \frac{\tau_{r\theta}}{r} + \rho F_z = \rho \frac{\partial^2 u_z}{\partial t^2},
\]

(7)

where \( F = (F_r, F_\theta, F_z) \) is the body force, \( \rho \), the density of the solid.

The equations of motion (5) and (7) are now satisfied identically, whilst, the equation of motion (6) becomes

\[
\frac{\partial \tau_{r\theta}}{\partial r} + \frac{\partial \tau_{\theta\theta}}{\partial z} + \frac{\tau_{r\theta}}{r} + F_\theta = \rho \frac{\partial^2 u_\theta}{\partial t^2},
\]

(8)

Expressing the equation of motion (8) in terms of displacement component \( v \), we get

\[
(1 + b \frac{\partial}{\partial t}) \left[ \frac{\partial^2 v}{\partial z^2} + \frac{1}{r} \frac{\partial v}{\partial r} - \frac{v}{r^2} + \frac{\partial^2 v}{\partial z^2} \right] + F_\theta
= \frac{\rho}{2 K_1} \left( 1 + a \frac{\partial}{\partial t} \right) \frac{\partial^2 v}{\partial t^2}.
\]

(9)
We shall consider the problem when the body force is of the type

$$F' = G(r, z) e^{-\omega t}, \quad \omega > 0 \quad (10)$$

i.e. the body force decaying with time.

Let us consider

$$\psi (r, z, t) = \int f(r, z) e^{-\omega t} \quad (11)$$

Substituting for \( \psi \) from (11) in (8), we get

$$\left( \frac{1}{r^2} - b \omega \right) \left[ \frac{\partial^2 f}{\partial r^2} + \frac{1}{r} \frac{\partial f}{\partial r} + \frac{\partial^2 f}{\partial z^2} - \left( \frac{1}{r^2} + \frac{\omega^2}{2 \mu_1 (1 - b \omega)} \right) \frac{\partial G}{\partial z} \right] = 0 \quad (12)$$

To solve (12), we apply HANKEL transform and FOURIER's cosine transform, defined by

$$\left[ f_1, \mathcal{G}_1 \right] = \int_{r = 0}^{\infty} \left[ f_1, \mathcal{G} \right] r J_1 (r \mathcal{G}) \, dr \quad (13)$$

and

$$\left[ f_2, \mathcal{G}_2 \right] = \int_{z = 0}^{\infty} \left[ f_2, \mathcal{G} \right] G_1 r \xi \, d \xi \quad (14)$$

where \( J_1 (r \mathcal{G}) \) is Bessel function of first kind and of order one.
Taking the HANKEL transform of the equation (12), we get

\[- \left[ \xi + \frac{\xi \omega^2 (1 - a_1 \omega)}{2 \kappa_1 (1 - b_1 \omega)} \right] f_1 (\xi, z) + \frac{\partial f_1}{\partial z^2} + \frac{G}{2 \kappa_1 (1 - b_1 \omega)} = 6 \quad (15)\]

The plane boundary of the half-space is stress-free so \( \frac{\partial f_1}{\partial z} = 0 \) on \( z = 0 \). Let us assume \( \frac{\partial f_1}{\partial z} \rightarrow 6 \) as \( z \rightarrow \infty \).

Applying FOURIER's cosine transform on (15), we have

\[- \left[ \xi + \xi^2 + \frac{\xi \omega^2 (1 - a_1 \omega)}{2 \kappa_1 (1 - b_1 \omega)} \right] f_2 (\xi, \zeta) + \frac{G_2 (\xi, \zeta)}{2 \kappa_1 (1 - b_1 \omega)} = 6 \quad (16)\]

Therefore,

\[f_2 (\xi, \zeta) = \frac{G_2}{2 \kappa_1 (1 - b_1 \omega) \left[ \xi^2 + \xi + \frac{\xi \omega^2 (1 - a_1 \omega)}{2 \kappa_1 (1 - b_1 \omega)} \right]} \quad (17)\]

By inversion Theorems for HANKEL and FOURIER's transform, from equation (17), we get

\[f (x, z) = \sqrt{\frac{2}{\pi}} \int_0^\infty \int_0^\infty \frac{G_2}{2 \kappa_1 (1 - b_1 \omega) \left[ \xi^2 + \xi + \frac{\xi \omega^2 (1 - a_1 \omega)}{2 \kappa_1 (1 - b_1 \omega)} \right]} J_1 (\xi r) \cos \zeta d\xi d\zeta \quad (18)\]
The solution for the displacement $v$, is therefore

$$v(r, z, t) = e^{-\omega t} \int \int_{\mathbb{R}^2} \mathcal{G}(\xi, \eta) \cos \xi z \, d\xi \, d\eta$$

$$= \frac{e^{-\omega t}}{\sqrt{\pi} \, \pi_1 (1 - b_1 \omega)} \int_{\xi = 0}^{\infty} \int_{\eta = 0}^{\infty} \mathcal{G}_2(\xi, \eta) \cos \xi z \, d\xi \, d\eta$$

$$= \frac{e^{-\omega t}}{\sqrt{\pi} \, \pi_1 (1 - b_1 \omega)} \int_{\xi = 0}^{\infty} \int_{\eta = 0}^{\infty} \mathcal{G}_2(\xi, \eta) \cos \xi z \, d\xi \, d\eta$$

$$= \frac{e^{-\omega t}}{\sqrt{\pi} \, \pi_1 (1 - b_1 \omega)} \int_{\xi = 0}^{\infty} \int_{\eta = 0}^{\infty} \mathcal{G}_2(\xi, \eta) \cos \xi z \, d\xi \, d\eta$$

$$= \frac{e^{-\omega t}}{\sqrt{\pi} \, \pi_1 (1 - b_1 \omega)} \int_{\xi = 0}^{\infty} \int_{\eta = 0}^{\infty} \mathcal{G}_2(\xi, \eta) \cos \xi z \, d\xi \, d\eta$$

