3

Composition Operators on Various Function Spaces

3.1 Introduction

The main focus of this chapter is to make a study of composition operators on the weighted Lorentz-Karamata-Bochner spaces and weighted Bergman-Nevanlinna spaces to Zygmund spaces. Accordingly the chapter contains two sections. In the first section we explore the boundedness of composition operators on the weighted Lorentz-Karamata-Bochner spaces and discuss isometry of composition operators.
The second section contains the study of composition operator preceding by differentiation on weighted Bergman-Nevanlinna spaces to Zygmund spaces and prove the following main results:

Let $\alpha > -1$ and $T$ be a holomorphic self map of $\mathbb{D}$. Then the following are equivalent:

(i) $DC_T : A^\alpha_N \to Z$ is bounded.

(ii) $DC_T : A^\alpha_N \to Z$ is compact.

(iii) For all $c > 0$,

\[
\lim_{|T(z)| \to 1} (1 - |z|^2)|T'(z)|^3(1 - |T(z)|^2)^3 \exp[c(1 - |Tz|^2)^{\alpha+2}] = 0
\]

\[
\lim_{|T(z)| \to 1} (1 - |z|^2)|T'(z),T''(z)|(1 - |T(z)|^2)^2 \exp[c(1 - |Tz|^2)^{\alpha+2}] = 0
\]

\[
\sup_{z \in \mathbb{D}} (1 - |z|^2)|T'(z)|^3 < \infty
\]

\[
\sup_{z \in \mathbb{D}} (1 - |z|^2)|T'(z),T''(z)| < \infty
\]

\[
\sup_{z \in \mathbb{D}} (1 - |z|^2)|T'''(z)| < \infty.
\]

### 3.2 Composition Operators on Weighted Lorentz-Karamata-Bochner Spaces

Let $(X, A, \mu)$ be a $\sigma$-finite measure space. A measurable transformation $T : X \to X$ is called non-singular if $\mu(T^{-1}(E)) = 0$ whenever $\mu(E) = 0$. Non-singularity is essential for the mapping $C_T(f \mapsto f \circ T)$ to be well defined. In this section, our aim is to discuss the boundedness of $C_T$, the composition operators on $L_{(p,q,\omega;b)}$, where $1 < p \leq \infty, 1 \leq q \leq \infty$ and $\omega$ is a non-negative measurable functions on a $\sigma$-finite measure space $(X, A, \mu)$.

**Theorem 3.2.1.** A non-singular measurable transformation $T : X \to X$ induces a composition operator $C_T : L_{(p,q,\omega;b)} \to L_{(p,q,\omega;b)}, 1 < p \leq \infty, 1 \leq q \leq \infty$ if there exist a constant $k > 0$ such that $(\omega_\mu \circ T^{-1})(E) \leq k\omega_\mu(E)$ for all $E \in A$. 
Proof. Suppose there is a constant \( k > 0 \) such that for all \( E \in A \),

\[
(\omega_\mu \circ T^{-1})(E) \leq k \omega_\mu(E).
\]

For \( f \) in \( L_{(p,q,\omega,b)}(X,A,\mu) \), the distribution of \( f \circ T \) satisfies

\[
\mu_{(f \circ T),\omega}(s) = \omega_\mu\{x \in X : \|f \circ T\|(x) > s\}
\]

\[
= (\omega_\mu \circ T^{-1})\{x \in X : \|f\|(x) > s\}
\]

\[
\leq k \omega_\mu\{x \in X : \|f\|(x) > s\} = k \mu_{f,\omega}(s).
\]

Therefore

\[
\{s > 0 : \mu_{f,\omega}(s) \leq t\} \subseteq \{s > 0 : \mu_{(f \circ T),\omega}(s) \leq kt\}.
\]

This gives

\[
\|f \circ T\|_\omega^*(kt) \leq \|f\|_\omega^*(t)
\]

and consequently

\[
\|f \circ T\|_\omega^{**}(kt) \leq \|f\|_\omega^{**}(t),\ t > 0.
\]

Now for \( f \) in \( L_{(p,q,\omega,b)} \), \( 1 < p < \infty, 1 \leq q < \infty \),

\[
\|C_T f\|_{p,q,\omega;b}^q = \int_0^\infty \left( t^{\frac{1}{p}} \gamma_b(t) \|(f \circ T)\|^{**}_\omega(t)\right)^q \frac{dt}{t}
\]

\[
= k^{\frac{2}{p}} \int_0^\infty \left( t^{\frac{1}{p}} \gamma_b(kt) \|(f \circ T)\|^{**}_\omega(kt)\right)^q \frac{dt}{t}
\]

\[
\leq k^{\frac{2}{p}} \int_0^\infty \left( t^{\frac{1}{p}} \gamma_b(t) \|(f)\|^{**}_\omega(t)\right)^q \frac{dt}{t}
\]

\[
= k^{\frac{2}{p}} \|f\|_{p,q,\omega;b}^q.
\]

