Chapter 6

The Shadow of Operators on Frames

6.1 Introduction

In this chapter, we discuss the behaviour of a bounded linear operator on retro Banach frames for Hilbert (Banach) spaces. It is well known that the image of a Hilbert frame (or Banach frame) under a bounded linear operator on the given Hilbert (or Banach) space need not be a Hilbert frame (or Banach frame) for the underlying space. Aldroubi in [1], gave two fundamental methods for generating Hilbert frames for a Hilbert space $\mathcal{H}$. In one of his method, one approach is to construct frames for $\mathcal{H}$ which are images of a given frame for $\mathcal{H}$ under $T \in B(\mathcal{H}, \mathcal{H})$. The other method uses bounded linear operator on $\ell^2$ to generate frames of $\mathcal{H}$. Motivated by the work of Aldroubi, in this chapter we discuss construction of retro Banach frames which are also images of given retro Banach frames under bounded linear operators on Hilbert (Banach) spaces. In Section 6.2, we discuss the construction of retro Banach frames for Hilbert spaces from bounded linear operators on Hilbert spaces. A necessary and sufficient condition in this direction is given. A characterization
for the construction of retro Banach frames from bounded linear operators on the underlying space is obtained. Section 6.3 is devoted to the study of compact linear operator from given retro Banach frames for Banach spaces. It is proved that the image of a retro Banach frame under a compact linear operator generated by the given retro Banach frame is a retro Banach frame. This is not true, in general, for arbitrary compact linear operator. Some examples are given in this direction. In Section 6.4, we discuss Schauder frames associated with eigenfunctions of a boundary value problem (BVP). We show that a compact linear operator can be generated if, Schauder frames associated with BVP are “sufficiently closed”. A perturbation problem related to Schauder frames is discussed. A sufficient condition for the existence of the Schauder frame with respect to the perturbed system associated with a given Schauder frame for a Banach space is given. The contents of this chapter are published in [46].

6.2 Operators on $\mathcal{H}$ for the Construction of Retro Banach Frames

Aldroubi in [1], gave fundamental methods for generating Hilbert frames for a Hilbert space $\mathcal{H}$. Let $\{f_n\} \subset \mathcal{H}$ be a frame for $\mathcal{H}$. In one of his method Aldroubi considered the sequence $\{\Theta_n = T(f_n)\}$, where $T \in \mathcal{B}(\mathcal{H}, \mathcal{H})$. Aldroubi proved that the system $\Theta_n = T(f_n)$ ($n \in \mathbb{N}$) is a frame for $\mathcal{H}$ if and only if the adjoint operator $T^*$ is coercive.

**Theorem 6.2.1.** [1] Let $\{f_n\}$ be a frame for $\mathcal{H}$ with frame bounds $0 < A \leq B$. If $T$ is a bounded linear operator from $\mathcal{H}$ into $\mathcal{H}$, then $\{\Theta_n = T(f_n)\}$ is a frame for $\mathcal{H}$ if and only if there exists a positive constant $\gamma$ such that the adjoint operator $T^*$
satisfies

\[ \|T^*(f)\|^2 \geq \gamma \|f\|^2, \text{ for all } f \in \mathcal{H}. \]