If, in particular, the body force acts inside the half space at a depth $z = b$ ($\forall \omega$) and is given by

$$F(r, z, t) = G(r, z) e^{-\omega t}$$

$$= p(r) \delta(z - b) e^{-\omega t}$$

Then,

$$G_1(\xi, \tau) = \int_{\tau = 0}^{\infty} \mathcal{G}_1(\tau) \delta(z - b) \mathcal{J}_1(\tau r) \, d\tau$$

$$G_2(\xi, \tau) = \sqrt{\frac{2}{\pi}} \int_{z = 0}^{\infty} \mathcal{G}_2(\xi, \tau) \delta(z - b) \cos \xi z \, dz$$

$$= \sqrt{\frac{2}{\pi}} \mathcal{G}_2(\xi, \tau) \cos \xi b, \left[ \Re e - (\Re e) \right]$$

The first derivative

$$\Phi_1(\xi) = \int_{r = 0}^{\infty} \mathcal{J}_1(\xi r) \, dr.$$
Therefore,

\[ V(r, z, t) = f(r, z) e^{-\omega t} \]

\[ = \frac{e^{-\omega t}}{\pi R_1 (1-b,\omega)} \int_{-\infty}^{\infty} \int_{0}^{\infty} \frac{p_i(\xi) \cos \xi b \cdot \xi J_1(\xi r) d\xi d\xi}{\xi^2 + \xi^2 + \frac{\omega\xi^3 (1-a,\omega)}{2 R_1 (1-b,\omega)}} \]

\[ = \frac{e^{-\omega t}}{\pi R_1 (1-b,\omega)} \int_{-\infty}^{\infty} \int_{0}^{\infty} \frac{p_i(\xi) \left\{ \frac{1}{2} \int_{0}^{\infty} \frac{\cos \xi (\xi+b) + \cos \xi (\xi-b)}{\xi^2 + \xi^2 + \frac{\omega\xi^3 (1-a,\omega)}{2 R_1 (1-b,\omega)}} d\xi \right\} \xi J_1(\xi r) d\xi \]

where

\[ \eta^2 = \xi^2 + \frac{\omega\xi^3 (1-a,\omega)}{2 R_1 (1-b,\omega)} \]

**PARTICULAR CASE**:

If \( p(r) = Q r \), \( r \leq a \)

\[ = 0 , \quad r > a \quad (26) \]

where \( Q \) and \( a \) are constants.

\[ \bar{p}_i(\xi) = \int_{0}^{a} Q r J_1(\xi r) dr \]

\[ = Q \frac{\alpha}{\xi} J_2(\xi a) \]

(Ref. -(66)) \quad (27)
Substituting (27) in (25), we get the displacement $v$ for a particular type of decaying torsional body force, in the following integral form

$$
\Psi (\gamma, z, t) = \frac{a^2 e^{-\omega t}}{k_1 (1- b, \omega)} \int_0^\infty \frac{e^{-\eta(z + b)} - e^{-\eta|z - b|}}{\eta} J_1(\xi) J_1(\xi) d\xi,
$$

where

$$
\xi = \frac{\omega}{\omega^* + \frac{J(\omega^* (1 - a, \omega))}{a \eta_1 (1 - b, \omega)}}.
$$

**Numerical Calculations:**

For Voigt solid, $a_1 = 0$, $k_1 = \mu$, $b_1 = \mu'$, where $\mu_1$ is LAMBE's constant and $\mu'$, the viscous constant.

Displacement $v$ is then given by

$$
\frac{\Psi (1 - b, \omega)}{Q a^2} e^{i \omega t} = \Gamma (\text{say})
$$

$$
= \int_0^\infty \frac{e^{-(z+b)\sqrt{\xi^2+1}} - e^{-|z-b|\sqrt{\xi^2+1}}}{\sqrt{\xi^2+1}} J_1(\xi) J_1(\xi) d\xi
$$

where $\frac{\omega^*}{2(\mu - \mu')}$ is assumed to be unity.
On the surface \( z = 0 \),

\[
[I]_{z=0} = 2 \int_0^\infty \frac{e^{-b/\sqrt{\xi+1}}}{\sqrt{\xi+1}} J_1(\xi y) J_1(\xi a) \, d\xi,
\]

assuming \( \frac{a^2}{2(\mu - \mu' \omega)} \) to be unity.

Considering \( a = 1 \) and assuming the body force to be acting at a depth \( z = b \), the proportional displacement \( I \) on the surface \( z = 0 \) is given by the integral

\[
(i) \, [I]_{z=0} = 2 \int_0^\infty \frac{e^{-b/\sqrt{\xi+1}}}{\sqrt{\xi+1}} J_1(\xi b) J_1(\xi) \, d\xi
\]

Taking different values of \( b \), we get corresponding displacement component \( I \) in the form of infinite integral (30). Also values of proportional displacements \( I \) at different depths \( z = bm \), \( m \) being a positive integer greater than one, for different radial distances are given by integrals of the type

\[
(ii) \, [I]_{z=bm} = \int_0^\infty \frac{e^{-(m+1)b/\sqrt{\xi+1}}}{\sqrt{\xi+1}} + \frac{e^{-(m-1)b/\sqrt{\xi+1}}}{\sqrt{\xi+1}} J_1(\xi b) J_1(\xi) \, d\xi
\]
**TABLE 1.**

Different values of $[I]_{z=0}$ for two specific values of $b$

<table>
<thead>
<tr>
<th>$r$</th>
<th>$b = 1$</th>
<th>$b = 2$</th>
<th>$b = 1$</th>
<th>$b = 2$</th>
<th>$b = 1$</th>
<th>$b = 2$</th>
<th>$b = 1$</th>
<th>$b = 2$</th>
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<td></td>
<td>.2</td>
<td>.4</td>
<td>.5</td>
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<td>1.5</td>
<td>2</td>
<td>.2</td>
<td>.4</td>
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<td></td>
<td></td>
<td></td>
<td></td>
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<td>.00025</td>
</tr>
<tr>
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<td>.00055</td>
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<tr>
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<td>.00019</td>
<td>.00098</td>
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### Table 2

Values of $I$ at different depths for different radial distances.

