5

Non-Normal Composition Operators on $L^2(\mu)$

5.1 Introduction

The study of non-normal classes of composition operators initiated by Singh [112] in 1994 have achieved tremendous importance for its applications in function spaces. In this chapter our aim is to make a study of non-normal classes of composition operators on $L^2(\mu)$. This chapter is divided into three sections. In the first section, we establish an operator $T$, called skew n-normal operator if $(T^n T^*)T = T(T^* T^n)$, for all natural numbers $n$. In this section, the condition under which composition
operators and weighted composition operators become skew n-normal operators have been obtained in terms of Radon-Nikodym derivative $h_n$. We investigate some basic properties of such operators and study the relation among non-normal composition operators and the skew n-normal composition operators.

In the second section, we study class $A(n)$ composition and weighted composition operators and $*A(n)$ composition and weighted composition operators. An operator $T$ is called $A(n)$ operator if $(|T^{n+1}|^{\frac{1}{n}} - |T|^2) \geq 0$ and $*A(n)$ operator if $(|T^{n+1}|^{\frac{1}{n}} - |T^*|^2) \geq 0$, for all natural numbers $n$. In this section, the conditions under which composition operators and weighted composition operators become $A(n)$ operators and $*A(n)$ operators, have been obtained in terms of Radon-Nikodym derivative $h_n$.

In the third section, our aim is to study quasi-$A(n)$ composition and weighted composition operators, quasi-$*A(n)$ composition and weighted composition operators, $k$-quasi-$A(n)$ composition and weighted composition operators and $k$-quasi-$*A(n)$ composition and weighted composition operators on $L^2(\mu)$. In this section, we discuss the conditions under which composition operators and weighted composition operators become quasi-$A(n)$ operators, quasi-$*A(n)$ operators, $k$-quasi-$A(n)$ operators and $k$-quasi-$*A(n)$ operators in terms of Radon-Nikodym derivative $h_n$.

5.2 Skew n-Normal Composition Operators

Let $C_T$ be the composition operator and $C_T^*$ be its adjoint which is given by $C_T^*f = h.E(f) \circ T^{-1}$.

Lemma 5.2.1. [20, 53] Let $P$ be the projection on $L^2(X, A, \mu)$ onto $\overline{R(C_T)}$. Then

(i) $C_T^*C_T f = hf$ and $C_T C_T^* f = (h \circ T) P f$ for all $f \in L^2(\mu)$.

(ii) $\overline{R(C_T)} = \{ f \in L^2(\mu) : f \text{ is } T^{-1}(A) \}$ measurable.

(iii) If $f$ is $T^{-1}(A)$ measurable and $g$ and $fg$ belong to $L^2(\mu)$, then $P(fg) = fP(g)$,
5.2 Skew n-Normal Composition Operators

\( f \) need not be in \( L^2(\mu) \).

(iv) \((C^* TC)^k f = h^k f\) for all \( k \in \mathbb{N} \).

(v) \((C TC^*)^k f = (h \circ T)^k P(f)\).

(vi) \(E\) is the identity operator on \( L^2(\mu) \) if and if \( T^{-1}(A) = A \).

Theorem 5.2.2. Let \( C_T \) be a composition operator on \( L^2(\mu) \). Then the following statements are equivalent:

(i) \( C_T \) is skew n-normal operator.

(ii) \( h \circ T^n = h \circ T \).

Proof. For \( f \in L^2(\mu) \)

\[
(C^n_T C^*_T)C_T f = (C^n_T C^*_T) f \circ T \\
= C^n_T (h.E(f \circ T) \circ T^{-1}) \\
= (h.E(f \circ T) \circ T^{-1}) \circ T^n \\
= h \circ T^n.E(f \circ T^n).
\]

Also,

\[
C_T(C^n_T C^*_T) f = C_T C^n_T(f \circ T^n) \\
= C_T h.E(f \circ T^n) \circ T^{-1} \\
= h.E(f \circ T^{n-1}) \circ T \\
= h \circ T.(E(f \circ T^n)).
\]

If \( C_T \) is skew n-normal operator then

\[
(C^n_T C^*_T)C_T = C_T(C^n_T C^*_T) \iff h \circ T^n = h \circ T.
\]

□
**Corollary 5.2.3.** Let $C_T$ be a composition operator on $L^2(\mu)$. Then the following statements are equivalent:

(i) $C_T$ is skew $n$-normal operator.

(ii) $\|\sqrt{h \circ T^n}\| = \|\sqrt{h \circ T}\|$.

**Corollary 5.2.4.** An operator $C_T$ is skew $n$-normal operator if and only if $C_T^*$ is skew $n$-normal operator.

**Proof.** Let $C_T$ be a skew $n$-normal operator therefore $(C_T^n C_T^*) C_T = C_T (C_T^* C_T^n)$. Taking adjoint on both side, we have

$$C_T^* (C_T C_T^n) = (C_T^n C_T) C_T^*$$

Therefore $C_T^*$ is skew $n$-normal operator. \qed

The following theorem, we explain the condition under which the adjoint of $C_T$ is skew $n$-normal operator.

**Theorem 5.2.5.** An operator $C_T^* \in L^2(\mu)$ is skew $n$-normal operator if and only if

$$h_n h = h_n h \circ T^{-n+1}$$

**Proof.** Suppose $C_T^*$ is skew $n$-normal operator. Since

$$C_T^* (C_T C_T^n) = (C_T^n C_T) C_T^*,$$

we have

$$C_T^* (C_T C_T^n) f = C_T^* C_T (h_n E(f) \circ T^{-n} )$$

$$= C_T^* (h_n E(f) \circ T^{-n} \circ T)$$

$$= h E(h_n \circ T E(f) \circ T^{1-n} ) \circ T^{-1}$$

$$= h E(h_n, E(f) \circ T^{-n} )$$

$$= h h_n f \circ T^{-n}.$$
Also,
\[
(C_T^n C_T^*) f = C_T^n C_T (h \circ T^{-1})
\]
\[
= C_T^n (h \circ T.E(f) \circ T^{-1} \circ T)
\]
\[
= h_n (h \circ T.E(f) \circ T^{-n})
\]
\[
= h_n h \circ T^{-n+1} f \circ T^{-n}.
\]

If $C_T^*$ is skew n-normal operator then
\[
C_T^* (C_T C_T^n) = (C_T^n C_T^*) C_T^* \iff h_n = h_n h \circ T^{-n+1}.
\]

**Theorem 5.2.6.** If $C_T$ is skew n-normal operator on $L^2(\mu)$. Then $\alpha C_T$ is skew n-normal operator for every complex number $\alpha$.

**Proof.** Consider
\[
((\alpha C_T)^n(\alpha C_T)^*)(\alpha C_T) = \alpha^n \sigma \alpha (C_T^n C_T^*) C_T.
\]
\[
= \alpha^n \sigma \alpha C_T (C_T^* C_T^n)
\]
\[
= (\alpha C_T)((\alpha C_T)^*(\alpha C_T)^n)
\]
so that $\alpha C_T$ is skew n-normal operator. \qed

**Theorem 5.2.7.** Let $C_T$ be the skew n-normal composition operator on a Hilbert space $L^2(\mu)$. If $(C_T)^{-1} = C_{T^{-1}}$ then $C_{T^{-1}}$ is skew n-normal composition operator.

**Proof.** Since $C_T$ is skew n-normal composition operator, therefore
\[
C_T (C_T^* C_T^n) = (C_T^n C_T^*) C_T.
\]
Taking inverse on both sides, we get
\[
C_T^{-1} C_{T^{-1}} C_{T^{-1}} = C_{T^{-1}} C_{T^{-1}} C_{T^{-1}}.
\]
Hence $C_{T^{-1}}$ is skew $n$-normal composition operator.

A composition operator $C_S$ is unitarily equivalent to $C_T$ if $C_S = UC_TU^*$, where $U$ is some unitary operator.

**Theorem 5.2.8.** If $C_T$ is skew $n$-normal operator on a Hilbert space $\mathcal{H}$ and $C_S$ is unitarily equivalent to $C_T$, then $C_S$ is skew $n$-normal operator.

