Chapter 1

Historical Background

In this section, we first give an introduction of frames from historical point of view. “A fundamental approach to signal decomposition in terms of elementary signals was introduced by Gabor’s [77]. In signal representation we require some basic building blocks which could be effectively used for computational purposes. Gabor’s approach quickly became a paradigm for the spectral analysis associated with time-frequency methods, such as the short-time Fourier transform and the Wigner transform. Gabor’s idea required a tiling of the time-frequency domain (also called the information plane or phase space depending upon the area you work in) by non-overlapping half open rectangles. Gabor reasoned that certain optimal elementary signals should provide an efficient decomposition of the information plane. This decomposition should determine countably many components of the information plane with each component sufficiently localized in time and frequency so that a coefficient c associated with a component $R$ would characterize the amount of information from R in the signal. Moreover, it should not require smaller components to distinguish different types of information from $R$.”
Gabor considered a function
\[ g(t) = \pi^{-\frac{1}{2}} e^{-\frac{t^2}{2}} \in L^2(\mathbb{R}). \]

We call the function \( g \) as the window function.

Fix \( a, b \in \mathbb{R}^+ \). Then, the signals which are elementary in nature are given by
\[ \{ E_{mb}T_{na} \}_{m,n \in \mathbb{Z}}, \]
where \( E \) and \( T \) are modulation and translation operator on \( L^2(\mathbb{R}) \).

Duffin and Schaeffer [61] studied Gabor’s idea and abstracted it for studying signal processing and proposed frames (or Hilbert frames) for Hilbert spaces. A frame can be looked upon as a family of vectors (not necessarily linearly independent) which serves as building blocks for other vectors of the space and the members of the frame are able to recover every vector of the space fully.

The idea of Duffin and Schaeffer [61] did not generate much general interest outside of nonharmonic Fourier series. However, after the publication of the landmark paper in 1986 by Daubechies, Grossmann and Meyer [54], many researchers started noticing the importance and potential of the theory of frames and henceforth it began to be more widely studied.

“A sequence \( \{ f_k \} \) in a separable Hilbert space \( \mathcal{H} \) is called a frame (or Hilbert frame) for \( \mathcal{H} \) if there exists positive constants \( A \) and \( B \) such that
\[ A \| f \|^2 \leq \sum_{k=1}^{\infty} | \langle f, f_k \rangle |^2 \leq B \| f \|^2, \text{ for all } f \in \mathcal{H}. \]  

(1.0.1)

The positive constants \( A \) and \( B \) ensure the stability in the reconstruction of the vector under consideration.

“Traditionally, frames were used in signal and image processing, nonharmonic Fourier series, data compression, and sampling theory. But today, frame theory
has ever-increasing applications to problems in both pure and applied mathematics, physics, engineering, and computer science, to name a few. Since applications mainly require frames in finite-dimensional spaces, this will be our focus. In this situation, a frame is a spanning set of vectors which are generally redundant (overcomplete), requiring control of its condition numbers. Thus, a typical frame possesses more frame vectors than the dimension of the space, and each vector in the space will have infinitely many representations with respect to the frame. It is this redundancy of frames which is key to their significance for applications. The role of redundancy varies depending on the requirements of the applications at hand. First, redundancy gives greater design flexibility, which allows frames to be constructed to fit a particular problem in a manner not possible by a set of linearly independent vectors. For instance, in areas such as quantum tomography, classes of orthonormal bases with the property that the modulus of the inner products of vectors from different bases are a constant are required. A second example comes from speech recognition, when a vector needs to be determined by the absolute value of the frame coefficients (up to a phase factor). A second major advantage of redundancy is robustness. By spreading the information over a wider range of vectors, resilience against losses (erasures) can be achieved. A further advantage of spreading information over a wider range of vectors is to mitigate the effects of noise in the signal. The area of frame theory is very closely related to other research fields in both pure and applied mathematics. Hilbert (or Banach) frame theory- in particular, including the infinite dimensional situation - intersects functional analysis and operator theory. It also bears close relations to the area of applied harmonic analysis, in which the design of representation systems, typically by a careful partitioning of the Fourier domain, is one major objective. Some
mathematicians even consider frame theory as belonging to this area. Restricting to the finite-dimensional situation - in which customarily the term finite frame theory is used - the classical areas of matrix theory and numerical linear algebra have close intersections, but also, for instance, the novel area of compressed sensing, as already pointed out.

For a nice introduction to frames one may refer to [3,4,6,9-13,18,19,21,22,24,26,28, 31-36,38,39,42,43,47-50,52,53,55-57,61-63,67,68,72-75,83,84,91-95, 107,110,111,122, 125, 128,132,135,136,143,145,146,150,151,167,169,172-175].

