STABILITY OF TRIBONACCI AND k-TRIBONACCI
FUNCTIONAL EQUATIONS IN MODULAR SPACES

Our goal in this chapter is to study the Hyers-Ulam-Rassias stability of Tribonacci and k-Tribonacci functional equations in Modular Spaces. This chapter has been divided into three sections. In the first section, we recall some basic definitions, notations and facts related to Modular Spaces. In 2011, M. Bidkhan and M. Hosslini [10] proved the stability of k-Fibonacci functional equation and in 2012, M. Gordji, M. Naderi and Th. M. Rassias [63] also proved the stability of Tribonacci functional equation in Non-Archimedean Spaces. Motivated by these results, in the next two sections of this chapter, we introduce the stability of Tribonacci and k-Tribonacci functional equations in Modular spaces, respectively. The result of these sections have been published in International Journal of Pure and Applied Mathematics, (104) (2) (2015), pp. 265-284.

2.1. Modular Spaces

At present, theory of modular and modular spaces is extensively applied, in particular, in the study of various W. Orlicz spaces [102] and interpolation theory [102, 112] which in their turn have broad applications [134]. The importance of applications consists in the richness of structure of modular function spaces,
that besides being Banach spaces (or F spaces in more general setting)-are equipped with modular equipped with modular of norm or metric notations. Several researchers like Iz. El. Fassi and S. Kabbaj, M. N. Parizi, M. E. Gordji [35,38] etc. used the concept of Modular Spaces in proving the stability results. They recalled the following definitions:

**Definition 2.1.1.[38]:** “Suppose \( Y \) be an arbitrary vector space.

(a) A functional \( \rho: Y \to [0, \infty] \) is called a modular if for arbitrary \( x, y \in Y \),

   (i) \( \rho(y) = 0 \) if and only if \( y = 0 \),

   (ii) \( \rho(\alpha y) = \rho(y) \) for every scalar \( \alpha \) with \( |\alpha| = 1 \),

   (iii) \( \rho(\alpha x + \beta y) \leq \rho(x) + \rho(y) \) if and only if \( \alpha + \beta = 1 \) and \( \alpha, \beta \geq 0 \),

(b) if (iii) is replaced by

   (iii) \( \rho(\alpha x + \beta y) \leq \alpha \rho(x) + \beta \rho(y) \) if and only if \( \alpha + \beta = 1 \) and \( \alpha, \beta \geq 0 \),

then we say that \( \rho \) is a convex modular.

(c) A modular \( \rho \) defines a corresponding modular space, i.e., the vector space \( Y_\rho \) given by

\[
Y_\rho = \{ y \in Y : \rho(\lambda y) \to 0 \text{ as } \lambda \to 0 \}.
\]

**Definition 2.1.2.[125]:** “Let \( \rho \) be a convex modular, the modular space \( Y_\rho \) can be equipped with a norm called the Luxemburg norm, defined by

\[
\|y\|_\rho = \inf\{ \lambda > 0 : \rho\frac{y}{\lambda} \leq 1 \}
\]

A function modular is said to be satisfy the \( \Delta_2 \)-condition if there exit \( k > 0 \) such that \( \rho(2y) \leq k \rho(y) \) for all \( y \in Y_\rho \).”
In general the modular $\rho$ does not behave as a norm or as a distance because it is not sub-additive.

**Definition 2.1.4.** [138]: “Let $\{y_n\}$ and $y$ be in $Y_\rho$. Then

(i) we say $\{y_n\}$ is a $\rho$-convergent to $y$ and write $y_n \xrightarrow{\rho} y$ if and only if $\rho(y_n - y) \rightarrow 0$ as $n \rightarrow 0$,

(ii) the sequence $\{y_n\}$, with $y_n \in Y_\rho$, is called $\rho$ - Cauchy if $\rho(y_n - y_m) \rightarrow 0$ as $m, n \rightarrow \infty$,

(iii) a subset $S$ of $Y_\rho$ is called $\rho$ - complete if and only if any $\rho$ - Cauchy sequence is $\rho$ - convergent to an element of $S$. ” For further details and proofs, we refer the reader to [144].

**Remark 2.1.5.** [38]: “ If $y \in Y_\rho$ then $\rho(ay)$ is a non decreasing function of $a \geq 0$. Suppose that $0 < a < b$, then property (iii) of definition (2.1) with $y = 0$ shows that

$$\rho(ay) = \rho\left(\frac{a}{b}by\right) \leq \rho(by).$$

Moreover, if $\rho$ is convex modular on $Y$ and $|\alpha| \leq 1$ then, $\rho(\alpha y) \leq |\alpha|\rho(y)$ and also

$$\rho(y) \leq \frac{1}{2} \rho(2y) \leq \frac{k}{2} \rho(y)$$

if $\rho$ satisfy the $\Delta_2$- condition for all $y \in Y$. ”

### 2.2. Hyers-Ulam-Rassias Stability of Tribonacci Functional Equation in Modular Spaces

Recently, In 2014, M. N. Parizi et al.[135] and In 2015, Iz. El.-Fassi and S. Kabbaj [38] proved the stability of Fibonacci functional equation and orthogonal quadratic functional equation in Modular spaces, respectively.