**Proof.** Since $C_S$ is unitarily equivalent to $C_T$, then there exists an unitary operator $U$ such that $C_S = UC_TU^*$.

Now

$$(C^n_S C^*_S)C_s = (UC^n_T U^* UC^*_T U^*)UC_T U^* = UC^n_T C_T U^* = U C_T (C^*_T C^n_T) U^*$$

because $C_T$ is skew $n$-normal operator. On the other hand

$$C_S (C^*_S C^n_S) = UC_T U^* (UC^*_T U^* UC^n_T U^*) = UC_T (C^*_T C^n_T) U^*.$$ 

This implies that $(C^n_S C^*_S)C_s = C_S (C^*_S C^n_S)$. Hence $C_S$ is skew $n$-normal operator.

**Theorem 5.2.9.** Let $C_S$ be normal operator and $C_T$ be skew $n$-normal operator. If $C_S$ and $C_T$ commute, then $C_S C_T$ is skew $n$-normal operator.

**Proof.** Consider

$$((C_SC_T)^n (C_SC_T)^*)C_S C_T = (C^n_S C^*_T C^n_T C^*_S) C_S C_T.$$ 

Since $C_S$ is normal operator which commutes with $C_T$, then by Fuglede-Putnam.
Theorem, $C_T$ commutes with $C_S^*$ therefore

$$
(C_S^n C_T^n C_S^*) C_S C_T = C_S^n (C_T^n C_S^*) C_T C_S^* C_S
$$
$$= C_S^n C_T (C_S^* C_T^n) C_S^* C_S
$$
$$= C_S T C_S^* C_T^{n-1} C_T C_S^* C_S
$$
$$= C_S T (C_S^* C_S C_T^{n-1} C_T C_S)
$$
$$= C_S T (C_S^* C_S C_T^{n-1} C_T C_S)
$$
$$= C_S T ((C_S T)^n C_S (C_S T))^n).
$$

Thus $C_S C_T$ is skew n-normal operator.

\[ \square \]

**Theorem 5.2.10.** Let $C_T$ be the skew n-normal operator, then $C_T$ is skew $n + k(n - 1)$-normal operator, for every natural number $k$.

**Proof.** We prove this result by using the method of induction for every natural number $k$.

(Base case): when $k = 1$

$$
(C_T^{n+(n-1)} C_S^*) C_T = C_T^{n-1} (C_T^n C_S^*) C_T
$$
$$= C_T^{n-1} C_T (C_S^* C_T^n)
$$
$$= (C_T^n C_S^*) C_T C_T^{n-1}
$$
$$= C_T (C_T^n C_S^*) C_T^{n-1}
$$
$$= C_T (C_T^n C_T^{n+(n-1)}).
$$

(Inductive step): Suppose the result is true for $n = k$.

To prove the result for $n = k + 1$
\[(C_T^{n+(k+1)(n-1)}C_T^*)C_T = C_T^{n-1}(C_T^{n+k(n-1)}C_T^*)C_T\]
\[= C_T^{n-1}C_T(C_T^*C_T^{n+k(n-1)})\]
\[= (C_T^*C_T)C_T^{n+k(n-1)-1}\]
\[= C_T(C_T^*C_T)C_T^{(n-1)(k+1)}\]
\[= C_T(C_T^*C_T^{n+(k+1)(n-1)}).\]

Therefore \(C_T\) is skew \(n + k(n - 1)\)-normal operator.

\(\square\)

**Theorem 5.2.11.** Every \(n\)-normal composition operator is skew \(n\)-normal composition operator.

**Proof.** Let \(C_T\) be \(n\)-normal operator therefore \((C_T^*C_T) = (C_T^*C_T)\) and since every \(n\)-normal operator is quasi \(n\)-normal operator, therefore \(C_T(C_T^*C_T) = (C_T^*C_T)C_T\). Then

\[C_T(C_T^*C_T) = (C_T^*C_T)C_T = (C_T^*C_T)C_T.\]

Thus \(C_T\) is skew \(n\)-normal operator.

\(\square\)

**Theorem 5.2.12.** Every quasi normal composition operator is skew \(n\)-normal composition operator.

**Proof.** Suppose that \(C_T\) is quasi normal composition operator. Then

\[C_T(C_T^*C_T) = (C_T^*C_T)C_T\]

Now

\[C_T^2(C_T^*C_T) = C_T[(C_T^*C_T)C_T]\]
\[= C_T(C_T^*C_T)C_T\]
\[= (C_T^*C_T)C_T^2.\]
Similarly $C_{n}^{*}C_{T}^{*}C_{T}$ commutes with $C_{T}^{*}C_{T}$ for every $n$, so that

$$(C_{n}^{*}C_{T}^{*})C_{T} = C_{T}C_{n}^{*}C_{T}^{*} = C_{T}(C_{n}^{*}C_{T})C_{T}^{*} = C_{T}C_{T}^{*}C_{n}^{*}. $$

Thus $C_{T}$ is skew $n$-normal composition operator.

5.2.1 Skew n-Normal Weighted Composition Operators

Let $(X, A, \mu)$ be a $\sigma$-finite measure space and $W = W_{u,T}$ be the weighted composition operator on $L^{2}(\mu)$ induced by the complex valued function $u$ and a measurable transformation $T$. The adjoint $W^{*}$ of $W$ is given by $W^{*}f = hE(u)f \circ T^{-1}$ for $f \in L^{2}(\mu)$. We put $u_{n} = u.(u \circ T)(u \circ T^{2}).....(u \circ T^{n-1})$, where $n$ be any natural number. For $f \in L^{2}(\mu), W^{n}f = u_{n}f \circ T^{n}$ and $W^{*n}f = h_{n}.E(u_{n}.f) \circ T^{-n}$.

**Theorem 5.2.13.** Let $W$ be a weighted composition operator on $L^{2}(\mu)$. Then the following statements are equivalent:

(i) $W$ is skew $n$-normal operator.

(ii) $u_{n}.h \circ T^{n}E(u^{2}) = u.h \circ TE(uu_{n})$.

**Proof.** For $f \in L^{2}(\mu)$

$$(W^{n}W^{*})Wf = (W^{n}W^{*})(u.f \circ T)$$

$= W^{n}(h.E(u^{2}.f \circ T) \circ T^{-1})$

$= u_{n}(h.E(u^{2}f) \circ T^{n})$

$= u_{n}h \circ T^{n}E(u^{2})f \circ T^{n}.$

Also,

$$W(W^{*}W^{n})f = WW^{*}(u_{n} \circ T^{n})$$

$= W(hE(uu_{n}.f \circ T^{n}) \circ T^{-1})$

$= u(h.E(uu_{n})f \circ T^{n-1}) \circ T$

$= uh \circ TE(uu_{n})f \circ T^{n}.$
Suppose that $W$ is a skew $n$-normal operator then

$$(W^nW^*)Wf = W(W^*W^n)f \iff u_n.h \circ T^n E(u^2) = u.h \circ TE(uu_n).$$

\[ \Box \]

**Theorem 5.2.14.** Let $W$ be a weighted composition operator on $L^2(\mu)$. Then the following statements are equivalent:

(i) $W^*$ is skew $n$-normal operator.

(ii) $hE(u^2h_nE(u_nf)) = h_n E(u_nuh \circ T^{-n+1}E(uf))$.

**Proof.**

\[
W^*(WW^*)f = W^*W(h_nE(u_nf) \circ T^{-n}) \\
= W^*u(h_nE(u_nf) \circ T^{-n}) \circ T \\
= hE(u.uh_n \circ TE(u_nf) \circ T^{-n+1}) \circ T^{-1} \\
= hE(u^2h_nE(u_nf)) \circ T^{-n}.
\]

Also,

\[
(W^nW)W^*f = (W^nW)(hE(uf) \circ T^{-1}) \\
= W^n(u.h \circ T)E(uf) \\
= h_n E(u_nu.h \circ TE(uf)) \circ T^{-n} \\
= h_n E(u_nuh \circ T^{-n+1}E(uf) \circ T^{-n}).
\]

If $W^*$ is a skew $n$-normal operator. Then

$W^*(WW^*) = (W^nW)W^* \iff hE(u^2h_nE(u_nf)) = h_n E(u_nuh \circ T^{-n+1}E(uf))$.

\[ \Box \]
5.3 Class $A(n)$ Composition Operators

Let $C_T$ be the composition operator and $C^*_T$ be its adjoint which is given by $C_T f = h.E(f) \circ T^{-1}$. Further, $C^*_T C_T = M_h$ and $C^*_T C_T^2 = M_{h_k}$. In general $C^*_T C^k_T = M_{h_k}$, where $M_{h_k}$ is the multiplication operators on $L^2(\mu)$ induced by complex-valued measurable function $h_k$.