This prove that \( C_T \) is bounded on \( L_{(p,q,\omega,b)} \), \( 1 < p \leq \infty, 1 \leq q \leq \infty \). For \( q = \infty, 1 <
$p \leq \infty$, we have

$$\|C_T f\|_{p,\infty,\omega;b} = \sup_{t>0} t^{\frac{1}{p}} \gamma_b(t) \| (f \circ T) \|_{\omega}^{**}(t)$$

$$\approx \sup_{t>0} t^{\frac{1}{p}} \gamma_b(kt) \| (f \circ T) \|_{\omega}^{**}(kt)$$

$$\leq k^{\frac{1}{p}} \sup_{t>0} t^{\frac{1}{p}} \gamma_b(t) \| (f) \|_{\omega}^{**}(t) = k^{\frac{1}{p}} \| f \|_{(p,\infty,\omega;b)}.$$  

This completes the proof.  

For the converse, we define a non-negative real valued function $\hat{\omega}_{\mu}$ on $\mathcal{A}$ as

$$\hat{\omega}_{\mu}(E) = \int_{0}^{\omega_{\mu}(E)} t^{\frac{1}{p}} \gamma_b(t) dt$$  

for $E \in \mathcal{A}$. Now using the properties of $\gamma_b$ functions, we find the following.

**Theorem 3.2.2.** If a non-singular measurable transformation $T : X \to X$ induces a composition operator $C_T : L_{(p,q,\omega;b)} \to L_{(p,q,\omega;b)}$, $1 < p \leq \infty, 1 \leq q \leq \infty$ then $\left( \omega_{\mu} \circ T^{-1} \right)(E) \leq k \hat{\omega}_{\mu}(E)$ for all $E \in \mathcal{A}$, for some $k > 0$.

**Proof.** First we assume that $C_T$ is a composition operators induced by $T$ on $L_{(p,q,\omega;b)}$, $1 < p < \infty, 1 \leq q < \infty$ and let $E \in \mathcal{A}$, $\omega_{\mu}(E) < \infty$. Let $\omega_0$ be a fixed element of $\Omega$ with $\| \omega_0 \| = 1$ and consider the function $\chi_E$ given by

$$\chi_E(x) = \begin{cases} \omega_0, & \text{if } \omega \in E \\ 0, & \text{otherwise} \end{cases}.$$  

The non-increasing rearrangement of the $\chi_E$ is given by

$$\| \chi_E \|^{**}(t) = \chi_{[0,\omega_0(\omega_{\mu}(E))]}(t).$$
Thus
\[
\|\chi_E\|_{\omega}^*(t) = \frac{1}{t} \int_0^t \|\chi_E\|_{\omega}^*(s) \, ds
\]
\[
= \begin{cases} 
1, & \text{if } 0 \leq t < \omega_\mu(E) \\
\frac{1}{t} \omega_\mu(E), & \text{if } t \geq \omega_\mu(E).
\end{cases}
\]

Therefore, using the properties of gamma function, we have
\[
\|\chi_E\|_{(p,q,\omega;b)} = \left( \int_0^\infty t^{\frac{2}{p}} \gamma_b(t) \|\chi_E\|_{\omega}^*(t) \right)^{\frac{1}{q}}
\]
\[
= \left( \int_0^{\omega_\mu(E)} t^{\frac{2}{p}-1} \gamma_b(t)^q \, dt + (\omega_\mu(E))^q \int_{\omega_\mu(E)}^\infty t^{-q(1-\frac{1}{p})-1} \gamma_b(t)^q \, dt \right)^{\frac{1}{q}}
\]
\[
\approx \left( (\omega_\mu(E))^{\frac{2}{p}} (\gamma_b(\omega_\mu(E)))^q + (\omega_\mu(E))^q (\omega_\mu(E))^{-q(1-\frac{1}{p})-1} (\gamma_b(\omega_\mu(E)))^q \right)^{\frac{1}{q}}
\]
\[
= 2^{\frac{1}{q}} (\omega_\mu(E))^{\frac{1}{p}} (\gamma_b(\omega_\mu(E))) = 2^{\frac{1}{q}} \omega_\mu(T^{-1} E)
\]
\[
= 2^{\frac{1}{q}} \omega_\mu(E).
\]