Throughout this section, we denote by $T_n$ the $n$th Tribonacci number where

$$T_n = T_{n-1} + T_{n-2} + T_{n-3} \text{ for } n \geq 3$$

with initial conditions $T_0 = 0$, $T_1 = 1$, $T_2 = 1$. From this, we may derived a functional equation

$$f(x) = f(x-1) + f(x-2) + f(x-3) \quad (*)$$

which is called the Tribonacci functional equation if a function $f : N \times R \to X$ satisfies the above equation for all $x \in R$. We denote the roots of equation $x^3 - x^2 - x - 1 = 0$ by $p$, $q$ and $r$ where $q$, $r$ are complex, $|q| = |r|$ and $p$ is greater than one. We obtain

$$p + q + r = 1, \quad pq + qr + pr = -1, \quad pqr = 1.$$ 

Now it follows from (*) that

$$f(x) - p(f(x-1) - r f(x-2)) - r f(x-1) = q [f(x-1) - (r + p) f(x-2) + pr f(x-3)]$$

for all $x \geq 0$. By mathematical induction, we verify that for all $x \geq 0$ and all $m$ belonging to the set \{0, 1, 2, \ldots\}, we obtain,

$$f(x) - p(f(x-1) - r f(x-2)) - r f(x-1) = q^m [f(x-m) - r f(x-m-1) + pr f(x-m-2) - p f(x-m-1)]$$

$$f(x) - r[f(x-1) - q f(x-2)] - q f(x-1) = p^m [f(x-m) - r f(x-m-1) + qr f(x-m-2) - q f(x-m-1)]$$

$$f(x) - q [f(x-1) - p f(x-2)] - p f(x-1) =$$
for all $x \geq 0$ and all $m \in \{0, 1, 2, \ldots\}$.

**Theorem 2.2.1.** Let $(X, \rho)$ be a Banach Modular Space. If a function $f : \mathbb{R} \to X$ satisfies the inequality

$$\rho(f(x) - f(x-1) - f(x - 2) - f(x - 2)) \leq \varepsilon$$

for all $x \in \mathbb{R}$, and for some $\varepsilon > 0$, then there exists a Tribonacci function $H : \mathbb{N} \times \mathbb{R} \to X$ such that

$$\rho(f(x) - H(x)) \leq \frac{2(1 +|q| + |q|^2)}{|q^2(r-p)+r^2(p-q)+p^2(q-r)|} \times \frac{\varepsilon}{1-|q|^1}.$$  

**Proof:** It follows from (2.2.1) that

$$\rho(f(x) - p\{f(x - 1) - r f(x - 2)\} - r f(x - 1)$$

$$- q [f(x - 1) - (r + p) f(x - 2) + pr f(x - 3)]) \leq \varepsilon$$

If we replace $x$ by $x - \alpha$ in the last inequality, then we get

$$\rho(f(x - \alpha) - p\{f(x - \alpha - 1) - r f(x - \alpha - 2)\} - r f(x - \alpha - 1)$$

$$- q p f(x - \alpha - 1) - (r + p) f(x - \alpha - 2) + pr f(x - \alpha - 3)) \leq \varepsilon$$

for all $x \in \mathbb{R}$.

Now multiplying both sides by $q^\alpha$,

$$\rho(q^\alpha \{f(x - \alpha) - p[f(x - \alpha - 1) - r f(x - \alpha - 2)] - r f(x - \alpha - 1)\}$$

$$- q^{\alpha+1} \{f(x - \alpha - 1) - (r + p) f(x - \alpha - 2) + pr f(x - \alpha - 3))\})$$

$$\leq |q^\alpha| \rho(\{f(x - \alpha) - p[f(x - \alpha - 1) - r f(x - \alpha - 2)] - r f(x - \alpha - 1)\}$$

61
\[ -q^{\alpha+1} \{ f(x - \alpha - 1) - (r + p) f(x - \alpha - 2) + pr f(x - \alpha - 3) \} \leq |q^\alpha| \varepsilon \]  

(2.2.3)

for all \( x \in \mathbb{R} \) and \( \alpha \in \mathbb{N} \). Furthermore, we have

\[
\begin{align*}
\rho(\{ f(x) - p \{ f(x - 1) - r f(x - 2) \} - r f(x-1) \}) \\
- q^m [f(x - m) - (r + p) f(x - m - 1) + pr f(x - m - 2)]
\end{align*}
\]

\[
\leq \rho\left( \sum_{\alpha=0}^{n-1} q^\alpha [f(x - \alpha) - p\{f(x - \alpha - 1) - r f(x - \alpha - 2)\} - r f(x - \alpha - 1)]
\right)
\]

\[
\leq \sum_{\alpha=0}^{n-1} q^\alpha (\rho ([f(x - \alpha) - p\{f(x - \alpha - 1) - r f(x - \alpha - 2)\} - r f(x - \alpha - 1)])
\]

\[
\leq \sum_{\alpha=0}^{n-1} |q|^\alpha \varepsilon \leq \frac{\varepsilon}{1-|q|} \]  

(2.2.4)

for all \( x \in \mathbb{R} \), \( m \in \mathbb{N} \). Let \( x \in \mathbb{R} \) be fixed, then (2.2.3) implies that \( \{ q^m [f(x-m) - p \{ f(x-m-1) - r f(x-m-2) \} - r f(x - m - 1)] \} \) is a Cauchy sequence ( \(|q| < 1\) ). So by the completeness of \( X \), we may define a function \( H_1 : \mathbb{R} \rightarrow X \) such that

\[
H_1(x) = \lim_{m \to \infty} q^m [f(x - m) - (p + r) f(x - m - 1) + pr f(x - m - 2)] \text{ for all } x \in \mathbb{R}.
\]

Applying the definition of \( H_1 \), we introduce the Tribonacci function

\[
H_1(x - 1) + H_1(x - 2) + H_1(x - 3) =
\]

\[
q^{-1} \lim_{m \to \infty} q^{m+1} [ f(x - (m+1)) - (p + r) f(x - (m +1) -1) + pr f(x - (m+1)-2) ]
\]