**Remark 6.2.1.** Let \( \mathcal{F} = (\{f_n\}, \Theta) \) be a retro Banach frame for \( \mathcal{H}^* \) with respect to \( Z_d = \ell^2 \) with frame bounds \( 0 < A, B < \infty \) and let \( Q \in \mathcal{B}(\mathcal{H}, \mathcal{H}) \). Then, Theorem 6.2.1, provides necessary and sufficient conditions for the existence of the reconstruction operator \( \hat{\Theta} \) such that \( \mathcal{G} \equiv (\{Q(f_n)\}, \hat{\Theta}) \) is a retro Banach frame for \( \mathcal{H}^* \) with respect to \( Z_d = \ell^2 \). More precisely, from the idea given in the proof of sufficient part of Theorem 6.2.1, one of the choice for retro frame bounds of \( \mathcal{G} \) are found to be \( \sqrt{\gamma A}, \sqrt{B}\|Q^*\| \). Aldroubi [1], gave construction of Hilbert frames for subspaces. By using certain ideas given in the proof of Theorem 2 in [1, p. 1663], we can construct a retro Banach frame for subspaces \( \mathcal{H}_1 \subset \mathcal{H} \) (from a given retro Banach frame for \( \mathcal{H} \)). Other results proved by Aldroubi in [1], are also useful in the construction of retro Banach frames for \( \mathcal{H}^* \) with respect to \( Z_d = \ell^2 \). Aldroubi in [1], gave applications of construction of frames from bounded linear operators in frames, e.g. affine frames, multiresolution and wavelet theory.

In this section, we discuss the construction of retro Banach frames for \( \mathcal{H}^* \) as image of a given retro Banach frame under \( Q \in \mathcal{B}(\mathcal{H}, \mathcal{H}) \). Let \( \mathcal{F} = (\{f_n\}, \Theta) \) be a retro Banach frame for \( \mathcal{H}^* \) with respect to \( Z_d \) and let \( Q \in \mathcal{B}(\mathcal{H}, \mathcal{H}) \). Then, in general, there exists no reconstruction operator \( \hat{\Theta} \) such that \( (\{Q(f_n)\}, \hat{\Theta}) \) is a retro Banach frame for \( \mathcal{H}^* \). First we give an example which shows that the image of a given retro Banach frame under bounded linear operator on \( \mathcal{H} \), need not be a retro Banach frame.

**Example 6.2.2.** Consider the discrete signal space \( \mathcal{H} = L^2(\Omega, \mu) \), where \( \Omega \) is the set of all positive integers and \( \mu \) is the counting measure.
Let $\{f_k\} \subset \mathcal{H}$ be a sequence given by

$$f_1 = \chi_1 \text{ and } f_k = \chi_{k-1}, \ k > 1,$$

where $\chi_k = \{0, 0, 0, \ldots, 1, 0, 0, \ldots\}$. Choose $Z_d = \{\{f^*(f_k)\} : f^* \in \mathcal{H}^*\}$. Then, $Z_d$ is an associated Banach space with norm

$$\|\{f^*(f_k)\}\|_{Z_d} = \|f^*\|_{\mathcal{H}^*}, \ f^* \in \mathcal{H}^*.$$

Define $\Theta : Z_d \to \mathcal{H}^*$ by

$$\Theta(\{f^*(f_k)\}) = f^*, \ f^* \in \mathcal{H}^*.$$

Then, $\mathcal{F} \equiv (\{f_k\}, \Theta)$ is a retroBanachframe for $\mathcal{H}^*$.

Let $\mathcal{Q} \in \mathcal{B}(\mathcal{H}, \mathcal{H})$ be given by

$$\mathcal{Q}(f) = \mathcal{Q}(\{\eta_1, \eta_2, \eta_3, \ldots\}) = \{0, 0, \eta_3, \ldots\}, \ f = \{\eta_j\} \in \mathcal{H}.$$

Then, there exists no reconstruction operator $\hat{\Theta}$ due to which $\mathcal{G} \equiv (\{\mathcal{Q}(f_k)\}, \hat{\Theta})$ is a retroBanachframe for $\mathcal{H}^*$. Indeed, let $A^0, \ B^0$ be a choice of retro frame bounds for retro Banach frame $(\{\mathcal{Q}(f_k)\}, \hat{\Theta})$.

Then

$$A^0\|f^*\| \leq \|\{f^*(\mathcal{Q}(f_k))\}\|_{Z_d} \leq B^0\|f^*\|, \ \text{for all } f^* \in \mathcal{H}^*. \quad (6.2.1)$$

In particular for $f^* = \chi_1$, we have $f^*(\mathcal{Q}(f_k)) = 0, k \in \mathbb{N}$. Therefore, by in view of the retroframe inequality (6.2.1), we obtain $f^* = 0$, a contradiction.