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<th>4</th>
<th>5</th>
<th>6</th>
</tr>
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<td>0.00254</td>
<td>0.00017</td>
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<td>0.00001</td>
</tr>
<tr>
<td>3</td>
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<td>0.00166</td>
<td>-0.00059</td>
<td>0.00012</td>
<td>0.00004</td>
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<table>
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<th>$r$</th>
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<th>2</th>
</tr>
</thead>
<tbody>
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<td>0.00000</td>
</tr>
<tr>
<td>8</td>
<td>0.00001</td>
<td>0.00000</td>
</tr>
</tbody>
</table>

Here we have assumed $b = 1$, $m = 2, 3, 4, 5, 6, 7, 8, 9$; $r = 2 \text{m}, 3 \text{m}$.

These values are exhibited in following figures.

Numerical values of integrals (31) and (32) (which are evidently convergent) are calculated by means of High-Speed Digital Computer viz., I.C.T's ATLAS installed at London University Computer Centre. The program on ATLAS was written by Mr. B. Biswas.
CHAPTER III.

2. DISTURBANCES IN AN INFINITE VISCO-ELASTIC SLAB OF THE LINEAR TYPE DUE TO TRANSIENT TORSIONAL FORCES APPLIED ON A CYLINDRICAL CAVITY *

INTRODUCTION:

Propagation of waves in an elastic medium having a cylindrical hole in which an exciting source is operating over the inner surface of the hole was investigated by SEZAWA (63) and CHAKRABORTY (12). Biot (6) has also considered the problem of propagation of waves in an infinite solid having cylindrical hole. Recently SCOTT and MIKLOWITZ (61) have studied the transient compressional waves in an infinite elastic plate with a circular cylindrical cavity. The problem treated here is that of an infinite free slab with a circular cylindrical hole subjected to a transient torsional force applied on the inner boundary of the hole, the medium being viscoelastic.

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FORMULATION OF THE PROBLEM:

The cylindrical coordinates \((r, \theta, z)\) is used, \(r\) being the spatial coordinate in the plane of the slab. The homogeneous viscoelastic slab is bounded by two parallel planes \(Z = \pm H\), the thickness of the infinite slab is \(2H\) and median plane is \(Z = 0\). From the problem it is clear that the \(u_r = u_z = 0\) and \(u_\theta = u\) (say) is independent of \(\theta\). The disturbance is determined by only one differential equation. The stress–strain relations for homogeneous viscoelastic solid are

\[
(1 + a_1 \frac{\partial}{\partial t}) \gamma_{ij} = 2 k_1 (1 + b_1 \frac{\partial}{\partial t}) \varepsilon_{ij},
\]

\((i, j = 1, 2, 3)\).

where \(a_1, k_1, b_1\) are material constants.

The only equation of motion not identically satisfied is

\[
\frac{\partial \tau_{rr}}{\partial r} + \frac{1}{r} \frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\partial \tau_{\theta\theta}}{\partial \theta} + \frac{\partial \tau_{\theta z}}{\partial z} + \frac{\partial \tau_{r z}}{\partial r} = \rho \frac{\partial^2 u}{\partial t^2},
\]

\(\rho\) being the density of the medium.

Here \(\tau_{rr} = \tau_{r\theta} = \tau_{zz} = \tau_{rz} = 0\),

and

\[
(1 + a_1 \frac{\partial}{\partial t}) \tau_{r\theta} = 2 k_1 (1 + b_1 \frac{\partial}{\partial t}) \varepsilon_{r\theta},
\]

\[
= k_1 (1 + b_1 \frac{\partial}{\partial t}) (\frac{\partial u}{\partial r} - \frac{u}{r}),
\]

\[
(1 + a_1 \frac{\partial}{\partial t}) \tau_{\theta z} = k_1 (1 + b_1 \frac{\partial}{\partial t}) \frac{\partial u}{\partial z}.
\]
The equation of motion (2) thus becomes

\[ (1 + b \frac{\partial}{\partial t}) \left[ \frac{\partial^2 u}{\partial r^2} \right] \frac{1}{r} \frac{\partial u}{\partial r} - \frac{u}{r^2} + \frac{\partial u}{\partial z} \right] = \frac{\sigma}{r}, \quad \left(1 + a \frac{\partial}{\partial t} \right) \frac{\partial u}{\partial z} \]

(5)

**BOUNDARY CONDITIONS**

The plane surfaces of the slab are taken to be stress free, so that \( \tau_{rz} = 0 \) at \( Z = \pm H \).

The surface of the cylindrical hole is subjected to an exponentially decaying torsional force over a portion of it so that

\[ \tau_{r\theta} \right|_{r=a} = -Q e^{-\omega t}, \quad \omega > 0, \quad t > 0, \quad \text{for} \ 0 < |z| < h \]

\[ = 0, \quad \text{for} \ h < |z| < H \]

\( a \) being radius of circular cylindrical hole.