62
\[+q^2 \lim_{m \to \infty} q^{m+2}[f(x - (m+2)) - (p + r) f(x - (m+2) - 1) + pr f(x - (m+2) - 2)]\]

\[+q^3 \lim_{m \to \infty} q^{m+3}[f(x - (m+3)) - (p + r) f(x - (m+3) - 1) + pr f(x - (m+3) - 2)]\]

\[= q^{-1}H_1(x) + q^{-2}H_1(x) + q^{-3}H_1(x)\]

\[= H_1(x) \text{ for all } x \in \mathbb{R}. \text{ Hence } H_1 \text{ is a Tribonacci function.}\]

If \(m \to \infty\), then from (2.2.4), we obtain

\[\rho(f(x) - (p + r) f(x - 1) + pr f(x - 2) - H_1) \leq \frac{1}{1 - |q|} \varepsilon \] (2.2.5)

for all \(x \in \mathbb{R}\). Furthermore, it follows from (2.2.1) that

\[\rho([f(x) - q \{f(x - 1) - p f(x - 2)\} - p f(x - 1)]\]

\[- r[f(x - 1) - p f(x - 2) + pq f(x - 3) - q f(x - 2)]) \leq \varepsilon\]

for all \(x \in \mathbb{R}\). Now, we replace \(x\) by \(x - \alpha\) in above inequality, we have

\[\rho([f(x - \alpha) - q \{f(x - \alpha - 1) - p f(x - \alpha - 2)\} - p f(x - \alpha - 1)]\]

\[- r[f(x - \alpha - 1) - p f(x - \alpha - 2) + pq f(x - \alpha - 3) - q f(x - \alpha - 2)]) \leq \varepsilon\]

and now multiplying by \(r^\alpha\) on both sides

\[\rho( r^\alpha [f(x - \alpha) - q \{f(x - \alpha - 1) - p f(x - \alpha - 2)\} - p f(x - \alpha - 1)]\]

\[- r^{\alpha+1}[f(x - \alpha - 1) - p f(x - \alpha - 2) + pq f(x - \alpha - 3) - q f(x - \alpha - 2)])\]

\[\leq |r^\alpha| (\rho ([f(x - \alpha) - q \{f(x - \alpha - 1) - p f(x - \alpha - 2)\} - p f(x - \alpha - 1)]\]

\[- r^{\alpha+1}[f(x - \alpha - 1) - p f(x - \alpha - 2) + pq f(x - \alpha - 3) - q f(x - \alpha - 2)])\]

\[\leq |r^\alpha| \varepsilon \] (2.2.6)
for all $x \in \mathbb{R}$, $\alpha \in \mathbb{Z}$. Now, we have

$$\rho \left( [f(x) - q\{f(x - 1) - p f(x - 2)}] - p f(x - 1) \right)$$

$$- r^m [f(x-m) - (q + p) f(x-m-1) + p q (f(x-m-2))]$$

$$\leq \rho \left( \sum_{k=1}^{m} (r^\alpha [f(x - \alpha) - q\{f(x - \alpha - 1) - p f(x - \alpha - 2)}] - p f(x - \alpha - 1))$$

$$- r^{\alpha+1} [f(x - \alpha - 1) - (p + q) f(x - \alpha - 2) + p q (f(x - \alpha - 3))] \right)$$

$$\leq \sum_{k=1}^{m} |r|^\alpha \left( \rho([f(x - \alpha) - q\{f(x - \alpha - 1) - p f(x - \alpha - 2)}] - p f(x - \alpha - 1))$$

$$- r[f(x - \alpha - 1) - (p + q) f(x - \alpha - 2) + p q (f(x - \alpha - 3))] \right)$$

$$\leq \sum_{k=1}^{m} |r|^\alpha \epsilon \leq \frac{\epsilon}{1-|r|} \quad (2.2.7)$$

for all $x \in \mathbb{R}$ and $m \in \mathbb{N}$.

We have $\{ r^m [f(x-m) - (q + p) f(x-m-1) + p q f(x-m-2)] \}$ is a Cauchy sequence ($|r| < 1$) for all $x \in \mathbb{R}$. Hence, we can define a function $H_2 : \mathbb{R} \rightarrow \mathbb{X}$ by

$$H_2(x) = \lim_{m \rightarrow \infty} r^m [f(x-m) - (q + p) f(x-m-1) + p q f(x-m-2)]$$

for all $x \in \mathbb{R}$. Using the above definition of $H_2$, we have

$$H_2(x - 1) + H_2(x - 2) + H_2(x - 3) =$$

$$r^{-1} \lim_{m \rightarrow \infty} r^{m+1} [f(x- (m+1)) - (q + p) f(x - (m+1) - 1) + p q f(x - (m+1) - 2)]$$

$$+ r^{-2} \lim_{m \rightarrow \infty} r^{m+2} [f(x- (m+2)) - (q + p) f(x - (m+2) - 1) + p q f(x - (m+2) - 2)]$$

$$+ r^{-3} \lim_{m \rightarrow \infty} r^{m+3} [f(x- (m+3)) - (q + p) f(x - (m+3) - 1) + p q f(x - (m+3) - 2)]$$

$$= r^{-1} H_2(x) + r^{-2} H_2(x) + r^{-3} H_2(x)$$

64
\[ = H_2(x) \text{ for all } x \in \mathbb{R}. \]

So, we can say that \(H_2\) is also a Tribonacci function. If \(m\) tends to \(\infty\), then from (2.2.7), we have

\[
\rho(f(x) - (q + p) f(x - 1) + qp f(x - 2) - H_2(x)) \leq \frac{1}{1 - |r|} \varepsilon = \frac{1}{1 - |q|} \varepsilon. \tag{2.2.8}
\]

for all \(x \in \mathbb{R}\). Finally, Analogous to (2.2.1), we obtain

\[
\rho([f(x) - r \{f(x - 1) - q f(x - 2)\} - q f(x - 1)]

- p[f(x - 1) - r f(x - 2) + q r f(x - 3) - q f(x - 2)]) \leq \varepsilon
\]

for all \(x \in \mathbb{R}\).