Thus it follows
\[
\|C_T \chi_E\|_{(p,q,\omega;b)} = \|\chi_{T^{-1}(E)}\|_{(p,q,\omega;b)} = 2^{\frac{1}{q}} \omega_\mu(T^{-1} E).
\]

Hence
\[
2^{\frac{1}{q}} \omega_\mu(T^{-1} E) = \|C_T \chi_E\|_{(p,q,\omega;b)} \leq k \|\chi_E\|_{(p,q,\omega;b)} = k 2^{\frac{1}{q}} \omega_\mu(E)
\]
that is
\[
(\omega_\mu(T^{-1} E)) \leq k \omega_\mu(E).
\]

A measurable transformation \( T : X \to X \) is said to be measure preserving if
\( \mu T^{-1}(E) = \mu(E) \) for all \( E \in \mathcal{A} \). We introduce the following notation.

**Definition 3.2.1.** A measurable transformation \( T : X \to X \) is called \( \omega \)-measure preserving if \( \omega_\mu T^{-1}(E) = \omega_\mu(E) \) for all \( E \in \mathcal{A} \).

With the following examples we show that there is no relation between the above two notations. We first verify that a \( \omega \)-measure preserving measurable transformation need not be measure preserving.

**Example 3.2.2.** Let \( X = [0, \infty) \) and \( \mu \) be the Lebesgue measure. Consider a measurable transformation \( T : [0, \infty) \to [0, \infty) \) given by \( T(x) = 2x \) for all \( x \in [0, \infty) \). Let \( \omega : [0, \infty) \to \mathbb{R}^+ \) be a non-negative measurable function such that \( \omega(x) = 0 \) for all \( x \in [0, \infty) \). If we take \( A = [0, 1] \) then \( \mu T^{-1}(A) = \frac{1}{2} \neq 1 = \mu(A) \), which implies that \( T \) is not measure preserving. But \( \omega_\mu T^{-1}(A) = \int_{T^{-1}(A)} \omega(z) d\mu = 0 = \int_{(A)} \omega(z) d\mu = \omega_\mu(A) \) for all \( A \in \mathcal{A} \). As a consequence, \( T \) is a \( \omega \)-measure preserving measurable transformation.

**Example 3.2.3.** Let \( X = S^1 \), the unit circle in the complex plane and \( \mu \) be the Lebesgue measure. Let \( T : S^1 \to S^1 \) be given by \( T(z) = e^{-i\frac{\pi}{2}}z \) for all \( z \in S^1 \). Clearly \( T \) is a measure preserving transformation. Define a non-negative measurable function \( \omega : S^1 \to \mathbb{R}^+ \) as \( \omega(z) = 2|\text{Re}z| \) for each \( z \in S^1 \). A direct computation shows that for \( A = \{ e^{i\theta} : 0 \leq \theta \leq \frac{\pi}{6} \}, T^{-1}(A) = \{ e^{i\theta} : \frac{\pi}{2} \leq \theta \leq \frac{2\pi}{3} \} \) and put \( z = e^{i\theta} \)

\[
\omega_\mu(A) = \int_A \omega(z) d\mu = \int_0^{\frac{\pi}{6}} 2\cos^2 \theta d\theta - \int_0^{\frac{\pi}{6}} \sin 2\theta d\theta = i\left(\frac{\pi}{6} + \frac{\sqrt{3}}{4}\right) - \frac{1}{4}.
\]

Also

\[
\omega_\mu T^{-1}(A) = \int_{T^{-1}(A)} \omega(z) d\mu = \int_{\frac{\pi}{2}}^{\frac{2\pi}{3}} 2\cos^2 \theta d\theta - \int_{\frac{\pi}{2}}^{\frac{2\pi}{3}} \sin 2\theta d\theta = i\left(\frac{\pi}{6} - \frac{\sqrt{3}}{4}\right) + \frac{1}{4}.
\]

This implies that \( \omega_\mu T^{-1}(A) \neq \omega_\mu(A) \). Hence \( T \) is a measure preserving transformation but not \( \omega \)-measure preserving.

Following is immediate from Theorem 3.2.1.
Corollary 3.2.3. Every non-singular measurable transformation $T : X \to X$ which is $\omega$-measure preserving inducing a composition operators $C_T$ on Weighted Lorentz-Karamata-Bochner spaces $L_{(p,q,\omega;b)}$, $1 < p \leq \infty$ and $1 \leq q \leq \infty$.

Corollary 3.2.4. If $T$ is a $\omega$-measure preserving then the induced composition operator $C_T$ is an isometry.