**Theorem 5.3.1.** An operator $C_T \in \mathcal{B}L^2(\mu)$ is class $A(n)$ operator if and only if $h_{n+1} \geq h^{n+1}$.

**Proof.** Let $C_T$ be class $A(n)$ operator therefore $(|C_T^{n+1}|^{\frac{2}{n+1}} - |C_T|^2) \geq 0$ trivially $C^*_T C_T^{n+1} - (C^*_T C_T)^{n+1} \geq 0$. Thus

$$\langle (C^*_T C_T^{n+1} - (C^*_T C_T)^{n+1})x, x \rangle \geq 0 \text{ for all } x \in L^2(\mu)$$

if and only if $\langle (C^*_T C_T^{n+1} - (C^*_T C_T)^{n+1})\chi_E, \chi_E \rangle \geq 0$ for every characteristic function $\chi_E$ in $A$ such that $\mu(E) < \infty$.

Since $C^*_T C_T^{n+1} = M_{h_{n+1}}$ and $(C^*_T C_T)^{n+1} = h^{n+1}$. Therefore,

$$\langle (M_{h_{n+1}} - h^{n+1})\chi_E, \chi_E \rangle \geq 0 \quad \iff \quad \int_E (M_{h_{n+1}} - h^{n+1})\chi_E d\mu \geq 0$$

$$\iff \quad \int_E (h_{n+1} - h^{n+1})d\mu \geq 0$$

$$\iff \quad h_{n+1} - h^{n+1} \geq 0$$

$$\iff \quad h_{n+1} \geq h^{n+1}.$$ 

Hence $C_T$ is class $A(n)$ operators if and only if $h_{n+1} \geq h^{n+1}$.

**Theorem 5.3.2.** An operator $C_T \in \mathcal{B}L^2(\mu)$ is class $A(n)$ operator if and only if $h_{n+1} \circ T^{n+1}P_{n+1} \geq (h \circ T)^{n+1}P$.

**Proof.** Let $C_T$ be class $A(n)$ operator therefore

$$\langle (C_T^{n+1} C_T^{n+1} - (C_T C_T)^{n+1})\chi_E, \chi_E \rangle \geq 0$$
for every characteristic function $\chi_E$ in $\mathcal{A}$ such that $\mu(E) < \infty$. Since $C_T^{n+1}C_T^{*n+1} = M_{h_{n+1}} \circ T^{n+1}P_{n+1}$ and $(C_TC_T^*)^{n+1} = M_{(h\circ T)^{n+1}}P_n$. Therefore,

\[
\langle (M_{h_{n+1}} \circ T^{n+1}P_{n+1} - M_{(h\circ T)^{n+1}P_n})\chi_E, \chi_E \rangle \geq 0
\]

\[
\iff \int_E (M_{h_{n+1}} \circ T^{n+1}P_{n+1} - M_{(h\circ T)^{n+1}P_n})\chi_E d\mu \geq 0
\]

\[
\iff \int_E (h_{n+1} \circ T^{n+1}P_{n+1} - (h \circ T)^{n+1}P_n)\chi_E d\mu \geq 0
\]

\[
\iff h_{n+1} \circ T^{n+1}P_{n+1} \geq (h \circ T)^{n+1}P_n.
\]

Hence $C_T^*$ is class $A(n)$ operators if and only if $h_{n+1} \circ T^{n+1}P_{n+1} \geq (h \circ T)^{n+1}P_n$.  

**Definition 5.3.3.** Let $(X, \mathcal{A}, \mu)$ be a $\sigma$-finite measure space and $Wf = u(f \circ T)$ be the weighted composition operator on $L^2(\mu)$ induced by the complex valued function $u$ and a measurable transformation $T$. The adjoint $W^*$ of $W$ is given by $W^*f = hE(uf) \circ T^{-1}$ for $f \in L^2(\mu)$. We put $u_n = u.(u \circ T).(u \circ T^2) .......(u \circ T^{n-1})$, where $n$ be any natural number. For $f \in L^2(\mu), W^n f = u_n(f \circ T^n)$ and $W^*n f = h_n.E(u_n.f) \circ T^{-n}$.

**Proposition 5.3.4.** [20] For $u \geq 0$

(i) $W^*Wf = h[E(u^2)] \circ T^{-1}f$

(ii) $WW^*f = u(h \circ T)E(uf)$

In general $W^*kW^k f = h_k[E(u_k^2)] \circ T^{-k}f$ and $W^kW^*k f = u_k(h_k \circ T_k)E(u_kf)$ for every $f \in L^2(\mu)$.

**Theorem 5.3.5.** An operator $W$ is a weighted composition class $A(n)$ operator if and only if $h_{n+1}E(u_{n+1}^2) \circ T^{-(n+1)} \geq (h[E(u^2)] \circ T^{-1})^{n+1}$.

**Proof.** $W$ be a weighted composition class $A(n)$ operator, therefore

\[
W^*n+1W^{n+1} - (W^*W)^{n+1} \geq 0
\]
\[ \Leftrightarrow \int_E (h_{n+1}E(u_{n+1}^2) \circ T^{-(n+1)} - (h[E(u^2)] \circ T^{-1})^{n+1}) \mu \geq 0 \quad \text{a.e.} \]

for every \( \mu > 0 \) and for every \( E \in A \) with \( \mu(E) < \infty \) i.e.

\[ \Leftrightarrow h_{n+1}E(u_{n+1}^2) \circ T^{-(n+1)} - (h[E(u^2)] \circ T^{-1})^{n+1} \geq 0 \quad \text{a.e.} \]

\[ \Leftrightarrow h_{n+1}E(u_{n+1}^2) \circ T^{-(n+1)} \geq (h[E(u^2)] \circ T^{-1})^{n+1}. \]

Hence \( W \) is weighted composition class \( A(n) \) operator if and only if \( h_{n+1}E(u_{n+1}^2) \circ T^{-(n+1)} \geq (h[E(u^2)] \circ T^{-1})^{n+1}. \)

**Corollary 5.3.6.** Let \( T^{-1}A = A \). Then \( W \) is weighted composition class \( A(n) \) operator if and only if \( h_{n+1}(u_{n+1}^2) \circ T^{-(n+1)} \geq (h(u^2) \circ T^{-1})^{n+1}. \)

**Theorem 5.3.7.** An operator \( W \) is a weighted composition operator then \( W^* \) is a weighted composition class \( A(n) \) operator if and only if \( u_{n+1}(h_{n+1} \circ T^{n+1})E(u_{n+1}) \geq [u(h \circ T)E(u)]^{n+1}. \)

**Proof.** \( W^* \) is weighted composition class \( A(n) \) operator therefore

\[ W^{n+1}W^{*n+1} - (WW^*)^{n+1} \geq 0. \]

Thus \( \langle (W^{n+1}W^{*n+1} - (WW^*)^{n+1}) x, x \rangle \geq 0 \) for all \( x \in L^2(\mu) \) if and only if \( \langle W^{n+1}W^{*n+1} - (WW^*)^{n+1} \rangle \chi_E, \chi_E \rangle \geq 0 \), for every \( \chi_E \in L^2(\mu) \) and for every \( \mu > 0 \).

\[ \Leftrightarrow \int_E (u_{n+1}(h_{n+1} \circ T^{n+1})E(u_{n+1}) - (u(h \circ T)E(u))^{n+1}) \chi_E d\mu \geq 0 \quad \text{a.e.} \]

\[ \Leftrightarrow \int_E (u_{n+1}(h_{n+1} \circ T^{n+1})E(u_{n+1}) - (u(h \circ T)E(u))^{n+1}) d\mu \geq 0 \quad \text{a.e.} \]
for every \( \mu > 0 \) and for every \( E \in A \) with \( \mu(E) < \infty \) i.e.

\[
\Leftrightarrow u_{n+1}(h_{n+1} \circ T^{n+1})E(u_{n+1}) - (u(h \circ T)E(u))^{n+1} \geq 0 \quad \text{a.e.}
\]

\[
\Leftrightarrow u_{n+1}(h_{n+1} \circ T^{n+1})E(u_{n+1}) \geq (u(h \circ T)E(u))^{n+1}.
\]

Hence \( W^* \) is weighted composition class \( A(n) \) operator if and only if \( u_{n+1}(h_{n+1} \circ T^{n+1})E(u_{n+1}) \geq [u(h \circ T)E(u)]^{n+1} \).