Now we replace \(x\) by \(x + \alpha\) in above inequality, that we have

\[
\rho(f(x + \alpha) - r \{f(x + \alpha - 1) - q f(x + \alpha - 2)\} - q f(x + \alpha - 1)

- p[f(x + \alpha - 1) - (r + q) f(x - \alpha - 2) + qr f(x + \alpha - 3)]) \leq \varepsilon
\]

and

\[
\rho(p^{-\alpha}[f(x + \alpha) - r \{f(x + \alpha - 1) - q f(x + \alpha - 2)\} - q f(x + \alpha - 1)]

- p^{-\alpha+1}[f(x + \alpha - 1) - (r + q) f(x - \alpha - 2) + qr f(x + \alpha - 3)])

\leq |\alpha^{-1}|^k \varepsilon \tag{2.2.9}
\]

for all \(x \in \mathbb{R}\) and \(\alpha \in \mathbb{Z}\). Applying (2.2.9), we obtain that

\[
\rho(p^{-m}[f(x + m) - r \{f(x + m - 1) - q f(x + m - 2)\} - q f(x + m - 1)]

- [f(x) - (r + q) f(x - 1) + rq f(x - 2)])

\leq \sum_{\alpha=1}^{m} \rho(p^{-\alpha}[f(x + \alpha) - r \{f(x + \alpha - 1) - q f(x + \alpha - 2)\} - q f(x + \alpha - 1)]

\]

65
\[- p^{-\alpha+1} [f(x + \alpha - 1) - (r + q) f(x + \alpha - 2) + qr f( x + \alpha - 3)] \]

\[ \leq \sum_{\alpha=1}^{m} p^{-\alpha} ( [f(x + \alpha) - r \{ f(x + \alpha - 1) - q f(x + \alpha - 2) \} - q f(x + \alpha - 1) ]
\]

\[ - p [f(x + \alpha - 1) - (r + q) f(x + \alpha - 2) + qr f( x + \alpha - 3)] \) \]

\[ \leq \sum_{\alpha=1}^{m} p^{-\alpha} \varepsilon \] \hspace{1cm} (2.2.10)

for all \( x \in \mathbb{R}, \ m \in \mathbb{N} \). We obviously have

\( \{ p^{m} [f(x + m) - (r + q) f(x + m - 1) + qr f(x + m - 2)] \} \) is a Cauchy sequence by definition of completeness for a fixed \( x \in \mathbb{R} \). Hence, we may define a function \( H_3 : \mathbb{R} \rightarrow X \) by

\[ H_3(x) = \lim_{m \to \infty} p^{m} [f(x + m) - (r + q) f(x + m - 1) + qr f(x + m - 2)] \]

for all \( x \in \mathbb{R} \). In view of above definition of \( H_3 \), we obtain

\[ H_3(x - 1) + H_3(x - 2) + H_3(x - 3) \]

\[ = p^{-1} \lim_{m \to \infty} p^{-(m-1)} [f(x + m - 1) - (r + q) f(x+(m-1) -1) + qr f(x+(m-1) -2)] \]

\[ + p^{-2} \lim_{m \to \infty} p^{-(m-2)} [f(x + m - 2) - (r + q) f(x+(m-2) -1) + qr f(x+(m-2) -2)] \]

\[ + p^{-3} \lim_{m \to \infty} p^{-(m-3)} [f(x + m - 3) - (r + q) f(x+(m-3) -1) + qr f(x+(m-3) -2)] \]

\[ = p^{-1} H_3(x) + p^{-2} H_3(x) + p^{-3} H_3(x) \]

\[ = H_3(x) \text{ for all } x \in \mathbb{R}. \]

Hence, we can say that \( H_3 \) is also a Tribonacci function. If we suppose, \( m \) tends to infinity in (2.2.10) then we have
\[ \rho(\mathcal{H}_3(x) - f(x) + (r + q) f(x - 1) - qr f(x - 2)) \leq \frac{\alpha^{-1}}{1 - |\alpha^{-1}|} \varepsilon \]  

(2.2.11)

for all \( x \in \mathbb{R} \). From (2.2.5), (2.2.7) and (2.2.11), we observe that

\[ \rho\left( f(x) - \left[ \frac{q^2(r-p)\mathcal{H}_1(x) + r^2(p-q)\mathcal{H}_2(x) - p^2(q-r)\mathcal{H}_3(x)}{q^2(r-p) + r^2(p-q) + p^2(q-r)} \right] \right) \]

\[ = \frac{1}{|q^2(r-p) + r^2(p-q) + p^2(q-r)|} \]

\[ \rho(\{q^2(r-p)+r^2(p-q)+p^2(q-r)\} f(x) - q^2(r-p)\mathcal{H}_1(x) - r^2(p-q) \mathcal{H}_2(x) + p^2(q-r) \mathcal{H}_3(x)) \]