**Proof.** Suppose that $T$ is $\omega$-measure preserving then $\omega \mu T^{-1}(A) = \omega \mu(A)$ for all $A \in \mathcal{A}$. The weighted distribution function of $C_T$ becomes

$$
\mu_{C_T f, \omega}(s) = \omega \mu \{ x \in X : \|C_T f(x)\| > s \} \\
= \omega \mu \{ x \in X : \|f \circ T(x)\| > s \} \\
= \omega \mu T^{-1} \{ x \in X : \|f(x)\| > s \} \\
= \omega \mu \{ x \in X : \|f(x)\| > s \}.
$$

Hence, for each $t > 0$,

$$
\|(f \circ T)\|_{\omega}^*(t) = \|f\|_{\omega}^*(t)
$$

so that

$$
\|(f \circ T)\|_{\omega}^*(t) = \|f\|_{\omega}^*(t).
$$

This gives $\|C_T f\|_{p,q,\omega;b} = \|f\|_{p,q,\omega;b}$. \qed

### 3.3 Composition Operators on Weighted Bergman-Nevanlinna Spaces to Zygmund Spaces

In this section, we explained Compactness and Boundedness of $DC_T$ from weighted Bergman-Nevanlinna spaces to Zygmund spaces. Recall that a linear map $L : A_N^p \to$
Z is bounded if $L(E) \subset Z$ is bounded for every bounded subset $E$ of $A_N^\alpha$. In addition, we indicate that $L$ is compact if $L(E) \subset Z$ is relatively compact for each bounded set $E \subset A_N^\alpha$.

Now, we state and prove the main result of this section.

**Theorem 3.3.1.** Let $\alpha > -1$ and $T$ be a holomorphic self map of $\mathbb{D}$. Then the following are equivalent:

(i) $DC_T : A_N^\alpha \to Z$ is bounded.

(ii) $DC_T : A_N^\alpha \to Z$ is compact.

(iii) For all $c > 0$,

\[
\lim_{|T(z)| \to 1} (1 - |z|^2)|T'(z)|^3(1 - |T(z)|^2)^3\exp[c(1 - |Tz|^2)^{\alpha + 2}] = 0
\]

\[
\lim_{|T(z)| \to 1} (1 - |z|^2)|T'(z).T''(z)|(1 - |T(z)|^2)^2\exp[c(1 - |Tz|^2)^{\alpha + 2}] = 0
\]

\[
\lim_{|T(z)| \to 1} (1 - |z|^2)|T''(z)|(1 - |T(z)|^2)^2\exp[c(1 - |Tz|^2)^{\alpha + 2}] = 0
\]

\[
\sup_{z \in \mathbb{D}} (1 - |z|^2)|T'(z)|^3 < \infty
\]

\[
\sup_{z \in \mathbb{D}} (1 - |z|^2)|T'(z).T''(z)| < \infty
\]

\[
\sup_{z \in \mathbb{D}} (1 - |z|^2)|T''(z)| < \infty.
\]

**Proof.** It is clear that (ii) $\Rightarrow$ (i). Thus, we only need to check two implications (i) $\Rightarrow$ (ii) and (iii) $\Rightarrow$ (ii), (i) $\Rightarrow$ (iii): Suppose that (i) holds. By taking $f(z) = z$ and by the boundedness of $DC_T : A_N^\alpha \to Z$ we have

\[
\sup_{z \in \mathbb{D}} (1 - |z|^2)|(DC_T f)''| < \infty
\]

that is

\[
\sup_{z \in \mathbb{D}} (1 - |z|^2)|f''(T(z)).(T'(z))^3 + 3f''(T(z)).T'(z).T''(z)) + f'(T(z))T'''(z)| < \infty.
\]

But $f'(z) = 1, f''(z) = 0, f'''(z) = 0$.

Therefore, we get

\[
\sup_{z \in \mathbb{D}} (1 - |z|^2)|T(z).T'''(z)| < \infty.
\]

Since $|T(z)| < 1$, therefore

\[
\sup_{z \in \mathbb{D}} (1 - |z|^2)|T''(z)| < \infty.
\]

Again, by taking $f(z) = \frac{z}{2}$ in $A_N^\alpha$ and the boundedness of $DC_T : A_N^\alpha \to Z$ we get
3.3 Composition Operators on Weighted Bergman-Nevanlinna Spaces to Zygmund Spaces

\[ \sup_{z \in D} (1 - |z|^2)|f'''(T(z)).(T'(z))^3 + 3f''(T(z)).T'(z).T''(z)) + f'(T(z))T'''(z)| < \infty. \]