Equation (5) has a solution of the form

\[ U_\theta = U = \phi_n(r) \alpha_\xi k_n z \cdot e^{-\omega t} \]

(7)

where, by (5), \( k_n \) is given by \( \frac{mn}{H} \), \( n \) being an integer and \( \phi_n(r) \) satisfies the equation.
\[
\frac{d^2 \phi_n}{d \gamma^2} + \frac{1}{\gamma} \frac{d \phi_n}{d \gamma} - \left( \lambda_n^2 + \frac{1}{\gamma^2} \right) \phi_n = 0 \quad (8)
\]

where
\[
\lambda_n^2 = k_n^2 + \frac{\rho \omega^2 (1 - a_i \omega)}{1 - b_i \omega},
\]

\[
= k_n^2 + \ell_n^2, \quad (9)
\]

and
\[
\ell_n^2 = \frac{\rho \omega^2 (1 - a_i \omega)}{1 - b_i \omega}. \quad (10)
\]

The complete solution of (8) is

\[
\phi_n(\gamma) = \bar{A}_n \mathcal{I}_1(\lambda_n \gamma) + \bar{B}_n k_1(\lambda_n \gamma) \quad (11)
\]

Where \( I_1 \) and \( K_1 \) are modified Bessel function of first and second kind and of degree one.

Condition that stress and displacement should tend to zero at infinity implies that \( A_n = 0 \). For the solution of the problem we, therefore, have

\[
\mathcal{U} = B_0 k_1(\ell \gamma) e^{-\omega t} + \sum_{n=1}^{\infty} B_n k_1(\lambda_n \gamma) e^{-\omega t}. \quad (12)
\]

Where
\[
K_n = \frac{n \pi}{H}.
\]

Constants \( B_n \) are evaluated from the boundary conditions (6).
From (6), we get

\[ \begin{align*}
B_0 \left\{ \ell \left. K_i' (\ell a) \right| - \frac{1}{\alpha} & K_i (\ell a) \right\} \\
&+ \sum_{n=1}^{\infty} B_n \left\{ \lambda_n K_i' (\lambda a) - \frac{1}{\alpha} K_i (\lambda a) \right\} \cos k_n z \end{align*} \]

(13)

\[ = - \frac{Q}{\ell} \left( - a, \omega \right) \left[ \frac{h}{H} + 2 \sum_{n=1}^{\infty} \frac{1}{\lambda_n} \sin k_n \theta \cos k_n z \right] \]

(13)

For equation (13) to be true for all \( z \), we must have,

\[ B_0 = \frac{h Q (1-a, \omega)}{\ell H K_i (1-b, \omega) K_2 (\ell e)} \]

(14)

\[ B_n = \frac{2 Q \sin \lambda_n \theta}{\lambda_n \pi K_2 (\lambda a)} \cdot \frac{(1-a, \omega)}{K_1 (1-b, \omega)} \]

(15)

\[ \cos \theta \cdot U = \frac{h Q (1-a, \omega)}{\ell K_2 (\ell a)} \frac{1-a, \omega}{\lambda_n (1-b, \omega)} K_1 (\ell r) + \sum_{n=1}^{\infty} \frac{2 Q (1-a, \omega)}{\pi K_1 (1-b, \omega)} \frac{\sin \lambda_n \theta \cos k_n z}{\lambda_n K_2 (\lambda a)} \]

or

\[ \frac{K_1 (1-b, \omega)}{Q (1-a, \omega)} \cos \theta \cdot U = \frac{h}{\ell K_2 (\ell a)} K_1 (\ell r) + \sum_{n=1}^{\infty} \frac{\sin \lambda_n \theta \cos k_n z}{\lambda_n K_2 (\lambda a)} k_n (\lambda r), \]

Hence

\[ \frac{K_1 (1-b, \omega)}{Q (1-a, \omega)} \cos \theta \cdot \left[ U \right]_{z=\pm \frac{H}{\ell K_2 (\ell a)}} = - \frac{h}{\ell K_2 (\ell a)} K_1 (\ell r) + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\sin \lambda_n \theta \cos k_n z}{\lambda_n K_2 (\lambda a)} K_1 (\lambda r), \]

(17)
PARTICULAR CASES:

I. If \(a_1 = 0\), \(k_1 = \mu\) and \(b, k_1 = \mu'\), then the corresponding solid is of the Voigt-type, \(\mu\) being LAME's constant, \(\mu'\) being viscosity constant.

In this case

\[
I = L [u]_{z=\pm H} = \frac{8}{\lambda K_2 (\lambda)} K_1 (\lambda r) + \frac{2}{\Pi} \sum_{n=1}^{\infty} (-1)^n \frac{\sin k_n r}{n \lambda_n K_2 (\lambda_n r)} K_1 (\lambda_n r)
\]

(18)

where

\[
L = \frac{\mu - \mu' \omega}{\varrho} e^{\omega t}
\]

\[
\lambda_n = \frac{n^2 \pi^2}{H^2} + \ell^2
\]

\[
\ell^2 = \frac{\rho \omega^2}{\mu - \mu' \omega}
\]

(19)

II. If \(a_1 = b_1 = 0\) and \(k_1 = \mu\), we get the displacement \(u\) in a homogeneous elastic solid given by

\[
I = \frac{\bar{K}}{\varrho} [u]_{z=\pm H}
\]

\[
= \frac{8}{H \ell_0 K_2 (\lambda_0)} + \frac{2}{\Pi} \sum_{n=1}^{\infty} (-1)^n \frac{\sin \frac{n \pi \ell}{H}}{n \lambda_n K_2 (\lambda_n r)} K_1 (\lambda_n r)
\]

where \(\ell^2 = \frac{n^2 \pi^2}{H^2} + \ell_0^2\), \(\ell_0^2 = \frac{\rho \omega^2}{\mu}\).
NUMERICAL CALCULATIONS:

Numerical values of I in viscoelastic (voigt-type) and elastic solid for $H = 1000$, $a = 3$, $w = 1$, $lo = 1$ (in viscoelastic solid), $r$ (in elastic solid) at different radial distances on the free surface are shown in the table and also exhibited in the figure.