Now, Let us suppose that

\[ \frac{1}{|q^2(r-p) + r^2(p-q) + p^2(q-r)|} = \frac{1}{|A|} \]  

(2.2.12)

\[ \leq \frac{1}{|A|} \rho(\{q^2(r-p)f(x) - q^2(r^2 - p^2)f(x - 1) + q^2(r-p)pr f(x - 2) - q^2(r-p)\mathcal{H}_1(x)\}) \]

\[ + [r^2(p-q) f(x) - r^2(p^2 - q^2) f(x - 1) + r^2(p-q)qp f(x - 2) - r^2(p-q)\mathcal{H}_2(x)] \]

\[ + [p^2(q-r) f(x) - p^2(q^2 - r^2) f(x - 1) + p^2(q-r)qr f(x - 2) - p^2(q-r)\mathcal{H}_3(x)]) \]

\[ \leq \frac{1}{|A|}\left[ \frac{1}{1 - |q|} + \frac{1}{1 - |q|} + \frac{|q^2|}{1 - |q^2|} \right] \varepsilon \]

\[ = \frac{1}{|A|}\left[ \frac{2}{1 - |q|} + \frac{|q^2|}{1 - |q^2|} \right] \varepsilon \]

\[ = \frac{1}{|A|} \left[ \frac{2(1 + |q|) + |q|^2}{1 - |q^2|} \right] \varepsilon \]

Putting the value of \( |A| \) from (2.2.12) we get the required result.

Hence,
for all \( x \in \mathbb{R} \). It is not difficult to show that \( H \) is a Tribonacci function satisfying (2.2.2).

### 2.3 Hyers-Ulam-Rassias Stability of \( k \)-Tribonacci Functional Equation in Modular Spaces

In 2011, M. Bidkhan and M. Hosslini [10] proved the stability of \( k \)-Fibonacci functional equation. Later on, M. Bidkhan et.al.[11] succeed to prove the Hyers-Ulam stability of \((k,s)\)-Fibonacci functional equation and in 2012, M. Gordji, M. Naderi and Th. M. Rassias [63] also proved the stability of Tribonacci functional equation in Non-Archimedean spaces. On the behalf of these results, we established the \( k \)-Tribonacci functional equation and proved their stability. Throughout the following section, we denote by \( F_{k,n} \) the \( n \)th \( k \)-Tribonacci number where

\[
F_{k,n} = k F_{k, n-1} + F_{k, n-2} + F_{k, n-3} \quad \text{for } n \geq 3
\]

with initial conditions \( F_{k,0} = 0, F_{k,1} = 1, F_{k,2} = 1 \). From this, we may derive a functional equation

\[
f(k, x) = k f(k, x - 1) + f(k, x - 2) + f(k, x - 3)
\]

which if called the \( k \)-Tribonacci functional equation if a function \( f : \mathbb{N} \times \mathbb{R} \rightarrow X \) satisfies the above equation for all \( x \in \mathbb{R}, k \in \mathbb{N} \), characteristic equation of the \( k \)th-Tribonacci sequence is \( x^3 - kx^2 - x - 1 = 0 \), and \( p, q, r \) denote the roots of characteristic equation where \( p \) is greater than one and \( q, r \in \mathbb{C} \) and \(|q| = |r|\).

We know that \( p + q + r = k, \quad pq + qr + pr = -1, \quad pqr = 1 \).

For each \( x \in \mathbb{R} \), \([x]\) stands for the largest integer that does not exceed \( x \).
**Theorem 2.3.1:** Let \((X, \rho)\) be a Banach Modular Space. If a function \(f : \mathbb{R} \to X\) satisfies the inequality

\[
\rho \left( f(k, x) - k f(k, x-1) - f(k, x-2) - f(k, x-2) \right) \leq \varepsilon \tag{2.3.1}
\]

for all \(x \in \mathbb{R}, \ k \in \mathbb{N}\) and for some \(\varepsilon > 0\), then there exists a \(k\)-Tribonacci function \(H : \mathbb{N} \times \mathbb{R} \to X\) such that

\[
\rho(f(k, x) - H(k, x)) \leq \frac{2(1 + |q|) + |q|^2}{\|q^2(r-p) + r^2(p-q) + p^2(q-r)\|} \times \frac{\varepsilon}{1-|q|^2}. \tag{2.3.2}
\]

**Proof:** Since, \(p + q + r = k, \ pq + qr + pr = -1\) and \(pqr = 1\). So from (**), we obtain

\[
\rho(f(k, x) - (p + q + r) f(k, x-1) + (pq + qr + pr) f(k, x-2) - pqr f(k, x-3)) \leq \varepsilon. \tag{2.3.3}
\]

for all \(x \in \mathbb{R}, \ k \in \mathbb{N}\). Now it follows from (2.3.1) that

\[
f(k, x) - p(f(k, x-1) - r f(k, x-2)) - r f(k, x-1) -
q[f(k, x-1) - (r + p) f(k, x-2) + pr f(k, x-3)] \leq \varepsilon \tag{2.3.4}
\]

for all \(k \in \mathbb{N}, \ x \geq 0\)

If we replace \(x\) by \(x - \alpha\) in inequality (2.3.4), then we get

\[
\rho(f(k, x - \alpha) - p[f(k, x - (\alpha - 1) - r f(k, x - \alpha - 2))] - r f(k, x -\alpha -1)
-q p f(k, x - \alpha -1) - (r + p) f(k, x - \alpha - 2) + pr f(k, x - \alpha - 3)) \leq \varepsilon
\]

for all \(x \in \mathbb{R}, \ k \in \mathbb{N}\).