Since \( f'(z) = z, f''(z) = 1, f'''(z) = 0. \)

Therefore, we get
\[ \sup_{z \in D} (1 - |z|^2)|3T'(z).T''(z)| < \infty. \]

that is
\[ \sup_{z \in D} (1 - |z|^2)|T'(z).T''(z)| < \infty. \]

Finally, by taking \( f(z) = \frac{z^a}{2^b} \) in \( A_N^a \) we get
\[ \sup_{z \in D} (1 - |z|^2)|f'''(T(z)).(T'(z))^3 + 3f''(T(z)).T'(z).T''(z)) + f'(T(z))T'''(z)| < \infty \]
but \( f'(z) = \frac{z^a}{2^b}, f''(z) = z, f'''(z) = 1. \)

Therefore, we get
\[ \sup_{z \in D} (1 - |z|^2)|(T'(z))^3 + 3T'(z).T''(z) + (T(z))^22T'''(z)| < \infty. \]

Since \( 2|(T(z))^2| < 1 \) and
\[ \sup_{z \in D} (1 - |z|^2)|3T(z).T''(z)| < \infty \]
\[ \sup_{z \in D} (1 - |z|^2)|T'''(z)| < \infty. \]

Therefore, we have
\[ \sup_{z \in D} (1 - |z|^2)|(T'(z))^3| < \infty. \]

Let \( \lambda \in D \) and \( c > 0 \), consider the function
\[ f_{\lambda}(z) = \{(3 + 2(\alpha + 2)).(1 - |T(\lambda)|^2)^{\alpha+2}(1 - \overline{T(\lambda)}z)^{2(\alpha+2)} \]
\[ - (4(\alpha + 2) + 4).(1 - |T(\lambda)|^2)^{(\alpha+3)}(1 - \overline{T(\lambda)}z)^{2(\alpha+2)+1} \]
\[ + (2(\alpha + 2) + 1).(1 - |T(\lambda)|^2)^{\alpha+4}(1 - \overline{T(\lambda)}z)^{2(\alpha+2)+2} \}
\[ \times \exp\{\alpha(2(\alpha + 2))^2 + 6(\alpha + 2) + 2).(1 - |T(\lambda)|^2)^{\alpha+2}(1 - \overline{T(\lambda)}z)^{2(\alpha+2)} \]
\[ - 2(2(\alpha + 2))^2 + 4(\alpha + 2).(1 - |T(\lambda)|^2)^{(\alpha+3)}(1 - \overline{T(\lambda)}z)^{2(\alpha+2)+1} \]
\[ + (2(\alpha + 2))^2 + 2(\alpha + 2).(1 - |T(\lambda)|^2)^{(\alpha+4)}(1 - \overline{T(\lambda)}z)^{2(\alpha+2)+2} \}. \]

Then
\[ |f_{\lambda}(z)| = \{(3 + 2(\alpha + 2)).(1 - |T(\lambda)|^2)^{\alpha+2}|1 - \overline{T(\lambda)}z|^{2(\alpha+2)} \]
\[ + (4(\alpha + 2) + 4).(1 - |T(\lambda)|^2)^{(\alpha+3)}|1 - \overline{T(\lambda)}z|^{2(\alpha+2)+1} \]
\[ + (2(\alpha + 2) + 1).(1 - |T(\lambda)|^2)^{\alpha+4}|1 - \overline{T(\lambda)}z|^{2(\alpha+2)+2} \}
\[ \times \exp\{\alpha(2(\alpha + 2))^2 + 6(\alpha + 2) + 2).(1 - |T(\lambda)|^2)^{\alpha+2}|1 - \overline{T(\lambda)}z|^{2(\alpha+2)} \]
\[-2(2(\alpha + 2))^2 + 4(\alpha + 2). (1 - |T(\lambda)|^2(\alpha + 3)|1 - T(\lambda)z|^2(\alpha + 2) + 1
+ (2(\alpha + 2))^2 + 2(\alpha + 2). (1 - |T(\lambda)|^2(\alpha + 4)|1 - T(\lambda)z|^2(\alpha + 2) + 2}\].