**Corollary 5.3.8.** Let \( T^{-1}A = A \). Then \( W^* \) is weighted composition class \( A(n) \) operator if and only if \( u_{n+1}(h_{n+1} \circ T^{n+1})(u_{n+1}) \geq [u(h \circ T)(u)]^{n+1} \).

### 5.4 Class \( * - A(n) \) Composition Operators

In this section, we obtain necessary and sufficient conditions for an operator to be \( * - A(n) \) composition operator and \( * - A(n) \) weighted composition operator.

**Theorem 5.4.1.** An operators \( C_T \in BL^2(\mu) \) is class \( * - A(n) \) operators if and only if \( h_{n+1} \geq (h \circ T)^{n+1}P \).

**Proof.** Let \( C_T \) be class \( * - A(n) \) composition operator therefore

\[
(|C^{m+1}_T|^{n+1} - |C^*_T|^2) \geq 0 \quad \text{trivially} \quad C^{m+1}_T C^m_T + (C_T C^*_T)^{n+1} \geq 0.
\]

Thus

\[
\langle (C^{m+1}_T C^m_T - (C_T C^*_T)^{n+1})x, x \rangle \geq 0 \quad \text{for all} \quad x \in L^2(\mu)
\]

if and only if \( \langle (C^{m+1}_T C^m_T - (C_T C^*_T)^{n+1})\chi_E, \chi_E \rangle \geq 0 \) for every characteristic function \( \chi_E \) in \( A \) such that \( \mu(E) < \infty \). Since \( C^{m+1}_T C^m_T = M_{h_{n+1}} \) and \( (C_T C^*_T)^{n+1} = \)
Therefore,

\[
\langle (M_{h_{n+1}} - M_{(h_0)_{n+1}}^p)\chi_E, \chi_E \rangle \geq 0
\]
\[
\Leftrightarrow \int_E (M_{h_{n+1}} - M_{(h_0)_{n+1}}^p)\chi_E d\mu \geq 0
\]
\[
\Leftrightarrow \int_E (h_{n+1} - (h \circ T)^{n+1} P) d\mu \geq 0
\]
\[
\Leftrightarrow h_{n+1} - (h \circ T)^{n+1} P \geq 0
\]
\[
\Leftrightarrow h_{n+1} \geq (h \circ T)^{n+1} P.
\]

Hence \( C_T \) is class \( * - A(n) \) composition operators if and only if 
\( h_{n+1} \geq (h \circ T)^{n+1} P \).

\[\square\]

**Theorem 5.4.2.** An operators \( C^*_T \in BL^2(\mu) \) is class \( * - A(n) \) operators if and only if 
\( h_{n+1} \circ T^{n+1} P_{n+1} \geq h^{n+1} \).

**Proof.** Let \( C^*_T \) be class \( * - A(n) \) operator therefore

\[
\langle (C^{n+1}_T C^{n+1}_T - (C^*_T C^*_T)^{n+1})\chi_E, \chi_E \rangle \geq 0
\]

for every characteristic function \( \chi_E \) in \( A \) such that \( \mu(E) < \infty \). Since \( C^{n+1}_T C^{n+1}_T = M_{h_{n+1}} \circ T^{n+1} P_{n+1} \) and \( (C^*_T C^*_T)^{n+1} = M_{h^{n+1}} \). Therefore,

\[
\langle (M_{h_{n+1}} \circ T^{n+1} P_{n+1} - M_{(h)^{n+1}})\chi_E, \chi_E \rangle \geq 0
\]
\[
\Leftrightarrow \int_E (M_{h_{n+1}} \circ T^{n+1} P_{n+1} - M_{(h)^{n+1}})\chi_E d\mu \geq 0
\]
\[
\Leftrightarrow \int_E (h_{n+1} \circ T^{n+1} P_{n+1} - h^{n+1}) d\mu \geq 0
\]
\[
\Leftrightarrow h_{n+1} \circ T^{n+1} P_{n+1} - h^{n+1} \geq 0
\]
\[
\Leftrightarrow h_{n+1} \circ T^{n+1} P_{n+1} \geq h^{n+1}.
\]

Hence \( C^*_T \) is class \( * - A(n) \) composition operators if and only if 
\( h_{n+1} \circ T^{n+1} P_{n+1} \geq h^{n+1} \).

\[\square\]
Now we obtain conditions from an operators to be \( \star - A(n) \) weighted composition operators and adjoint of Class \( \star - A(n) \) weighted composition operators using the terms of Radon-Nikodym derivative \( h_n \).

**Theorem 5.4.3.** Let \( W \in BL^2(\mu) \) be a weighted composition class \( \star - A(n) \) operator if and only if

\[
h_{n+1}E(u_{n+1}^2) \circ T^{-(n+1)} \geq (u(h \circ T)E(u))^{n+1}.
\]

**Proof.** \( W \) is a weighted composition class \( \star - A(n) \) operator therefore

\[
W^{*n+1}W^{n+1} - (WW^*)^{n+1} \geq 0
\]

\[
\Leftrightarrow \int_E (h_{n+1}E(u_{n+1}^2) \circ T^{-(n+1)} - ((u(h \circ T)E(u))^{n+1})d\mu \geq 0 \quad a.e.
\]

for every \( \mu > 0 \) and for every \( E \in \mathcal{A} \) with \( \mu(E) < \infty \) i.e.

\[
\Leftrightarrow h_{n+1}E(u_{n+1}^2) \circ T^{-(n+1)} - (u(h \circ T)E(u))^{n+1} \geq 0 \quad a.e.
\]

\[
\Leftrightarrow h_{n+1}E(u_{n+1}^2) \circ T^{-(n+1)} \geq (u(h \circ T)E(u))^{n+1}.
\]

Hence \( W \) is weighted composition class \( \star - A(n) \) operator if and only if \( h_{n+1}E(u_{n+1}^2) \circ T^{-(n+1)} \geq (u(h \circ T)E(u))^{n+1} \).

**Corollary 5.4.4.** Let \( T^{-1}\mathcal{A} = \mathcal{A} \). Then \( W \) is weighted composition class \( \star - A(n) \) operator if and only if \( h_{n+1}(u_{n+1}^2) \circ T^{-(n+1)} \geq (u(h \circ T)u)^{n+1} \).