<table>
<thead>
<tr>
<th>$\frac{r}{w}$</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>8</th>
<th>10</th>
</tr>
</thead>
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<td>VISCO-ELASTIC</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\frac{1}{2}$</td>
<td>.0125</td>
<td>.0044</td>
<td>.0019</td>
<td>.00059</td>
<td>.00004</td>
</tr>
<tr>
<td>$\frac{1}{3}$</td>
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<td>.00271</td>
<td>.00202</td>
<td>.00014</td>
<td>.000054</td>
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<td>ELASTIC SOLID</td>
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<tr>
<td>$\frac{1}{4}$</td>
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<td>-.00469</td>
<td>-.00109</td>
<td>.00059</td>
<td>.00008</td>
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<tr>
<td>$\frac{1}{5}$</td>
<td>.004099</td>
<td>.000886</td>
<td>.000706</td>
<td>-.000123</td>
<td>-.000059</td>
</tr>
</tbody>
</table>
Fig. 7

Ie stands for viscoelastic solid
Iv stands for elastic solid
CHAPTER III

3. DISTURBANCES IN A SEMI-INFINITE VISCO-ELASTIC MEDIUM DUE TO TRANSIENT NORMAL FORCE ON THE PLANE BOUNDARY.

INTRODUCTION:

The problem of disturbances generated in a semi-infinite perfectly elastic medium by a concentrated force periodic in time applied on the surface, was first considered by LAMB (39). In that classical paper the solution was obtained in integral forms. The solution in the integral form for a concentrated force which is periodic in time acting normal to the free surface of a transversely isotropic medium, was obtained by CHAKRABORTTY (9). Here the problem of the disturbances due to normal transient concentrated force acting on the surface of anisotropic viscoelastic semi-infinite medium of the VOIGT type, is discussed. Neglecting higher order effect of viscosity parameters and imposing certain restrictions on the solid approximate solutions are obtained.
FORMULATION OF THE PROBLEM:

Let \((r, \theta, z)\) be coordinates of any point of the semi-infinite medium with its plane surface as the plane \(z = 0\) and \((u_r, u_\theta, u_z)\) be the displacement components at the point \((r, \theta, z)\) of the medium. In this problem \(u_r\) and \(u_z\) are independent of \(\theta\) and \(u_\theta = 0\).

Stress-strain relations for a viscoelastic medium of the VOIGT type can be written, in our problem, as:

\[
\tau_{rr} = (\lambda + \lambda' \frac{\partial}{\partial r}) \Delta + 2 \left( \mu + \mu' \frac{\partial}{\partial r} \right) \frac{u_r}{r},
\]

\[
\tau_{\theta\theta} = (\lambda + \lambda' \frac{\partial}{\partial r}) \Delta + 2 \left( \mu + \mu' \frac{\partial}{\partial r} \right) \frac{u_\theta}{r}, \quad (1)
\]

\[
\tau_{zz} = (\lambda + \lambda' \frac{\partial}{\partial r}) \Delta + 2 \left( \mu + \mu' \frac{\partial}{\partial r} \right) \frac{u_z}{r}.
\]

\[
\tau_{rz} = (\lambda + \mu' \frac{\partial}{\partial r}) \left( \frac{\partial u_r}{\partial r} + \frac{\partial u_z}{\partial z} \right), \quad \tau_{rz} = \tau_{rz} = 0.
\]

where \(\Delta = \frac{\partial^2 u_r}{\partial r^2} + \frac{u_r}{r} + \frac{\partial u_z}{\partial z}; \lambda, \mu\) are LAME'S elastic constants; \(\lambda', \mu'\) are viscosity constants.

Equations of motion, in the absence of body force, not identically satisfied, are

\[
\frac{\partial \tau_{rr}}{\partial r} + \frac{\partial \tau_{rz}}{\partial z} + \tau_{rr} - \tau_{\theta\theta} = \varphi \frac{\partial^2 u_r}{\partial t^2}. \quad (2)
\]
\[
\frac{\partial \tau_{rz}}{\partial z} + \frac{\partial \tau_{rz}}{\partial z} + \frac{1}{r} \tau_{r z} = f \frac{\partial^2 u_z}{\partial r^2} \tag{3}
\]

Expressing (2) and (3) in terms of displacement.

Components \(u_r, u_z\)

\[
\left[(\lambda + 2 \mu) + (\lambda' + 2 \mu') \frac{\partial}{\partial r}\right] \frac{\partial^2 u_r}{\partial r^2} + \frac{1}{r} \frac{\partial u_r}{\partial r} - \frac{u_r}{r} \left[ \frac{\partial^2 u_r}{\partial r^2} + \frac{2}{r} \frac{\partial u_r}{\partial r} \right] + \left[(\lambda + \mu) + (\lambda' + \mu') \frac{\partial}{\partial z}\right] \frac{\partial^2 u_z}{\partial r^2} = f \frac{\partial^2 u_r}{\partial r^2} \tag{4}
\]

\[
\left[(\lambda + 2 \mu) + (\lambda' + 2 \mu') \frac{\partial}{\partial z}\right] \frac{\partial^2 u_z}{\partial z^2} + \left[(\lambda + \mu) + (\lambda' + \mu') \frac{\partial}{\partial r}\right] \frac{\partial^2 u_z}{\partial z^2} = f \frac{\partial^2 u_z}{\partial r^2} \tag{5}
\]