“Gröchenig in [82] generalized Hilbert frames to Banach spaces. Before the concept of Banach frames was formalized, it appeared in the foundational work of Feichtinger and Grochenig [69,70,71] related to atomic decompositions. Feichtinger and Grochenig studied the atomic decomposition via integrable group representation. Atomic decompositions appeared in the field of applied mathematics providing many applications [37,38,39,40,41]. An atomic decomposition allow a representation of every vector of the space via a series expansion in terms of a fixed sequence of vectors which we call atoms. On the other hand Banach frame for a Banach space ensure reconstruction via a bounded linear operator or synthesis operator. During the development of frames and expansions systems (redundant building blocks), in the later half of twentieth century, Coifman and Weiss in [51] introduced the notion of atomic decomposition for function spaces. This concept was further generalized by Grochenig [82] who introduced the notion of Banach frames for Banach spaces. Recently, various generalization of frames in Banach spaces have been introduced and studied. Han and Larson [90] defined a Schauder frame for a Banach space X to be
an inner direct summand (i.e. a compression) of a Schauder basis of X. Schauder frames were further studied in [121,160]. Casazza, Han and Larson [29] also carried out a study of atomic decompositions and Banach frames.’’

Regarding existence of Banach frames in Banach spaces, they proved that

“Every separable Banach space has a Banach frame with frame bounds $A = B = 1.$’’

This result was further improved in [101], where the separability for a Banach space to admit Banach frame is not required. Casazza et al. in [29], classify atomic decompositions in terms of bases for Banach spaces. They also gave a relationship between atomic decompositions and approximation property for Banach spaces.

“For a Banach space $X$, Casazza et al. proved that

(i) $X$ has an atomic decomposition.

(ii) $X$ has a finite dimensional expansion of the identity.

(iii) $X$ is complemented in a Banach space with a basis.

(iv) $X$ has the bounded approximation property.”

For recent development in frames in Banach spaces one may refer to [45,46,117,118, 119,120,121,160,161,162]. Retro Banach frames were introduced and studied in [100]. The reconstruction property in Banach spaces was introduced and studied by Casazza and Christensen in [30] and further studied in [117,163]. The reconstruction property is an important tool in several areas of mathematics and engineering. As the perturbation result of Paley and Wiener preserves reconstruction property, it becomes more important from an application. Construction of new redundant building blocks (e.g. frames) from a given redundant system attracts engineers and scientists [40].
Aldroubi in [1], considered a frame (Hilbert frame) \( \{ f_k \} \) for a Hilbert space \( \mathcal{H} \). He introduced two methods for generating frames of a Hilbert space \( \mathcal{H} \). In one of his methods, one approach is to construct frames for \( \mathcal{H} \) which are images of a given frame for \( \mathcal{H} \) under \( T \in \mathcal{B}(\mathcal{H}, \mathcal{H}) \). The other method uses bounded linear operator on \( \ell^2 \) to generate frames of \( \mathcal{H} \). He gave necessary and sufficient conditions for the image of a frame under bounded linear operator on \( \mathcal{H} \) to be a frame for \( \mathcal{H} \). More precisely, Aldroubi [1] proved that the sequence \( \{ \Theta_n = T(f_n) \} \) is a frame for \( \mathcal{H} \) if and only if the adjoint operator \( T^* \) is coercive. By using the idea of construction of frames from bounded linear operators, a construction to create frames for subspaces of the given Hilbert space can be found in [1].

Furthermore, Aldroubi proved that how to construct all frames of a given Hilbert space, starting from any given one. A characterization of all the mappings that transform frames into other frames for Hilbert spaces can be found in [1]. By using a result of Aldroubi [1] explaining how one can map a frame onto another using a bounded operator \( T \) on \( \ell^2 \), Casazza and Christensen [27] studied the overcompleteness of the frames in terms of various properties of the operator \( T \). Casazza [20] proved that

“Every frame for a Hilbert space \( \mathcal{H} \) can be written as a (multiple of a) sum of three orthonormal bases for \( H \).”

A sequence \( \{ f_k \} \) in a separable Hilbert space \( \mathcal{H} \) is said to be

(i) **Riesz basis** for \( \mathcal{H} \) if, it is complete in \( \mathcal{H} \) and there exists positive finite constants \( a_o, b_o \) such that

\[
 a_o \sum |c_k|^2 \leq \sum \| c_k f_k \|^2 \leq b_o \sum |c_k|^2
\]

for all finite scalar sequences \( \{ c_k \} \).
He also proved by using a result of Kalton [112] that

“A frame can be represented as a linear combination of two orthonormal bases if and only if it is a Riesz basis.”