**Theorem 5.4.5.** If \( W^* \in BL^2(\mu) \) be an adjoint weighted composition class \( \star - A(n) \) operator if and only if

\[
u_{n+1}(h_{n+1} \circ T^{n+1})E(u_{n+1}) \geq (h[E(u)^2] \circ T^{-1})^{n+1}.
\]

**Proof.** \( W^* \) is weighted composition class \( \star - A(n) \) operator therefore

\[
W^{n+1}W^{*n+1} - (W^*W)^{n+1} \geq 0.
\]
Thus \((W^{n+1}W^{*n+1} - (W^{*}W)^{n+1})x, x\) \(\geq 0\) for all \(x \in L^2(\mu)\) if and only if \((W^{n+1}W^{*n+1} - (W^{*}W)^{n+1})\chi_E, \chi_E\) \(\geq 0\), for every \(\chi_E \in L^2\mu\) and for every \(\mu > 0\).

\[
\Leftrightarrow \int_E (h_{n+1}^{T+1}E(u_{n+1}) - (h[E(u)^2] \circ T^{-1})^{n+1})d\mu \geq 0 \quad \text{a.e.}
\]

\[
\Leftrightarrow \int_E (u_{n+1}(h_{n+1} T^{n+1}E(u_{n+1}) - (h[E(u)^2] \circ T^{-1})^{n+1})d\mu \geq 0 \quad \text{a.e.}
\]

for every \(\mu > 0\) and for every \(E \in \mathcal{A}\) with \(\mu(E) < \infty\).

\[
\Leftrightarrow u_{n+1}(h_{n+1} T^{n+1}E(u_{n+1}) - (h[E(u)^2] \circ T^{-1})^{n+1} \geq 0 \quad \text{a.e.}
\]

Hence \(W^*\) is weighted composition class \(* - A(n)\) operator if and only if \(u_{n+1}(h_{n+1} T^{n+1}E(u_{n+1}) \geq (h[E(u)^2] \circ T^{-1})^{n+1}\). \(\square\)

**Corollary 5.4.6.** Let \(T^{-1}A = A\). Then \(W^*\) is weighted composition class \(* - A(n)\) operator if and only if \(u_{n+1}(h_{n+1} T^{n+1})(u_{n+1}) \geq (h(u)^2 \circ T^{-1})^{n+1}\).

### 5.5 Quasi-\(A(n)\) Composition Operators

Let \(C_T\) be the composition operator and \(C_T^*\) be its adjoint which is given by \(C_T^*f = h.E(f) \circ T^{-1}\).

**Theorem 5.5.1.** If \(C_T\) be the composition operators induced by \(T\) on \(L^2(\mu)\). Then the following statements are equivalent:

(i) \(C_T\) is quasi-\(A(n)\).

(ii) \(h_{n+2} \geq h^{n+2}\).

**Proof.** For every \(f \in L^2(\mu)\), directly follows from Lemma 5.2.1

\[
C_T^{n+2}C_T^{n+2}f = h_{n+2}f.
\]
Also,

\[ C_T^*(C_T^*C_T)^{n+1}C_T f = C_T^*(C_T^*C_T)^{n+1}(f \circ T) \]
\[ = C_T^*(h^{n+1} f \circ T) \]
\[ = h E(h^{n+1} f \circ T) \circ T^{-1} \]
\[ = h h^{n+1} E(f) \]
\[ = h^{n+2} f. \]

If \( C_T \) is quasi-\( A(n) \), then

\[ C_T^{n+2} C_T^{n+2} f \geq C_T^*(C_T^*C_T)^{n+1}C_T f \]
\[ \iff h_{n+2} f \geq h^{n+2} f \]
\[ \iff h_{n+2} \geq h^{n+2}. \]

\[ \square \]

**Theorem 5.5.2.** If \( C_T \) be the composition operators induced by \( T \) on \( L^2(\mu) \). Then the following statements are equivalent:

(i) \( C_T^* \) is quasi-\( A(n) \).

(ii) \( h_{n+2} \circ T^{n+2} \geq (h \circ T)^{n+2} E(h) \circ T \).

**Proof.** For every \( f \in L^2(\mu) \),

\[ C_T^{n+2} C_T^{n+2} f = C_T^{n+2}(h_{n+2} E(f) \circ T^{-(n+2)}) \]
\[ = (h_{n+2} E(f) \circ T^{-(n+2)}) \circ T^{n+2} \]
\[ = h_{n+2} \circ T^{n+2} E(f) \]
\[ = h_{n+2} \circ T^{n+2} f. \]
Also,

\[ C_T(C_TC_T^*)^{n+1}C_T^*f = C_T(C_TC_T^*)^{n+1}(h.E(f) \circ T^{-1}) \]

\[ = C_T[(h \circ T)^{n+1}.E(h.E(f) \circ T^{-1})] \]

\[ = [(h \circ T)^{n+1}.E(h.E(f) \circ T^{-1})] \circ T \]

\[ = (h \circ T)^{n+1}E(h).(E(f) \circ T^{-1}) \circ T \]

\[ = (h \circ T)^{n+1}E(h) \circ T.(E(f)) \]

\[ = (h \circ T)^{n+1}E(h) \circ T f. \]

If \( C_T \) is quasi-\( A(n) \), then

\[ C_T^{n+2}C_T^{*n+2}f \geq C_T(C_TC_T^*)^{n+1}C_T^*f \]

\[ \iff h_{n+2} \circ T^{n+2}f \geq (h \circ T)^{n+1}E(h) \circ T f \]

\[ \iff h_{n+2} \circ T^{n+2} \geq (h \circ T)^{n+1}E(h) \circ T. \]

\[ \square \]

**Definition 5.5.3.** Let \((X, \mathcal{A}, \mu)\) be a \(\sigma\)-finite measure space and \(W \equiv W_{u,T} \) be the weighted composition operator on \(L^2(\mu)\) induced by the complex-valued function \(u\) and a measurable transformation \(T\).

The adjoint \(W^*\) of \(W\) is given by \(W^*f = h.E(uf) \circ T^{-1}\) for \(f \in L^2(\mu)\). We put \(u_n = u.(u \circ T),(u \circ T^2),...,(u \circ T^{n-1})\), where \(n\) be any natural number. For \(f \in L^2(\mu)\), \(W^n f = u_n(f \circ T^n)\) and \(W^*n f = h_n.E(u_n,f) \circ T^{-n}\). Define

\[ J = h.E(|u|^2) \circ T^{-1}. \]

**Proposition 5.5.4.** [20] For \(u \geq 0\)

(i) \(W^*Wf = h[E(u^2)] \circ T^{-1}f\)

(ii) \(WW^*f = u(h \circ T)E(uf)\).
In general $W^kW^nf = h_k[E(u_k^2)] \circ T^{-k}f$, $W^kW^*f = u_k(h_k \circ T^k)E(u_kf)$ and $(W^*W)^k f = h^k[E(u^2)]^k \circ T^{-1}f = J^k f$ for every $f \in L^2(\mu)$.

**Lemma 5.5.5.** [62] Let $f \in L^2(\mu)$ and $WW^*f = u(h \circ T)E(uf)$. Then for all $k \in (0, \infty)$,

$$(WW^*)^k f = u(h[k \circ T[E(u^2)]^{k-1}]E(uf)).$$

**Theorem 5.5.6.** If $W$ be a weighted composition $A(n)$ operator on $L^2(\mu)$. Then the following statements are equivalent:

(i) $W$ is quasi-$A(n)$ operator. (ii) $h_{n+2}E(u_{n+2}^2) \circ T^{-(n+2)} \geq h.E(u^2J^{n+1})$.

*Proof.* For every $f \in L^2(\mu)$ and using the properties of conditional expectation operator $E$

$$W^*W^{n+2}W^{n+2}f = W^*W^{n+2}(u_{n+2}^2f \circ T^{n+2})$$

$$= h_{n+2}E(u_{n+2}^2f \circ T^{n+2}) \circ T^{-(n+2)}$$

$$= h_{n+2}E(u_{n+2}^2f \circ T^{-(n+2)})f$$

Also,

$$W^*(W^*W)^nWf = W^*(W^*W)^{n+1}(uf \circ T)$$

$$= W^*(J^{n+1}uf \circ T)$$

$$= h.E(u^2J^{n+1}f \circ T) \circ T^{-1}$$

$$= h.E(u^2J^{n+1})f.$$ 

If $W$ is quasi-$A(n)$ operator, then

$$W^{n+2}W^{n+2}f \geq W^*(W^*W)^{n+1}Wf$$

$$\iff h_{n+2}E(u_{n+2}^2) \circ T^{-(n+2)}f \geq h.E(u^2J^{n+1})f$$

$$\iff h_{n+2}E(u_{n+2}^2) \circ T^{-(n+2)} \geq h.E(u^2J^{n+1}).$$
Corollary 5.5.7. Let $T^{-1}A = A$. Then $W$ is weighted composition quasi-$A(n)$ operator if and only if $h_{n+2}(u_{n+2}^2) \circ T^{-(n+2)} \geq h. (u^2 J^{n+1})$.