**SOLUTION:**

For transient solution, let us assume

\[
\begin{align*}
U_r &= U(r) e^{-(\gamma z + \omega t)} \\
U_z &= W(r) e^{-(\gamma z + \omega t)}
\end{align*}
\tag{6}
\]
Equations (4) and (5) then assume the form

$$\left\{ (\lambda + 2\lambda') - (\lambda'+2\lambda') \omega \right\} \left\{ \frac{d^2w}{dr^2} + \frac{1}{r} \frac{dw}{dr} - \frac{u}{r^2} \right\}$$

$$- \Gamma \left\{ (\lambda + \lambda') - (\lambda'+\lambda') \omega \right\} \frac{d^2w}{dr^2} + (\mu - \mu' \omega) \gamma^2 u = S \omega^2 u$$

(7)

$$\left\{ (\lambda + 2\lambda') - (\lambda'+2\lambda') \omega \right\} \gamma^2 w - \Gamma \left\{ (\lambda + \lambda') - (\lambda'+\lambda') \omega \right\}$$

$$\cdot \left\{ \frac{dU}{dr} + \frac{U}{r} \right\} + (\mu - \mu' \omega) \left\{ \frac{d^2w}{dr^2} + \frac{1}{r} \frac{dw}{dr} \right\} = S \omega^2 W$$

(8)

Now \( U \) and \( W \) are assumed in the form

$$U(r) = AJ_1(\lambda r),$$

$$U(r) = H_\lambda(\mu r),$$

(9)

then from (7) and (8),

$$A \left\{ - (\alpha_2 - \alpha_1 \omega) \gamma^2 + (\beta_2 - \beta_1 \omega) \gamma \frac{1}{r} - \omega^2 \right\}$$

$$+ B \left\{ (\alpha_2 - \beta_2^2) - (\alpha'_2 - \beta'_2 \omega) \right\} \gamma k = 0$$

(10)
\[ A \left\{ \left( \alpha^2 - \beta^2 \right) + \left( \alpha'^2 - \beta'^2 \right) \right\} k \tau + B \left\{ \left( \alpha^2 - \alpha'^2 \right) \nu^2 - \left( \beta^2 - \beta'^2 \right) \right\} \kappa^2 - \omega^2 = 0, \tag{11} \]

where
\[ \alpha^2 = \frac{\lambda^2 + \frac{2}{3} \mu^2}{\nu^2}, \quad \alpha'^2 = \frac{\lambda'^2 + \frac{2}{3} \mu'^2}{\nu^2}, \quad \beta^2 = \frac{\lambda}{\nu^2}, \quad \beta'^2 = \frac{\lambda'}{\nu^2}. \tag{12} \]

Taking \( \gamma = k \varphi \) and \( w = \nu k \) and eliminating \( A \) and \( B \) from (10) and (11), we get the following quadratic equation in \( q_v^2 \):
\[ \left( q_1^2 - \ell_2^2 \right) \left( q_2^2 - \ell_1^2 \right) + \left( q_1^2 - \ell_2^2 \right) q_v^2 = 0 \tag{13} \]

where
\[ \ell_1^2 = \alpha^2 - \alpha'^2 \omega^2, \quad \ell_2^2 = \beta^2 - \beta'^2 \omega^2. \]

Roots of (13) are
\[ q_1^2 = 1 + \frac{\ell_1^2}{\ell_2^2}, \quad q_2^2 = 1 + \frac{\ell_2^2}{\ell_1^2}. \tag{14} \]

These will be positive and greater than one provided \( \ell_1^2 \) and \( \ell_2^2 \) are both positive.

Thus, \[ u_\tau = \left[ A_1 e^{-k q_1 \tau} + A_2 e^{-k q_2 \tau} \right] e^{-\frac{\omega \tau}{f_i(k \tau)}}. \tag{15} \]
\[
U_z = \left[ B_1 e^{-\kappa q_1 z} + B_2 e^{-\kappa q_2 z} \right] e^{-\omega t} \mathcal{J}_0(kr) \quad (16)
\]

represent the solution of (4) and (5) satisfying the condition that \( u_r \) and \( u_z \) both tend to zero as \( z \to \infty \) in infinity.

Also from (13) & (14),

\[ A_1 = m_1 \rho_1, \quad B_1 = m_2 \rho_2 \quad B_2 \; B_1 \quad \frac{\rho_2}{\rho_1}, \quad m_1 \quad \text{and} \quad m_2 \quad \text{are given by.} \]

\[
m_1 = \frac{\ell_1^2 - \ell_2^2}{\ell_1^2 - \ell_2^2 \rho_1^2 + \sigma^2}, \quad m_2 = \frac{\ell_1^2 - \ell_2^2}{\ell_1^2 - \ell_2^2 \rho_2^2 + \sigma^2}. \quad (17)
\]

Boundary conditions in the problem are

\[ \begin{align*}
\iota) \quad & \quad \mathcal{T}_{rz} \bigg|_{z=0} = 0, \\
\ii) \quad & \quad \mathcal{T}_{zz} \bigg|_{z=\delta} = -\frac{P e^{-\omega t} \delta(r)}{2\pi \delta^2},
\end{align*} \quad (18) \tag{19}
\]

where \( \frac{P e^{-\omega t} \delta(r)}{2\pi \delta^2} \) denotes the force per unit area,

\( \delta(r) \) is Dirac-delta function defined by
\( \mathcal{E}(x) \neq 0, \quad \int_{-\infty}^{\infty} \mathcal{E}(x) \, dx = 1 \),
and
\[ \frac{\mathcal{E}(r)}{2 \pi r} = \int_{0}^{\infty} J_0(\kappa r) \, d\kappa, \]
P being positive number.