Remark 6.2.2. Let $\mathcal{Q}_0 \in \mathcal{B}(\mathcal{H}, \mathcal{H})$ be the operator given by

$$\mathcal{Q}_0(f) = \mathcal{Q}_0(\{\eta_1, \eta_2, \eta_3, \ldots\}) = \left\{\frac{\eta_j}{j}\right\}, \ f = \{\eta_j\} \in \mathcal{H}.$$
Then, $\hat{\Theta}_0$ is a reconstruction operator such that $\mathcal{G}_0 \equiv (\{\mathcal{Q}_0(f_k)\}, \hat{\Theta}_0)$ is a retroBanachframe for $\mathcal{H}^*$ with respect to $Z_0 = \{f^*(\mathcal{Q}_0(f_k)) : f^* \in \mathcal{H}^*\}$.

The first result of this section provides a necessary condition for the construction of retroBanachframes for $\mathcal{H}^*$ from bounded linear operators on $\mathcal{H}$. This can be generalized to construction of Hilbert frames for Hilbert spaces.

**Theorem 6.2.3.** Let $\mathcal{F} \equiv (\{f_k\}, \Theta)$ be a retro Banach frame for $\mathcal{H}^*$ with respect to $Z_d$ and let $Q \in \mathcal{B}(\mathcal{H}, \mathcal{H})$. If there exists a reconstruction operator $\hat{\Theta}$ such that $\mathcal{G} \equiv (\{Q(f_k)\}, \hat{\Theta})$ is a retro Banach frame for $\mathcal{H}^*$, then

$$\sup_{\xi \in (0, \infty)} \|\xi(I + QQ^*)^{-1}\| \leq 1,$$

(6.2.2)

where $I$ is the identity operator on $\mathcal{H}$.

**Proof.** Let $\mathcal{G}$ be a retroBanachframe for $\mathcal{H}^*$ with respect to the associated Banachspace $Z_Q$, then one can find constants $A_0$ and $B_0$ with $0 \leq A_0, B_0 < \infty$ such that

$$A_0\|f^*\| \leq \|\{f^*(Q(f_k))\}\|_{Z_Q} \leq B_0\|f^*\|, \text{ for each } f^* \in \mathcal{H}^*.$$

(6.2.3)

If $\langle QQ^*f, f \rangle = 0$ for some $f \in \mathcal{H}$, then $\|Q^*f\| = 0$. Therefore, by using retro frame inequality (6.2.3), we have $f = 0$. Indeed, by using the fact that $(\text{Ker}Q^*)^\perp = [\text{Ran}Q]$, lower inequality in (6.2.3) gives $\text{Ker}Q^* = 0$, i.e. $Q^*$ is injective.

Thus, $\|Q^*f\| = 0$ implies that $f = 0$.

That is

$$\langle QQ^*f, f \rangle > 0, \text{ whenever } f \neq 0.$$

Therefore

$$\langle (QQ^* + \xi I)f, f \rangle = \langle QQ^*f, f \rangle + \xi\langle f, f \rangle$$
\[ \geq \xi \|f\|^2, \text{ whenever } f \neq 0, \ \xi \in (0, \infty). \quad (6.2.4) \]

By using Cauchy Schwartz inequality, (6.2.4) gives
\[
\| (QQ^* + \xi I) f \| \| f \| \geq \langle (QQ^* + \xi I) f, f \rangle
\]
\[
\geq \xi \| f \|^2, \text{ whenever } f \neq 0, \ \xi \in (0, \infty).
\]

Therefore
\[
\| (QQ^* + \xi I) f \| \geq \xi \| f \|, \text{ whenever } f \neq 0, \ \xi \in (0, \infty).
\]

This gives
\[
\xi \| (QQ^* + \xi I)^{-1} f \| \leq \| f \|, \text{ whenever } f \neq 0, \ \xi \in (0, \infty).
\]

Hence
\[
\sup_{\xi \in (0, \infty)} \|\xi I + QQ^*\|^{-1} \leq 1.
\]

Remark 6.2.3. The condition (6.2.2) in Theorem 6.2.3 is not sufficient. Indeed, let
\( Q = O \) be the zero operator on \( \mathcal{H} \).