Theorem 5.5.8. If $W$ be a weighted composition operator on $L^2(\mu)$. Then the following statements are equivalent:

(i) $W^*$ is quasi-$A(n)$ operator.

(ii) $u_{n+2}(h_{n+2} \circ T^{n+2}) E(u_{n+2}) \geq u^2(h^{n+1} \circ T^2) E(u^2)^{n+1} \circ Th$.

Proof.

$$W^{n+2}W^{*n+2}f = W^{n+2}(h_{n+2}.E(u_{n+2}f) \circ T^{-(n+2)})$$

$$= u_{n+2}(h_{n+2}.E(u_{n+2}f) \circ T^{-(n+2)}) \circ T^{n+2}$$

$$= u_{n+2}(h_{n+2} \circ T^{n+2}) E(u_{n+2}f).$$

Also,

$$W(WW^*)^{n+1}W^*f = W(WW^*)^{n+1}(h.E(u)f \circ T^{-1})$$

$$= W[u(h^{n+1} \circ T)(E(u^2))^n E(uh.E(u)f \circ T^{-1})]$$

$$= u[h(h^{n+1} \circ T)(E(u^2))^n E(uh.E(u)f \circ T^{-1})] \circ T$$

$$= u^2(h^{n+1} \circ T^2) E(u^2)^{n+1} f \circ T^{-1} \circ T$$

$$= u^2(h^{n+1} \circ T^2) E(u^2)^{n+1} \circ Th f$$

If $W^*$ is quasi-$A(n)$ operator, then

$$W^{n+2}W^{*n+2}f \geq W(WW^*)^{n+1}W^*f$$

$$\Leftrightarrow u_{n+2}(h_{n+2} \circ T^{n+2})(Eu_{n+2})f \geq u^2(h^{n+1} \circ T^2) E(u^2)^{n+1} \circ Th f$$

$$\Leftrightarrow u_{n+2}(h_{n+2} \circ T^{n+2})(Eu_{n+2}) \geq u^2(h^{n+1} \circ T^2) E(u^2)^{n+1} \circ Th.$$

\[\square\]
Corollary 5.5.9. Let $T^{-1}A = A$. Then $W^*$ is weighted composition quasi-$A(n)$ operator if and only if $(u_{n+2})^2(h_{n+2} \circ T^{n+2}) \geq (u^2)^{n+2}(h^{n+1} \circ T^2) \circ Th$.

5.6 Quasi- $* - A(n)$ Composition Operators

In this section, we obtain necessary and sufficient conditions for an operator to be quasi- $* - A(n)$ composition operator and quasi- $* - A(n)$ weighted composition operator.

Theorem 5.6.1. If $C_T$ be the composition operators induced by $T$ on $L^2(\mu)$. Then the following statements are equivalent:

(i) $C_T$ is quasi- $* - A(n)$. (ii) $h_{n+2} \geq hE(h \circ T)^{n+1}$.

Proof. For every $f \in L^2(\mu)$,

$$C_T^{*n+2}C_T^{m+2}f = h_{n+2}f.$$

Also,

$$C_T^*(C_T^*)^{n+1}C_Tf = C_T^*(C_T^*)^{n+1}(f \circ T)$$

$$= C_T^*(h \circ T)^{n+1}E(f \circ T)$$

$$= hE((h \circ T)^{n+1}E(f \circ T)) \circ T^{-1}$$

$$= hE((h \circ T)^{n+1})E(f)$$

$$= hE((h \circ T)^{n+1})f.$$
If $C_T$ is quasi--$*--A(n)$, then

$$C_T^{m+2}C_T^{n+2}f \geq C_T^*(C_T^nC_T^n)^{n+1}C_Tf$$

$$\iff h_{n+2}f \geq hE((h \circ T)^{n+1})f$$

$$\iff h_{n+2} \geq hE(h \circ T)^{n+1}.$$ 

\[ \square \]

**Theorem 5.6.2.** If $C_T$ be the composition operators induced by $T$ on $L^2(\mu)$. Then the following statements are equivalent:

(i) $C_T^*$ is quasi--$*--A(n)$.

(ii) $h_{n+2} \circ T^{n+2} \geq h^{n+2} \circ T$.

**Proof.** For every $f \in L^2(\mu)$,

$$C_T^{m+2}C_T^{n+2}f = h_{n+2} \circ T^{n+2}f.$$ 

Also,

$$C_T(C_T^nC_T)^{n+1}C_T^*f = C_T(C_T^nC_T)^{n+1}(h.E(f) \circ T^{-1})$$

$$= C_T[(h^{n+1}(h.E(f) \circ T^{-1})]$$

$$= h^{n+2}(E(f) \circ T^{-1}) \circ T$$

$$= h^{n+2} \circ TE(f)$$

$$= h^{n+2} \circ Tf.$$
If $C_T^*$ is quasi-$-A(n)$, then

$$C_T^{n+2}C_T^{n+2}f \geq C_T(C_T^*C_T)^{n+1}C_T^*f$$

$$\iff h_{n+2} \circ T^{n+2}f \geq h^{n+2} \circ T f$$

$$\iff h_{n+2} \circ T^{n+2} \geq h^{n+2} \circ T.$$

Now we obtain conditions for an operator to be quasi-$-A(n)$ weighted composition operator and adjoint of quasi-$-A(n)$ weighted composition operator using the terms of Radon-Nikodym derivative $h_n$.

**Theorem 5.6.3.** If $W$ be a weighted composition operator on $L^2(\mu)$. Then the following statements are equivalent:

(i) $W$ is quasi-$-A(n)$ operator.

(ii) $h_{n+2}E(u_{n+2}^2) \circ T^{-(n+2)} \geq h^{n+2}[E(u^2)]^{n+2} \circ T^{-1}$.

**Proof.** For every $f \in L^2(\mu)$,

$$W^{*n+2}W^{n+2}f = h_{n+2}E(u_{n+2}^2) \circ T^{-(n+2)} f.$$ 

Now, by using [Theorem 3.2, 47]

$$W^*(W^*W)^{n+1}Wf = h^{n+2}[E(u^2)]^{n+2} \circ T^{-1} f.$$
If \( W \) is quasi-\( * \)\( A(n) \) operator, then

\[
W^{*n+2}W^{n+2}f \geq W^*(W^*W)^{n+1}Wf
\]

\[
\Leftrightarrow h_{n+2}E(u_{n+2}^2) \circ T^{-(n+2)}f \geq h^{n+2}[E(u^2)]^{n+2} \circ T^{-1}f
\]

\[
\Leftrightarrow h_{n+2}E(u_{n+2}^2) \circ T^{-(n+2)} \geq h^{n+2}[E(u^2)]^{n+2} \circ T^{-1}.
\]

\[\square\]

**Corollary 5.6.4.** Let \( T^{-1}A = A \). Then \( W \) is weighted composition quasi-\( * \)\( A(n) \) operator if and only if \( h_{n+2}u_{n+2}^2 \circ T^{-(n+2)} \geq h^{n+2}[u^2]^{n+2} \circ T^{-1} \).

**Theorem 5.6.5.** If \( W \) be a weighted composition operator on \( L^2(\mu) \). Then the following statements are equivalent:

(i) \( W^* \) is quasi-\( * \)\( A(n) \) operator.

(ii) \( u_{n+2}(h_{n+2} \circ T^{n+2})(E u_{n+2}) \geq u.J^{n+1}h \circ T E(u) \).