Now multiplying both sides by \(q^\alpha\),

\[
\rho(q^\alpha [f(k, x - \alpha) - p(f(k, x - \alpha - 1) - r f(k, x - \alpha - 2))] - r f(k, x - \alpha - 1)]
\]
\[-q^{\alpha+1} [f(k, x - \alpha - 1) - (r + p) f(k, x - \alpha - 2) + pr f(k, x - \alpha - 3)]\]
\[\leq |q^\alpha| \rho ([f(k, x - \alpha) - p\{f(k, x - \alpha - 1) - r f(k, x - \alpha - 2)\} - r f(k, x - \alpha - 1)] \]
\[\quad - q[f(k, x - \alpha - 1) - (r + p) f(k, x - \alpha - 2) + pr f(k, x - \alpha - 3)])\]
\[\leq |q^\alpha| \varepsilon \tag{2.3.5}\]

for all \(x \in R, k \in N\). Furthermore, we have

\[\rho(f(k,x) - p\{f(k, x - 1) - r f(k, x - 2)\} - r f(k, x-1)\]
\[- q^m[f(k, x - m) - (r + p ) f(k, x - m-1) + pr f(k, x - m - 2)] )\]
\[\leq \rho \left( \sum_{\alpha=0}^{m-1} q^\alpha [f(k , x - \alpha) - p\{f(k, x - \alpha - 1) - r f(k, x - \alpha - 2)\} - r f(k, x - \alpha - 1)] \]
\[\quad - q^{\alpha+1} [f(k, x - \alpha - 1) - (r + p) f(k, x - \alpha - 2) + pr f(k, x - \alpha - 3))]]\]
\[\leq \sum_{\alpha=0}^{m-1} |q|^{\alpha} \rho ([f(k, x - \alpha) - p\{f(k, x - \alpha - 1) - r f(k, x - \alpha - 2)\} - r f(k, x - \alpha - 1)] \]
\[\quad - q[f(k, x - \alpha - 1) - (r + p) f(k, x - \alpha - 2) + pr f(k, x - \alpha - 3)])\]
\[\leq \sum_{\alpha=0}^{m-1} |q|^\alpha \varepsilon \leq \frac{\varepsilon}{1-|q|} \tag{2.3.6}\]

for all \(x \in R, m, k \in N\). Let \(x \in R\) be fixed, than (2.3.6) implies that \(\{q^m [f(k, x-m) - p (f(k, x-m-1) - r f(k, x-m-2)) - r f(k, x - m - 1)]\}\) is a Cauchy sequence (\(|q| < 1\)). So by the completeness of \(X\), we may define a function \(H_1 : R \to X\) such that

\[H_1(k, x) = \lim_{m \to \infty} q^m [f(k, x-m) - (p + r) f(k, x - m - 1) + pr f(k, x - m - 2)]\]

for all \(x \in R, k \in N\). Applying the definition of \(H_1\),we introduce the k-Tribonacci function
\[ k H_1(k, x - 1) + H_1(k, x - 2) + H_1(k, x - 3) = \\
\lim_{m \to \infty} k^{-1} q^{m+1} \left[f(k, x - (m+1)) - (p + r) f(k, x - (m+1)-1) + pr f(k, x - (m+1)-2)\right] \\
+ q^{-2} \lim_{m \to \infty} q^{m+2} \left[f(k, x - (m+2)) - (p + r) f(k, x - (m+1)-1) + pr f(k, x - (m+1)-2)\right] \\
+ q^{-3} \lim_{m \to \infty} q^{m+3} \left[f(k, x - (m+3)) - (p + r) f(k, x - (m+1)-1) + pr f(k, x - (m+1)-2)\right] \\
= k q^{-1} H_1(k, x) + q^{-2} H_1(k, x) + q^{-3} H_1(k, x) \\
= H_1(k, x) \text{ for all } x \in \mathbb{R}, k \in \mathbb{N}. \text{ Hence } H_1 \text{ is a } k\text{-Trionacci function.} \\

If \( m \to \infty \), then from (2.3.6) we obtain \\
\[ \rho(f(k, x) - (p + r) f(k, x - 1) + pr f(k, x - 2) - H_1(k, x)) \leq \frac{1}{1 - |q|} \varepsilon \quad (2.3.7) \]
for all \( x \in \mathbb{R}, k \in \mathbb{N} \). Furthermore, it follows from (2.3.1) that \\
\[ \rho(f(k, x) - q(f(k, x - 1) - p f(k, x - 2)) - p f(k, x - 1) \\
- r [f(k, x - 1) - p f(k, x - 2) + pq f(k, x - 3) - q f(k, x - 2)]) \leq \varepsilon \]
for all \( x \in \mathbb{R}, k \in \mathbb{N} \). Now, we replace \( x \) by \( x - \alpha \) in above inequality, we have \\
\[ \rho(f(k, x - \alpha) - q(f(k, x - \alpha - 1) - p f(k, x - \alpha - 2)) - p f(k, x - \alpha - 1) \\
- r [f(k, x - \alpha - 1) - p f(k, x - \alpha - 2) + pq f(k, x - \alpha - 3) - q f(k, x - \alpha - 2)]) \]
\[ \leq \varepsilon \]
and now multiplying by \( r^\alpha \) on both sides. \\
\[ \rho \left( r^\alpha [f(k, x - \alpha) - q(f(k, x - \alpha - 1) - p f(k, x - \alpha - 2)) - p f(k, x - \alpha - 1)] \\
- r^{\alpha+1} [f(k, x - \alpha - 1) - p f(k, x - \alpha - 2) + pq f(k, x - \alpha - 3) - q f(k, x - \alpha - 2)]) \]
\[
\leq \left| r^\alpha \right| \rho([f(k, x - \alpha) - q(f(k, x - \alpha - 1) - p f(k, x - \alpha - 2)) - p f(k, x - \alpha - 1)] \\
- r[f(k, x - \alpha - 1) - p f(k, x - \alpha - 2) + pq f(k, x - \alpha - 3) - q f(k, x - \alpha - 2)]) \\
\leq \left| r^\alpha \right| \varepsilon \quad (2.3.8)
\]
for all \(x \in \mathbb{R}, \alpha \in \mathbb{Z}\). Now, we have