Holub [98] introduced the notion of near-Riesz basis and Besselian frames. A frame \( \{f_k\} \) for a Hilbert space \( \mathcal{H} \) is said to be

(i) a near-Riesz basis for \( \mathcal{H} \) if, there is a finite set \( \sigma \) for which \( \{f_k\}_{n\notin \sigma} \) is a Riesz basis for \( \mathcal{H} \).

(ii) Besselian, if whenever \( \sum a_k f_k \) converges in \( H \), then \( \{a_n\} \in \ell^2 \).

Holub proved the following equivalent statements for a frame \( \{f_k\} \) for \( \mathcal{H} \).

(i) \( \{f_k\} \) is a near-Riesz basis for \( \mathcal{H} \).

(ii) \( \{f_k\} \) is Besselian.

(iii) \( \sum_{k=1}^{\infty} a_k f_k \) converges in \( H \) if and only if \( \{a_k\} \in \ell^2 \).

Walnut [164] discussed the stability of frame for Hilbert spaces. It was further studied by Cassazza et al [25,27], Favier and Zalik [66]. Casazza and Christensen [27] studied perturbation of frames in the sense that two frames are ‘close” if a certain operator is compact. This leads to an equivalence relation on the set of frames with the property that frames in the same equivalence class have the same overcompleteness. They also proved that Paley-Wiener type perturbation does not have this property. The stability for Banach frames in Banach spaces were studied in [102]. Balan [5] gave various stability results for Fourier frames and wavelet Riesz bases.

Casazza and Christensen in [30], introduced the reconstruction property in Banach spaces. Regarding the existence of Banach spaces which admits the reconstruction property, they proved that there exists a Banach space \( X \) with the following property.
There exists a sequence in $X$ such that each vector in $X$ can be represented as an infinite series with respect to the given sequence.

$X$ does not have the reconstruction property with respect to any sequence in $X$.

Casazza and Christensen pointed out a problem in [30]:

“Suppose that each vector of a Banach space $X$ is expressed as an infinite linear combination of a given sequence. What condition(s) on $X$ guarantee the existence of the reconstruction property for $X$”?

Kaushik et al. in [117], gave sufficient conditions for the existence of the reconstruction property in $X$. They, also gave some perturbation results for the reconstruction property in Banach spaces. Some new types of reconstruction system are presented in [117]. For recent development in the reconstruction property in Banach spaces one may refer to [117,163].

In the present thesis, our aim is to study and enhance frames for Banach spaces. The thesis as such has been divided into six chapters.

In Chapter 2, we present a background material from Hilbert space and Banach space theory. After this preparation, we give an overview of well-known results on frames for Hilbert spaces, Banach frames and retro Banach frames.

In Chapter 3, we study dual of retro Banach frames in Banach spaces. In this chapter we define a type of duality for retro Banach frames, namely $\Lambda$-type duality for retro Banach frames. Some examples and counterexamples regarding existence of $\Lambda$-dual retro Banach frames in Banach spaces are given. Necessary and sufficient conditions for the existence of dual of retro Banach frames for Banach spaces is given. Applications of the duality of retro Banach frames is discussed. Finally, we presents
an application of Young’s result in $\Lambda$-dual of retro Banach frames.

In Chapter 4, we study the concept of Banach $\Xi$-frame for operator spaces. Banach $\Xi$-frames can be generated from the reconstruction property in Banach spaces. There may be other sequence spaces with respect to which Banach $\Xi$-frames exists. Necessary and sufficient conditions for the existence of Banach $\Xi$-frame for operator spaces are given. A method of construction of Banach $\Xi$-frame for operator spaces from operators on associated Banach spaces of sequences is discussed. A Paley-Wiener type perturbation of Banach $\Xi$-frame for operator spaces is studied.

Chapter 5 is devoted to the study of excess of retro Banach frames in Banach spaces. A necessary and sufficient condition for the excesses of retro Banach frames in Banach spaces is obtained.

Finally in Chapter 6, we study construction of retro Banach frames for Banach spaces from operators on Hilbert (and Banach spaces). A necessary and sufficient condition for the existence of a retro Banach frame as the image of a given retro Banach frame under bounded linear operator on the underlying space is given. A characterization for the construction of retro Banach frames from bounded linear operators on the underlying space with respect to the same associated Banach space of scalar valued sequences is obtained. A compact linear operator generated from a given retro Banach frame is studied. It is proved that the image of a retro Banach frame under compact operator generated by the given retro Banach frame is a retro Banach frame for the underlying space. This is, not true, in general, for arbitrary compact operator. Some examples are given in this direction. Schauder frames associated with eigenfunctions of a boundary value problem (BVP) are discussed. Also, it has been proved that a compact operator can be generated if Schauder frames associated with
BVP are “sufficiently closed”. A perturbation problem related to Schauder frames is discussed. A sufficient condition for the existence of the Schauder frame with respect to the perturbed system associated with a given Schauder frame for a Banach space is given.