**Proof.**

\[
W^{n+2}W^{*n+2}f = u_{n+2}(h_{n+2} \circ T^{n+2})E(u_{n+2})f.
\]

Also,

\[
W(W^*W)^{n+1}W^*f = W(W^*W)^{n+1}(h.E(u)f \circ T^{-1})
\]

\[
= W[J^{n+1}h.E(u)f \circ T^{-1}]
\]

\[
= u[J^{n+1}h.E(u)f \circ T^{-1}] \circ T
\]

\[
= u.J^{n+1}h \circ T E(u)f \circ T^{-1} \circ T
\]

\[
= u.J^{n+1}h \circ T E(u)f.
\]
If \( W^* \) is quasi-\(-A(n)\) operator, then
\[
W^{n+2}W^*W^{n+2}f \geq W(W^*W)^{n+1}W^*f
\]
\[\iff u_{n+2}(h_{n+2}\circ T^{n+2})(Eu_{n+2})f \geq u.J^{n+1}h\circ TE(u)f\]
\[\iff u_{n+2}(h_{n+2}\circ T^{n+2})(Eu_{n+2}) \geq u.J^{n+1}h\circ TE(u).\]

Corollary 5.6.6. Let \( T^{-1}A = \mathcal{A} \). Then \( W^* \) is weighted composition quasi-\(-A(n)\) operator if and only if \((u_{n+2})^2(h_{n+2}\circ T^{n+2}) \geq (u)^2.J^{n+1}h\circ T.\)

### 5.7 \( k \)-Quasi-\( A(n) \) Composition Operators

In this section, we discuss conditions for an operator to be \( k \)-quasi-\( A(n) \) composition operator and \( k \)-quasi-\( A(n) \) weighted composition operator.

**Theorem 5.7.1.** If \( C_T \) be the composition operator induced by \( T \) on \( L^2(\mu) \). Then the following statements are equivalent:

(i) \( C_T \) is \( k \)-quasi-\( A(n) \).

(ii) \( h_{n+k+1} \geq h_k E(h^{n+1}) \).

**Proof.** For every \( f \in L^2(\mu) \),
\[
C_T^{n+k+1}C_T^{m+k+1}f = h_{n+k+1}f.
\]
Also,

\[ C_T^k (C_T^* C_T)^{n+1} C_T^k f = C_T^k (C_T^* C_T)^{n+1} (f \circ T^k) = C_T^k (h^{n+1} \circ T^k) = h_k E(h^{n+1} \circ T^k) \circ T^{-k} = h_k E(h^{n+1}) f = h_k E(h^{n+1}) f. \]

If \( C_T \) is \( k \)-quasi-\( A(n) \), then

\[ C_T^{n+k+1} C_T^{n+k+1} f \geq C_T^{n+k} (C_T^* C_T)^{n+1} C_T^k f \Leftrightarrow h_{n+k+1} \geq h_k E(h^{n+1}). \]

\[ \square \]

**Theorem 5.7.2.** If \( C_T \) be the composition operator induced by \( T \) on \( L^2(\mu) \). Then the following statements are equivalent:

(i) \( C_T^* \) is \( k \)-quasi-\( A(n) \).

(ii) \( h_{n+k+1} \circ T^{n+k+1} \geq (h \circ T)^{n+1} E(h_k) \circ T^k \).

**Proof.** For every \( f \in L^2(\mu) \),

\[ C_T^{n+k+1} C_T^{n+k+1} f = h_{n+k+1} \circ T^{n+k+1} f. \]
Also,

\[
C^k_T(C_T C^*_T)^{n+1} C^*_T = C^k_T(C_T C^*_T)^{n+1}(h_k. E(f) \circ T^{-k})
\]

\[
= C^k_T[(h \circ T)^{n+1} E(h_k. E(f) \circ T^{-k})]
\]

\[
= (h \circ T)^{n+1} E(h_k) \circ T^k \cdot (E(f) \circ T^{-k}) \circ T^k
\]

\[
= (h \circ T)^{n+1} E(h_k) \circ T^k \cdot (E(f))
\]

\[
= (h \circ T)^{n+1} E(h_k) \circ T^k f.
\]

If \( C_T \) is \( k \)-quasi-\( A(n) \), then

\[
C^{n+k+1}_T C_n^{n+k+1} f \geq C^k_T(C_T C^*_T)^{n+1} C^*_T f
\]

\[
\Leftrightarrow h_{n+k+1} \circ T^{n+k+1} f \geq (h \circ T)^{n+1} E(h_k) \circ T^k f
\]

\[
\Leftrightarrow h_{n+k+1} \circ T^{n+k+1} \geq (h \circ T)^{n+1} E(h_k) \circ T^k.
\]

\[\square\]

**Theorem 5.7.3.** If \( W \) be a weighted composition operator on \( L^2(\mu) \). Then the following statements are equivalent:

(i) \( W \) is \( k \)-quasi-\( A(n) \) operator.

(ii) \( h_{n+k+1} E(u_{n+k+1}^2) \circ T^{-(n+k+1)} \geq h_k. E((u_k)^2 J^{n+1}). \)

**Proof.** For every \( f \in L^2(\mu) \) and using the properties of conditional expectation operator E

\[
W^{n+k+1} W^{n+k+1} f = h_{n+k+1} E(u_{n+k+1}^2) \circ T^{-(n+k+1)} f
\]
Also,

\[ W^k(W^k W)^{n+1} W^k f = W^k(W^k W)^{n+1}(u_k f \circ T^k) \]

\[ = W^k(J^{n+1} u_k f \circ T^k) \]

\[ = h_k E((u_k)^2 J^{n+1} f \circ T^k) \circ T^{-k} \]

\[ = h_k E((u_k)^2 J^{n+1}) f \]

If \( W \) is \( k \)-quasi-\( A(n) \) operator, then

\[ W^{n+k+1} W^k W^{n+k+1} f \geq W^k(W^k W)^{n+1} W^k f \]

\[ \iff h_{n+k+1} E(u_{n+k+1}^2) \circ T^{-(n+k+1)} f \geq h_k E((u_k)^2 J^{n+1} f) \]

\[ \iff h_{n+k+1} E(u_{n+k+1}^2) \circ T^{-(n+k+1)} \geq h_k E((u_k)^2 J^{n+1}). \]

\[ \Box \]

**Corollary 5.7.4.** Let \( T^{-1} A = A \). Then \( W \) is weighted composition \( k \)-quasi-\( A(n) \) operator if and only if \( h_{n+k+1}(u_{n+k+1}^2) \circ T^{-(n+k+1)} \geq h_k ((u_k)^2 J^{n+1}). \)

**Theorem 5.7.5.** If \( W \) be a weighted composition operator on \( L^2(\mu) \). Then the following statements are equivalent:

(i) \( W^* \) is \( k \)-quasi-\( A(n) \) operator.

(ii) \( u_{n+k+1}(h_{n+k+1} \circ T^{n+k+1}) E(u_{n+k+1}) \geq u_k [u(h^{n+1} \circ T)] \circ T^k (E(u^2))^n E(uh_k).u_k. \)

**Proof.**

\[ W^{n+k+1} W^{n+k+1} f = u_{n+k+1}(h_{n+k+1} \circ T^{n+k+1}) E(u_{n+k+1}) f. \]
Also,
\[
W^k(WW^*)^{n+1}W^k f = W^k(WW^*)^{n+1}(h_k.E(u_k.f) \circ T^{-k})
\]
\[= W^k[u(h^{n+1} \circ T)(E(u^2))^n E(uh_k).E(u_k.f) \circ T^{-k})]
\[= u_k[u(h^{n+1} \circ T)(E(u^2))^n E(uh_k).(E(u_k.f) \circ T^-)] \circ T^k
\[= u_k[u(h^{n+1} \circ T)] \circ T^k(E(u^2))^n E(uh_k).u_k.f.
\]

If \(W^*\) is \(k-\text{quasi}-A(n)\) operator, then
\[
W^{n+k+1}W^{*n+k+1}f \geq W^k(WW^*)^{n+1}W^k f
\]
\[
\Leftrightarrow u_{n+k+1}(h_{n+k+1} \circ T^{n+k+1})E(u_{n+k+1})f \geq u_k[u(h^{n+1} \circ T)]
\circ T^k(E(u^2))^n E(uh_k).u_k f
\]
\[
\Leftrightarrow u_{n+k+1}(h_{n+k+1} \circ T^{n+k+1})E(u_{n+k+1}) \geq u_k[u(h^{n+1} \circ T)]
\circ T^k(E(u^2))^n E(uh_k).u_k
\]

**Corollary 5.7.6.** Let \(T^{-1}A = A\). Then \(W^*\) is weighted composition \(k-\text{quasi}-A(n)\) operator if and only if
\[
(u_{n+k+1})^2(h_{n+k+1} \circ T^{n+k+1}) \geq u_k[u(h^{n+1} \circ T)] \circ T^k u^{2n+1}h_k u_k.
\]

### 5.8 \(k-\text{Quasi}-* - A(n)\) Composition Operators

\(k-\text{Quasi}-* - A(n)\) operator a generalization of \(\text{Quasi}-* - A(n)\) operator and in this section, we explain necessary and sufficient conditions for an operator to be \(k-\text{quasi}-* - A(n)\) composition operator and \(k-\text{quasi}-* - A(n)\) weighted composition operator in terms of Radon-Nikodym derivative and expectation operators.
Theorem 5.8.1. If $C_T$ be the composition operator induced by $T$ on $L^2(\mu)$. Then the following statements are equivalent:

(i) $C_T$ is $k$–quasi–$*–A(n)$.