\[
\rho(f(k, x) - q\{f(k, x - 1) - p f(k, x - 2)\} - p f(k, x - 1) \\
- r^m [f(k, x - m) - (q + p) f(k, x - m - 1) + p q (f(k, x - m - 2))] \\
\leq \rho(\sum_{\alpha=0}^{m-1} r^\alpha [f(k, x - \alpha) - q\{f(k, x - \alpha - 1) - p f(k, x - \alpha - 2)\} - p f(k, x - \alpha - 1)]) \\
- r^{\alpha+1}[f(k, x - \alpha - 1) - (p + q) f(k, x - \alpha - 2) + pq f(k, x - \alpha - 3)]) \\
\leq \sum_{\alpha=0}^{m-1} \left| r \right|^\alpha \rho( [f(k, x - \alpha) - q\{f(k, x - \alpha - 1) - p f(k, x - \alpha - 2)\} - p f(k, x - \alpha - 1)] \\
- r[f(k, x - \alpha - 1) - (p + q) f(k, x - \alpha - 2) + pq f(k, x - \alpha - 3)]) \\
\leq \sum_{\alpha=0}^{m-1} \left| r \right|^\alpha \varepsilon \leq \frac{\varepsilon}{1-\left| r \right|} \quad (2.3.9)
\]
for all \(x \in \mathbb{R}\) and \(m \in \mathbb{N}\).

We have \(\{r^m[f(k, x - m) - (q + p) f(k, x - m - 1) + pq f(k, x - m - 2)]\}\) is a Cauchy sequence \(||r|| < 1\) for all \(x \in \mathbb{R}\). Hence, we can define a function \(H_2: \mathbb{R} \to X\) by

\[
H_2(k,x) = \lim_{m \to \infty} r^m[f(k, x - m) - (q + p) f(k, x - m - 1) + pq f(k, x - m - 2)]
\]
for all \(x \in \mathbb{R}\). Using the above definition of \(H_2\), we have

\[
kH_2(k, x - 1) + H_2(k, x - 2) + H_2(k, x - 3) =
\]

72
\[
\lim_{m \to \infty} r^{m+1} [f(k, x - (m+1)) - (q + p) f(k, x - (m+1) - 1) + pq f(k, x - (m+1) - 2)]
\]

\[
+ \lim_{m \to \infty} r^{m+2} [f(k, x - (m+2)) - (q + p) f(k, x - (m+2) - 1) + pq f(k, x - (m+2) - 2)]
\]

\[
+ \lim_{m \to \infty} r^{m+3} [f(k, x - (m+3)) - (q + p) f(k, x - (m+3) - 1) + pq f(k, x - (m+3) - 2)]
\]

\[
= kr^{-1} H_2(k, x) + r^2 H_2(k, x) + r^3 H_2(k, x)
\]

\[
= H_2(k, x) \text{ for all } x \in \mathbb{R}.
\]

So, we can say that \(H_2\) is also a \(k\)-Tribonacci function. If \(m\) tends to \(\infty\), then from (2.3.9), we have

\[
\rho(f(k, x) - (q + p) f(k, x - 1) + qp f(k, x - 2) - H_2(k, x)) \leq \frac{1}{1-|r|} \varepsilon = \frac{1}{1-|q|} \varepsilon
\]

(2.3.10)

for all \(x \in \mathbb{R}\). Finally, Analogous to (2.3.1), we obtain

\[
\rho(f(k, x) - r (f(k, x - 1) - q f(k, x - 2)) - q f(k, x - 1)
\]

\[
- p [f(k, x - 1) - r f(k, x - 2) + q r f(k, x - 3) - q f(k, x - 2))] \leq \varepsilon
\]

for all \(x \in \mathbb{R}\). Now we replace \(x\) by \(x + \alpha\) in above inequality, then we have

\[
\rho(f(k, x + \alpha) - r \{f(k, x + \alpha - 1) - q f(k, x + \alpha - 2)\} - q f(k, x + \alpha - 1)
\]

\[
- p \left[ f(k, x + \alpha - 1) - (r + q) f(k, x - \alpha - 2) + qr f(k, x + \alpha - 3) \right] \leq \varepsilon
\]

and

\[
\rho(p^{-\alpha} f(k, x + \alpha) - r (f(k, x + \alpha - 1) - q f(k, x + \alpha - 2)) - q f(k, x + \alpha - 1)
\]

\[
- p^{-\alpha+1} \left[ f(k, x + \alpha - 1) - (r + q) f(k, x - \alpha - 2) + qr f(k, x + \alpha - 3) \right]
\]

\[
\leq |\alpha^{-1}|^k \varepsilon
\]