**SOLUTION:**

To get the solution of the displacements \((u_x, u_z)\)
satisfying boundary conditions (18), (19), we take

\[
u_x = e^{-\omega t} \left[ m_1 B_1 q_1 e^{-k q_1 z} + m_2 B_2 q_1 e^{-k q_2 z} \right] J_1(\kappa r) \, d\kappa, \tag{21}
\]

\[
u_z = e^{-\omega t} \left[ B_1 e^{-k q_1 z} + B_2 e^{-k q_2 z} \right] J_0(\kappa r) \, d\kappa, \tag{22}
\]

where

\[
B_1 = - \frac{1 + m_2 q_2^2}{l + m_1 q_2^2} B_2, \tag{23}
\]

\[
B_2 = \frac{p}{\delta} \cdot \frac{1 + m_1 q_1^2}{q_2 \left[ l_1^2 - (l_1^2 - 2l_2^2) m_2 \right] \left[ 1 + m_1 q_1^2 \right]}
\]

\[
- q_1 \left[ l_1^2 - (l_1^2 - 2l_2^2) m_2 \right] \left[ 1 + m_2 q_2^2 \right] \tag{24}
\]
\[ \mathbf{B}_1 = \frac{-\frac{P}{\xi} (1 + m_2 q_2^2)}{\left\{ \ell_2^{\perp}((\ell_1^{\perp} - 2\ell_2^{\perp}) m_2) \right\} q_2^{(1 + m_1 q_1^2)} - \left\{ \ell_1^{\perp}((\ell_1^{\perp} - 2\ell_2^{\perp}) m_2) \right\} \cdot q_1^{(1 + m_2 q_2^2)}} \]  
\hfill (25)

\[ A_1 = \frac{\ell_1^{\perp} - \ell_2^{\perp}}{\ell_2^{\perp} - \ell_2^{\perp} q_1^{\perp} + e^2} \mathbf{B}_1 \mathbf{V}_1 \]  
\hfill (26)

\[ B_1 = \frac{\ell_1^{\perp} - \ell_2^{\perp}}{\ell_2^{\perp} - \ell_2^{\perp} q_2^{\perp} + c^2} \mathbf{B}_2 \mathbf{V}_2 \]  
\hfill (27)

**PARTicular CASE:**

The above solution is valid only if \( q_1 \neq q_2 \). If, however, \( \ell_1^{\perp} = \ell_2^{\perp} = \ell^{\perp}(A_2) \), then roots of equation (13) become equal. Let \( q \) be the common root. Here solutions can be written as

\[ e^{\omega t} \cdot \mathbf{u}_r = \frac{P}{\xi \ell^2} \int_{\alpha}^{\infty} \frac{1}{1 - q} e^{-kq} J_1(kr) dk \]  
\hfill (28)

\[ e^{\omega t} \cdot \mathbf{u}_x = \frac{P}{\xi \ell^2} \int_{\alpha}^{\infty} \frac{q}{1 - q} e^{-kq} J_0(kr) dk. \]  
\hfill (29)
where \( q^2 = 1 + \frac{\omega}{\ell^2 \kappa^2}, \quad \ell^2 = \frac{\lambda - \lambda'}{p' (\lambda' + \mu')}, \quad \omega = \frac{\lambda + \mu}{\lambda' + \mu'} \) \( (30) \)

If further restrictions be imposed on the solid viz.

i) \( \mu > \lambda \), ii) \( \lambda' = \mu' \) then

\[ \ell^2 = (\lambda - \lambda') / 2 \mu' \] \( (31) \)

\[ \omega = (\lambda + \mu) / 2 \mu', \] \( (32) \)

\[ q^2 = 1 + \frac{p(\lambda + \mu)^2}{2 (\lambda - \lambda') \mu'}, \quad \frac{1}{k^2} = 1 + \frac{x^2}{k^2}, \] \( (33) \)

where \[ x^2 = \frac{p(\lambda + \mu)^2}{2 (\lambda - \lambda') \mu'}, \] \( x^2 \)

Then displacement components \( w \) is given by

\[ e^{\omega t} \cdot U_r = - \frac{2}{\mu - \lambda} \cdot \frac{1}{x^2} \int_0^\infty k (k + \sqrt{k^2 + x^2}) e^{-\sqrt{k^2 + x^2}} \frac{J_1(kr)}{J_1(k1)} \, dk, \]

\[ e^{\omega t} \cdot (\lambda - \lambda') U_r = - \frac{2p}{x^2} \left[ \int_0^{\infty} k^2 e^{-\sqrt{k^2 + x^2}} \frac{J_1(kr)}{J_0(kr)} \, dk \right. \]

\[ + \int_0^{\infty} k \sqrt{k^2 + x^2} e^{-\sqrt{k^2 + x^2}} \frac{J_0(kr)}{J_1(kr)} \, dk \] \( (35) \)

\[ = -2p \left[ \sqrt{\frac{x}{\pi}} \cdot \frac{p \sqrt{z}}{(z^2 + x^2)^{3/4}} \cdot K_{3/2} (x \sqrt{r^2 + z^2}) \right. \]

\[ + \frac{1}{x^2} \int_0^{\infty} k (\sqrt{k^2 + x^2}) e^{-\sqrt{k^2 + x^2}} \frac{J_1(kr)}{J_1(kr)} \, dk \]
Making use of the integral
\[
\frac{1}{\sqrt{\pi}} \int_0^\infty \frac{k^\frac{3}{2}}{\sqrt{k^2 + x^2}} e^{-\sqrt{k^2 + x^2} z} (kr)^\frac{1}{2} J_1(kr) \, dk = \frac{1}{2\sqrt{\pi}} \frac{x^2 z}{(r^2 + z^2)^{\frac{3}{2}}} K_\frac{3}{4} \left( \sqrt{xr^2 + z^2} \right)
\]
(Ref ERDELYI, P-31, Vol II)