(ii) $h_{n+k+1} \geq h_k E((h \circ T)^{n+1}) \circ T^{-k}$.

Proof. For every $f \in L^2(\mu)$,

$$C_T^{*n+k+1}C_T^{m+k+1}f = h_{n+k+1}f.$$ 

Also,

$$C_T^{*k}(C_TC_T^n)^{n+1}C_T^k f = C_T^{*k}(C_TC_T^n)^{n+1}(f \circ T^k)$$

$$= C_T^{*k}(h \circ T)^{n+1}E(f \circ T^k)$$

$$= h_k E((h \circ T)^{n+1}E(f \circ T^k)) \circ T^{-k}$$

$$= h_k E((h \circ T)^{n+1}) \circ T^{-k} E(f)$$

$$= h_k E((h \circ T)^{n+1}) \circ T^{-k} f.$$ 

If $C_T$ is $k$–quasi–$*–A(n)$, then

$$C_T^{*n+k+1}C_T^{m+k+1}f \geq C_T^{*k}(C_TC_T^n)^{n+1}C_T^k f$$

$$\Leftrightarrow h_{n+k+1}f \geq h_k E((h \circ T)^{n+1}) \circ T^{-k} f$$

$$\Leftrightarrow h_{n+k+1} \geq h_k E((h \circ T)^{n+1}) \circ T^{-k}.$$ 

\hfill \Box

Theorem 5.8.2. If $C_T$ be the composition operators induced by $T$ on $L^2(\mu)$. Then the following statements are equivalent:
(i) $C^*_T$ is $k$-quasi-$-A(n)$.

(ii) $h_{n+k+1} \circ T^{n+k+1} \geq h^{n+1}(h_k \circ T^k)$.

**Proof.** For every $f \in L^2(\mu)$,

$$C^{n+k+1}_T C^{n+k+1}_T f = h_{n+k+1} \circ T^{n+k+1} f.$$

Also,

$$C^k_T (C^*_T C_T)^n + 1 C^*_T f = C^k_T (C^*_T C_T)^{n+1} (h_k E(f) \circ T^{-k})$$

$$= C^k_T [h^{n+1} (h_k E(f) \circ T^{-k})]$$

$$= [(h^{n+1} (h_k E(f) \circ T^{-k})) \circ T^k]$$

$$= h^{n+1} (h_k \circ T^k) E(f)$$

$$= h^{n+1} (h_k \circ T^k) f.$$

If $C^*_T$ is $k$-quasi-$-A(n)$, then

$$C^{n+k+1}_T C^{n+1} C^{n+k+1}_T f \geq C^k_T (C^*_T C_T)^{n+1} C^*_T f$$

$$\Leftrightarrow h_{n+k+1} \circ T^{n+k+1} f \geq h^{n+1} (h_k \circ T^k) f$$

$$\Leftrightarrow h_{n+k+1} \circ T^{n+k+1} \geq h^{n+1} (h_k \circ T^k).$$

\[ \square \]

**Theorem 5.8.3.** If $W$ be a weighted composition operator on $L^2(\mu)$. Then the following statements are equivalent:

(i) $W$ is $k$-quasi-$-A(n)$ operator.

(ii) $h_{n+k+1} E(u^2_{n+k+1}) \circ T^{-(n+k+1)} \geq h_k E[u_k (u h^{n+1} \circ T)] \circ T^{-k} [E(u^2)]^{n} E(u_k).$
Proof. For every $f \in L^2(\mu)$

$$W^{*n+k+1}W^{n+k+1}f = h_{n+k+1}E(u_{n+k+1}^2) \circ T^{-(n+k+1)}f.$$ 

Also,

$$W^{*k}(W^*W)^{n+1}W^kf = W^{*k}(W^*W)^{n+1}(u_k)f \circ T^k$$

$$= W^{*k}(u(h^{n+1} \circ T)[E(u^2)]^n E(u_k f \circ T^k))$$

$$= h_kE[u_k(u(h^{n+1} \circ T)[E(u^2)]^n E(u_k f \circ T^k))] \circ T^{-k}$$

$$= h_kE[u_k(u(h^{n+1} \circ T)) \circ T^{-k}[E(u^2)]^n E(u_k f).$$

If $W$ is $k-$quasi-$* - A(n)$ operator, then

$$W^{*n+k+1}W^{n+k+1}f \geq W^{*k}(W^*W)^{n+1}W^kf$$

$$\iff h_{n+k+1}E(u_{n+k+1}^2) \circ T^{-(n+k+1)}f \geq h_kE[u_k(u(h^{n+1} \circ T)]$$

$$\circ T^{-k}[E(u^2)]^n E(u_k f)$$

$$\iff h_{n+k+1}E(u_{n+k+1}^2) \circ T^{-(n+k+1)} \geq h_kE[u_k(uh^{n+1} \circ T)]$$

$$\circ T^{-k}[E(u^2)]^n E(u_k).$$

□

Corollary 5.8.4. Let $T^{-1}A = A$. Then $W$ is weighted composition $k-$quasi-$* - A(n)$ operator if and only if

$$h_{n+k+1}(u_{n+k+1}^2) \circ T^{-(n+k+1)} \geq h_k[u_k(uh^{n+1} \circ T)] \circ T^{-k}[E(u^2)]^n(u_k)$$
Theorem 5.8.5. If \( W \) be a weighted composition operator on \( L^2(\mu) \). Then the following statements are equivalent:

(i) \( W^* \) is \( k \)-quasi-*-\( A(n) \) operator. (ii) \( u_{n+k+1}(h_{n+k+1} \circ T^{n+k+1})E(u_{n+k+1}) \geq u_k J^{n+1} h_k \circ T^k E(u_k) \).

Proof.

\[
W^{n+k+1}W^* W^{n+k+1} f = u_{n+k+1}(h_{n+k+1} \circ T^{n+k+1})E(u_{n+k+1}) f.
\]

Also,

\[
W^k(W^* W)^{n+1}W^* k f = W^k(W^* W)^{n+1}(h_k E(u_k f) \circ T^{-k})
\]

\[
= W^k[J^{n+1} h_k E(u_k f) \circ T^{-k}]
\]

\[
= u_k[J^{n+1} h_k E(u_k f) \circ T^{-k}] \circ T^k
\]

\[
= u_k J^{n+1} h_k \circ T^k E(u_k f)
\]

\[
= u_k J^{n+1} h_k \circ T^k E(u_k f).
\]

If \( W^* \) is \( k \)-quasi-*-\( A(n) \) operator, then

\[
W^{n+k+1}W^* W^{n+k+1} f \geq W^k(W^* W)^{n+1}W^* k f
\]

\[
\Leftrightarrow u_{n+k+1}(h_{n+k+1} \circ T^{n+k+1})E(u_{n+k+1}) f \geq u_k J^{n+1} h_k \circ T^k E(u_k) f
\]

\[
\Leftrightarrow u_{n+k+1}(h_{n+k+1} \circ T^{n+k+1})E(u_{n+k+1}) \geq u_k J^{n+1} h_k \circ T^k E(u_k).
\]

Corollary 5.8.6. Let \( T^{-1} A = A \). Then \( W^* \) is weighted composition \( k \)-quasi-*-\( A(n) \) operator if and only if

\[
(u_{n+k+1})^2(h_{n+k+1} \circ T^{n+k+1}) \geq (u_k)^2 J^{n+1} h_k \circ T^k.
\]