(2.3.11)
for all $x \in \mathbb{R}$ and $\alpha \in \mathbb{Z}$. Applying (2.3.11), we obtain that

$$\rho(p^m[f(k, x + m) - r(f(k, x + m - 1) - qf(k, x + m - 2)) - qf(k, x + m - 1)]$$

$$- [f(k, x) - (r + q)f(k, x - 1) + rqf(k, x - 2)])$$

$$\leq \sum_{\alpha=1}^{m} \rho(p^{\alpha}[f(k, x + \alpha) - r(f(k, x + \alpha - 1) - qf(k, x + \alpha - 2)) - qf(k, x + \alpha - 1)]$$

$$- p^{-\alpha+1} [f(k, x + \alpha - 1) - (r + q)f(k, x + \alpha - 2) + qr f(k, x + \alpha - 3)])$$

$$\leq \sum_{\alpha=1}^{m} p^{-\alpha} \rho ([f(k, x + \alpha) - r(f(k, x + \alpha - 1) - qf(k, x + \alpha - 2)) - qf(k, x + \alpha - 1)]$$

$$- p^{-\alpha+1} [f(k, x + \alpha - 1) - (r + q)f(k, x + \alpha - 2) + qr f(k, x + \alpha - 3)])$$

$$\leq \sum_{\alpha=1}^{m} p^{-\alpha} \epsilon$$  \hspace{1cm} (2.3.12)

for all $x \in \mathbb{R}$, $m \in \mathbb{N}$. By using (2.3.12) we see that

$\{p^m[f(k, x + m) - (r + q)f(k, x + m - 1) + qr f(k, x + m - 2)]\}$ is a Cauchy sequence by definition of completeness for a fixed $x \in \mathbb{R}$. Hence, we may define a function $H_3 : \mathbb{R} \to X$ by

$$H_3(k,x) = \lim_{m \to \infty} p^m[f(k, x + m) - (r + q)f(k, x + m + 1) + qr f(k, x + m - 2)]$$

for all $x \in \mathbb{R}$. In view of above definition of $H_3$, we obtain

$$kH_3(k, x - 1) + H_3(k, x - 2) + H_3(k, x - 3)$$

$$= k p^{-1} \lim_{m \to \infty} p^{(m-1)}[f(k, x + m - 1) - (r + q)f(k, x + (m-1) - 1) + qr f(k, x + (m-1) - 2)]$$

$$+ p^{-2} \lim_{m \to \infty} p^{(m-2)}[f(k, x + m - 2) - (r + q)f(k, x + (m-2) - 1) + qr f(k, x + (m-2) - 2)]$$

$$+ p^{-3} \lim_{m \to \infty} p^{(m-3)}[f(k, x + m - 3) - (r + q)f(k, x + (m-3) - 1) + qr f(k, x+(m-3) - 2)]$$

74
\[ \begin{align*}
&= k p^{-1} H_3(k, x) + p^{-2} H_3(k, x) + p^{-3} H_3(k, x) \\
&= H_3(k, x) \text{ for all } x \in \mathbb{R}, k \in \mathbb{N}.
\end{align*} \]

Hence, we can say that $H_3$ is also a $k$-Tribonacci function. If we suppose, $m$ tends to infinity in (2.3.12) then we have

\[ \rho(H_3(k, x) - f(k, x) + (r + q)(f(k, x - 1) - qr f(k, x - 2))) \leq \frac{\alpha^{-1}}{1 - |\alpha^{-1}|} \varepsilon \quad (2.3.13) \]

for all $x \in \mathbb{R}$. From (2.3.7), (2.3.10) and (2.3.13), we observe that

\[ \begin{align*}
\rho &\left( f(k, x) - \left[ \begin{array}{c}
q^2(r-p)H_1(k, x) + r^2(p-q)H_2(k, x) - p^2(q-r)H_3(k, x) \\
q^2(r-p) + r^2(p-q) + p^2(q-r)
\end{array} \right] \right) \\
&= \frac{1}{|q^2(r-p) + r^2(p-q) + p^2(q-r)|} \\
&= \frac{1}{|A|} \\
&= \frac{1}{1 - |q|} + \frac{1}{1 - |q^2|} + |q^2| \varepsilon
\end{align*} \]

For convince, we suppose that

\[ \begin{align*}
&\leq \frac{1}{|A|} \rho\left( q^2(r-p)f(k, x) - q^2(r^2 - p^2)f(k, x - 1) + q^2(r - p)f(k, x - 2) - q^2(r - p)H_1(k, x) \right) \\
&\quad + (r^2(p-q)f(k, x) - r^2(p^2 - q^2)f(k, x - 1) + r^2(p - q)f(k, x - 2) - r^2(p - q)H_2(k, x)) \\
&\quad + (p^2(q-r)f(k, x) - p^2(q^2 - r^2)f(k, x - 1) + p^2(q - r)f(k, x - 2) - p^2(q - r)H_3(k, x)) \\
&\leq \frac{1}{|A|} \left[ \frac{1}{1 - |q|} + \frac{1}{1 - |q^2|} + |q^2| \right] \varepsilon \\
&= \frac{1}{|A|} \left[ \frac{2}{1 - |q|} + \frac{|q^2|}{1 - |q^2|} \right] \varepsilon
\end{align*} \]
\[
\frac{1}{|A|}\left[\frac{2(1+|q|)+|q|^2}{1-|q|^2}\right]\varepsilon
\]

Putting the value of $|A|$ from (2.3.14) we get the required result.

Hence,

\[
H(k, x) = \frac{q^2(r-p)H_1(k, x) + r^2(p-q)H_2(k, x) - p^2(q-r)H_3(k, x)}{q^2(r-p) + r^2(p-q) + p^2(q-r)}
\]

for all $x \in \mathbb{R}$. It is easy to show that $H$ is a $k$-Tribonacci function satisfying (2.3.1).