Neglecting higher order of \( \mu^2 \), \( \sqrt{k^2 + x^2} \) can be written as
\[
\sqrt{k^2 + x^2} \approx x + \frac{k}{2}\frac{x^2}{z} \quad (36)
\]

the second integral of (35) can be written as
\[
-\frac{2P}{x^2} \int_0^\infty \left( k \frac{x^2}{z} + \frac{k^3}{z^2} \right) e^{-\hat{\lambda} x z - \frac{k^2 z}{2z}} J_1(kr) \, dk
\]

Making use of the following integrals

ii) \( \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{\Gamma(\frac{3}{2})}{\Gamma(n+1)} \frac{k}{\sqrt{k^2 + x^2}} e^{-\hat{\lambda} x z - \frac{k^2 z}{2z}} J_1(kr) \, dk = \frac{1}{\sqrt{\pi}} \frac{\Gamma(n+1)}{\Gamma(n+2)} I_1(xr^2) - I_1(xr^2) \quad a > 0, \quad n \text{ being a positive integer.}
\]

(Ref. ERDELYI, P-29, Vol II)

the approximate value of \( e^{\omega t}, u_\tau \), \( (\mu - \lambda) \) can be expressed in the form
\[
e^{\omega t} \cdot u_\tau (\mu - \lambda) \approx -2P \left\{ \sqrt{\frac{x}{\pi^\frac{1}{2}}} \cdot \frac{r z}{(r^2 + z^2)^{\frac{3}{2}}} \cdot k^\frac{3}{2} \left( x \sqrt{r^2 + z^2} \right) \right\}
\]
\[
+ e^{-\frac{\tau z}{\sqrt{\pi}}} \left\{ \frac{1}{2} \sqrt{\frac{\tau}{2}} \cdot \frac{\tau}{\sqrt{\pi}} e^{-\frac{\tau^2 z}{4\pi}} \left( I_0(xr^2) - I_1(xr^2) \right) \right\}
\]
\[
+ e^{-\frac{\tau^2 z}{\sqrt{\pi}}} \left\{ \frac{3\tau}{16} \sqrt{\frac{\tau}{\pi}} \cdot F_1 \left( \frac{x^2}{z^2}, \frac{3\tau}{2\pi}; -\frac{\tau^2 z}{4\pi} \right) \right\}
\]
From (29)

\[ e^{\omega k \cdot u_z \cdot \left( \frac{\mu - \lambda}{2 \mathbf{r}} \right)} = \int_0^\infty e^{-\frac{kq^2}{4}} J_0(\kappa r) \, dk + \int_0^\alpha e^{-\kappa q^2} J_0(\kappa r) \, dk \]

\[ = I_1 + I_2 \text{ (say)} \]

Neglecting higher orders of \( \frac{1}{\alpha^2} \),

\[ I_1 \approx \frac{1}{\alpha^2} \int_0^\alpha (\kappa \mathbf{u} + \kappa \mathbf{v} + \frac{k^3}{2 \alpha}) e^{-(\alpha + \frac{k_1}{2 \alpha})} \frac{z}{J_0(\kappa r)} \, dk \]

\[ \approx \frac{e^{-\alpha z}}{\alpha^2} \left[ \frac{\alpha^2}{z} e^{-\frac{\alpha z}{2}} + \sqrt{\pi} \frac{\alpha^{3/2}}{z^{3/2}} F_1 \left( \frac{3}{2} ; 1 ; \frac{-\alpha^2}{2} \right) \right] \]

\[ \approx \frac{e^{-\alpha z}}{\alpha^2} \left[ \frac{\alpha^2}{z} e^{-\frac{\alpha z}{2}} + \sqrt{\pi} \frac{\alpha^{3/2}}{z^{3/2}} F_1 \left( \frac{3}{2} ; 1 ; \frac{-\alpha^2}{2} \right) \right] \]

\[ \omega \mu a \quad F_1 \left( \alpha ; b ; \mathbf{x} \right) = \sum_{p=0}^{\infty} \frac{\Gamma(a+r) \Gamma(b)}{\Gamma(a) \Gamma(b+r)} \frac{\mathbf{x}^p}{r!} \]

\[ I_2 = \int_0^\alpha e^{-\sqrt{k^2 + \alpha^2}} \frac{z}{J_0(\kappa r)} \, dk \]

\[ \approx e^{-\alpha z} \int_0^\alpha e^{-\frac{k^1}{2 \alpha} z} J_0(\kappa r) \, dk \]

\[ \approx e^{-\alpha z} \int_0^\alpha e^{-\frac{r^2}{2 \alpha} z} \frac{\mathbf{I}_0(\frac{r^2 \alpha}{4z})}{\mathbf{I}_0(\frac{r^2 \alpha}{4z})} \]
Therefore,

\[ e^\omega t \cdot u_z \cdot \left( \frac{\mu - \lambda}{2 \pi} \right) \approx e^{-xz} \left[ \frac{e^{-r^2x^2}}{2z} + \sqrt{\frac{\pi}{2x}} \cdot F_1 \left( \frac{3}{2} ; 1 ; -\frac{r^2x^2}{2z} \right) + \sqrt{\frac{\pi \nu x}{2z}} \cdot e^{-\frac{r^2x^2}{4z}} I_0 \left( \frac{r^2x^2}{4z} \right) \right] + \frac{1}{4xz^2} \cdot F_1 \left( 2 ; 1 ; -\frac{r^2x^2}{2z} \right) \]