Again, by using some elementary inequalities

\[\log(1 + xy) \leq \log(1 + x) + \log(1 + y)\]

\[\log(1 + ax) \leq a\log(1 + x)\]

\[\log(1 + x) \leq x\]

\[\log(1 + x) \leq 1 + \log(x), x, y > 0, \alpha \geq 1,\]

we have

\[\log(1 + |f_\lambda(z)|) = \log[1 + \{(3 + 2(\alpha + 2)). (1 - |T(\lambda)|^2(\alpha + 2)|1 - T(\lambda)z|^2(\alpha + 2)
+ (4(\alpha + 2) + 4). (1 - |T(\lambda)|^2(\alpha + 3)|1 - T(\lambda)z|^2(\alpha + 2) + 1
+ (2(\alpha + 2) + 1). (1 - |T(\lambda)|^2(\alpha + 4)|1 - T(\lambda)z|^2(\alpha + 2) + 2)]

+ \log[1 + \exp(c2\{(2(\alpha + 2))^2 + 6(\alpha + 2) + 2). (1 - |T(\lambda)|^2(\alpha + 2)|1 - T(\lambda)z|^2(\alpha + 2)
+ (2(\alpha + 2))^2 + 2(\alpha + 2). (1 - |T(\lambda)|^2(\alpha + 3)|1 - T(\lambda)z|^2(\alpha + 2) + 1
+ (2(\alpha + 2) + 1). (1 - |T(\lambda)|^2(\alpha + 4)|1 - T(\lambda)z|^2(\alpha + 2) + 2)]

+ c2\{(2(\alpha + 2))^2 + 6(\alpha + 2) + 2). (1 - |T(\lambda)|^2(\alpha + 2)|1 - T(\lambda)z|^2(\alpha + 2)
+ 2(\alpha + 2))^2 + 4(\alpha + 2). (1 - |T(\lambda)|^2(\alpha + 3)|1 - T(\lambda)z|^2(\alpha + 2) + 1
+ (2(\alpha + 2))^2 + 2(\alpha + 2). (1 - |T(\lambda)|^2(\alpha + 4)|1 - T(\lambda)z|^2(\alpha + 2) + 2)]
\leq 1 + C_\alpha|1 - |T(\lambda)|^2(\alpha + 2)|1 - T(\lambda)z|^2(\alpha + 2)\]

then \[\|f_\lambda(z)\|_{A_N^\alpha} \leq C_\alpha\] and so \(f_\lambda(z) \in A_N^\alpha\).

Also, one can easily check that

\[f_\lambda(T(z)) = 0\]
\[f'_\lambda(T(z)) = 0\]
\[f''_\lambda(T(z)) = c(1 - |T(\lambda)|^2(\alpha + 4)\exp\{c(1 - |T(\lambda)|^2(\alpha + 2)\}\]
\[f'''_\lambda(T(z)) = 0.\]

Since \(DC_T : A_N^\alpha \to Z\) is bounded therefore, there exists a constant \(m_1 > 0\) such
that

\[ m_1 \geq (1 - |\lambda|^2)|f'''(T(\lambda)).(T'(\lambda))^3 + 3f''(T(\lambda)).T'(\lambda).T''(\lambda)) + f'(T(\lambda))T'''(\lambda)| \]

\[ = (1 - |\lambda|^2)T'(\lambda).T'''(\lambda))3c(1 - |T(\lambda)|^2)^{\alpha+4}exp\{c(1 - |T(\lambda)|^2)^{\alpha+2}\} \]

In particular

\[ (1 - |\lambda|^2)T'(\lambda).T'''(\lambda))|(1 - |T(\lambda)|^2)^2 \leq m_13c(1 - |T(\lambda)|^2)^{\alpha+2}. \]

Taking the limit as \( |T(\lambda)| \to 1 \) on both sides in the above inequality we get

\[ \lim_{|T(\lambda)| \to 1} (1 - |\lambda|^2)T'(\lambda).T'''(\lambda))|(1 - |T(\lambda)|^2)^2exp\{c(1 - |T(\lambda)|^2)^{\alpha+2}] = 0 \]

Again, consider the function

\[ g_\lambda(z) = \{(4(\alpha + 2) + 4).\{(1 - |T(\lambda)|^2)^{\alpha+2}(1 - \overline{T(\lambda)}z)^{2(\alpha+2)} \]

\[ - (8(\alpha + 2) + 6).\{(1 - |T(\lambda)|^2)^{(\alpha+3)}(1 - \overline{T(\lambda)}z)^{2(\alpha+2)+1} \]

\[ + (4(\alpha + 2) + 2).\{(1 - |T(\lambda)|^2)^{\alpha+4}(1 - \overline{T(\lambda)}z)^{2(\alpha+2)+2} \}

\[ \times exp\{2\{(4(\alpha + 2)^2 + 6(\alpha + 2) + 2).\{(1 - |T(\lambda)|^2)^{\alpha+2}(1 - \overline{T(\lambda)}z)^{2(\alpha+2)} \]

\[ - (8(\alpha + 2)^2 + 8(\alpha + 2)).\{(1 - |T(\lambda)|^2)^{(\alpha+2)}(1 - \overline{T(\lambda)}z)^{2(\alpha+2)+2} \]

\[ + (4(\alpha + 2)^2 + 2(\alpha + 2)).\{(1 - |T(\lambda)|^2)^{(\alpha+4)}(1 - \overline{T(\lambda)}z)^{2(\alpha+2)+2}\} \].

Then it can easily prove that \( \|g_\lambda(z)\|_{A^0_N} \leq C_\alpha \) and so \( g_\lambda(z) \in A^0_N \).

Also, one can easily check that

\[ g_\lambda'(T(z)) = 0 \]

\[ g_\lambda''(T(z)) = 0 \]

\[ g_\lambda'''(T(z)) = c(1 - |T(\lambda)|^2)^{\alpha+5}exp\{c(1 - |T(\lambda)|^2)^{\alpha+2}\} \]

Since \( DC_T : A^0_N \to Z \) is bounded therefore, there exists a constant \( m_2 > 0 \) such that

\[ m_2 \geq (1 - |\lambda|^2)|g_\lambda'(T(\lambda)).(T'(\lambda))^3 + 3g_\lambda''(T(\lambda)).T'(\lambda).T''(\lambda)) + g_\lambda'(T(\lambda))T'''(\lambda)| \]

In particular

\[ (1 - |\lambda|^2)c.|T'(\lambda)|^3(1 - |T(\lambda)|^2)^{\alpha+5}exp\{c(1 - |T(\lambda)|^2)^{\alpha+2}\} \leq m_2 \]

hence

\[ (1 - |\lambda|^2)|T'(\lambda)|^3(1 - |T(\lambda)|^2)^{\alpha+3}exp\{c(1 - |T(\lambda)|^2)^{\alpha+2}\} \leq m_2c(1 - |T(\lambda)|^2)^2. \]

Taking the limit as \( |T(\lambda)| \to 1 \) on both sides in the above inequality we get

\[ (1 - |\lambda|^2)|T'(\lambda)|^3(1 - |T(\lambda)|^2)^{\alpha+3}exp\{c(1 - |T(\lambda)|^2)^{\alpha+2}\} = 0 \]
Finally, consider the function
\[ h_\lambda(z) = \{(7\alpha + 2 + 1). (1 - |T(\lambda)|^2)^{\alpha+2}(1 - \overline{T(\lambda)}z)^{2(\alpha+2)} - (12\alpha + 2 + 2). (1 - |T(\lambda)|^2)^{2(\alpha+2)}(1 - \overline{T(\lambda)}z)^{3(\alpha+2)} + (5\alpha + 2 + 1). (1 - |T(\lambda)|^2)^{3(\alpha+2)}(1 - \overline{T(\lambda)}z)^{4(\alpha+2)}\} \times \exp\{6c12.(1 - |T(\lambda)|^2)^{2(\alpha+2)}(1 - \overline{T(\lambda)}z)^{3(\alpha+2)} - 13(1 - |T(\lambda)|^2)^{2(\alpha+2)}(1 - \overline{T(\lambda)}z)^{3(\alpha+2)}\].

Now, it can easily prove that \( h_\lambda(z) \in A_N^\alpha \)

Also, one can easily check that
\[
\begin{align*}
  h'_\lambda(T(z)) &= -2(\alpha + 2)\overline{T(\lambda)}(1 - |T(\lambda)|^2)^{\alpha+3}\exp\{c(1 - |T(\lambda)|^2)^{\alpha+2}\} \\
  h''_\lambda(T(z)) &= 0 \\
  h'''_\lambda(T(z)) &= 0
\end{align*}
\]
Since \( DC_T : A_N^\alpha \to Z \) is bounded. Therefore we can find a constant \( m_3 > 0 \) such that
\[ m_3 \geq (1 - |z|^2)|g'''_\lambda(T(\lambda)).(T'(\lambda))^3 + 3h''_\lambda(T(\lambda)).T''(\lambda).T'(\lambda)| + h'_\lambda(T(\lambda))T''(\lambda)\]

In particular
\[ (1 - |\lambda|^2)^2|\alpha + 2||T(\lambda)||T''(\lambda)|(1 - |T(\lambda)|^2)^{\alpha+3}\exp\{c(1 - |T(\lambda)|^2)^{\alpha+2}\} \leq m_3. \]

Hence
\[ (1 - |\lambda|^2)|T''(\lambda)|(1 - |T(\lambda)|^2)\exp\{c(1 - |T(\lambda)|^2)^{\alpha+2}\} \leq m_3|\alpha + 2||(1 - |T(\lambda)|^2)^{\alpha+2} \]
Taking the limit as \(|T(\lambda)| \to 1\) on both sides in the above inequality we get
\[ (1 - |\lambda|^2)|T''(\lambda)|(1 - |T(\lambda)|^2)\exp\{c(1 - |T(\lambda)|^2)^{\alpha+2}\} = 0 \]

(iii) \implies (ii) Assume that (iii) holds for all \( c > 0 \). Note that if \( f \in A_N^\alpha \) then by Cauchy integral formula for derivatives, we have
\[ (1 - |z|^2)|f'(z)| \leq 2\pi \int_{\partial D} |f(z + 12(1 - |z|)\xi)|d\xi \leq \exp\{c_a\alpha m_0\|f\|_{A_N^\alpha}(1 - |z|^2)^{\alpha+2}\} \]
and
\[ (1 - |z|^2)|f'(z)| \leq 16\pi \int_{\partial D} |f(z + 12(1 - |z|)\xi)|d\xi \leq 8\exp\{c_a\alpha m_0\|f\|_{A_N^\alpha}(1 - |z|^2)^{\alpha+2}\} \]
and
\[ (1 - |z|^2)|f'(z)| \leq 96\pi \int_{\partial D} |f(z + 12(1 - |z|)\xi)|d\xi \leq 48\exp\{c_a\alpha m_0\|f\|_{A_N^\alpha}(1 - |z|^2)^{\alpha+2}\} \]
Choose for \( m > 0 \) any sequence \( \{f_n\} \) in \( A_N^\alpha \) such that \( \|f\|_{A_N^\alpha} \leq m \) and \( f_n \to 0 \)
locally uniformly on \( \mathbb{D} \). Then for each \( r \in (0, 1) \) we have

\[
\sup_{|T(z)| \leq r} (1 - |z|^2)|DC_T f|'' = \sup_{|T(z)| \leq r} (1 - |z|^2)|f''(T(z))||T'(z)|^3 + \sup_{|T(z)| \leq r} (1 - |z|^2)3|f''(T(z))||T'(z).T''(z)|
\]

\[
+ \sup_{|T(z)| \leq r} (1 - |z|^2)|f'(T(z))||T'''(z)|.
\]

\[
\leq A \sup_{|T(z)| \leq r} |f''(T(z))| + B \sup_{|T(z)| \leq r} |f''(T(z))|
\]

\[
+ C \sup_{|T(z)| \leq r} |f'(T(z))| \to 0 \text{ as } n \to \infty,
\]

where

\[
A = \sup_{z \in \mathbb{D}} (1 - |z|^2)|T'(z)|^3
\]

\[
B = \sup_{z \in \mathbb{D}} (1 - |z|^2)3T'(z)T''(z)
\]

\[
C = \sup_{z \in \mathbb{D}} (1 - |z|^2)|T'''(z)|.
\]

On the other hand, whenever \( r \to 1 \) we have

\[
\sup_{|T(z)| > r} (1 - |z|^2)|(DC_T f)|''
\]

\[
= \sup_{|T(z)| \leq r} (1 - |z|^2)|f''(T(z))||T'(z)|^3
\]

\[
+ \sup_{|T(z)| > r} (1 - |z|^2)3|f''(T(z))||T'(z).T''(z)|
\]

\[
+ \sup_{|T(z)| > r} (1 - |z|^2)|f'(T(z))||T'''(z)|.
\]

\[
= \sup_{|T(z)| > r} (1 - |z|^2)|T'(z)|^3(1 - |T'(z)|^2)3exp[c_\alpha m_\alpha]f_{A_N^\alpha} (1 - |z|^2)\alpha + 2
\]

\[
+ \sup_{|T(z)| > r} (1 - |z|^2)|T'(z)T''(z)|(1 - |T'(z)|^2)3exp[c_\alpha m_\alpha]f_{A_N^\alpha} (1 - |z|^2)\alpha + 2
\]

\[
+ \sup_{|T(z)| > r} (1 - |z|^2)|T'''(z)|(1 - |T'(z)|^2)3exp[c_\alpha m_\alpha]f_{A_N^\alpha} (1 - |z|^2)\alpha + 2
\]

\[
\to 0 \text{ as } n \to \infty.
\]

The above estimates together with the fact \( |DC_T f_n(0)| \to 0 \) and \( |(DC_T f_n)'(0)| \to 0 \) yields

\[
\|DC_T f_n\|_z \to 0 \text{ as } n \to \infty.
\]

Hence \( DC_T : A_N^\alpha \to Z \) is compact. \( \square \)