Then
\[
\sup_{\xi \in (0, \infty)} \|\xi I + QQ^*\|^{-1} = 1.
\]

But there exists no reconstruction operator \( \hat{\Theta} \) which makes the pair \( \{Q(f_k)\}, \hat{\Theta} \) is a retroBanachframe for \( \mathcal{H}^* \) with respect to any associatedBanachspace of scalar valued sequences.
Recall that if, \( F \equiv \{f_k\}, \Theta \) is a retro Banach frame for \( H^* \), then, in general, there exists no reconstruction operator \( \hat{\Theta} \) which makes the pair \((\{Q(f_k)\}, \hat{\Theta})\) a retro Banach frame for \( H^* \) with respect to some associated Banach space of scalar valued sequences (see Example 6.2.2). Regarding construction of Hilbert frames for Hilbert spaces from bounded linear operators, Aldroubi in one of his methods gave an explicit form of the operator under consideration in [1] (see at page 1663).

The following theorem gives sufficient condition for the existence of a reconstruction operator to construct a new retro Banach frame.

**Theorem 6.2.4.** Let \( F \equiv \{f_k\}, \Theta \) \((\{f_k\} \subset H, \Theta : Z_d \to H^*)\) be a retro Banach frame for \( H^* \) with respect to an associated sequence space \( Z_d \). Let \( T \in \mathcal{B}(H, H) \) be such that

\[
\|\xi(I + TT^*)^{-1}\| < 1, \text{ for some } \xi > 0,
\]

where \( I \) is the identity operator on \( H \). Then, there exists a \( Q \in \mathcal{B}(H, H) \) and a reconstruction operator \( \hat{\Theta} \) such that \((\{Q(f_k)\}, \hat{\Theta})\) is a retro Banach frame for \( H^* \) with respect to some associated Banach space of scalar valued sequences.

**Proof.** Assume that \( \|\xi(I + TT^*)\| < 1, \) for some \( \xi > 0. \)

We compute

\[
TT^*(\xi I + TT^*)^{-1} = ((\xi I + TT^*) - \xi I)(\xi I + TT^*)^{-1}
\]

\[
= I - \xi(I + TT^*)^{-1}.
\]

By using (6.2.5), the operator \( \xi(I + TT^*)^{-1} \) is a bounded linear operator on \( H \) such that

\[
\|\xi(I + TT^*)^{-1}\| < 1.
\]
Thus, $I - \xi(I + TT^*)^{-1}$ is an invertible operator on $\mathcal{H}$. Therefore, by using (6.2.6), we conclude that $TT^*(I + TT^*)^{-1}$ is invertible.

Choose $Q = TT^*(I + TT^*)^{-1}$.

Then, $Q$ is a bounded linear invertible operator on $\mathcal{H}$. By the invertibility of $Q$ and the fact that $\mathcal{F}$ is a retro Banach frame for $\mathcal{H}^*$, it is easy to verify that $\mathcal{Z}_Q = \{f^*(Q(f_k)) : f^* \in \mathcal{H}^*\}$ is a Banach space with the norm given by

$$\|\{f^*(Q(f_k))\}\| = \|f^*\|, \quad f^* \in \mathcal{X}^*.$$ 

Define $\widehat{\Theta} : \mathcal{Z}_Q \to \mathcal{H}^*$ by

$$\widehat{\Theta}(|f^*(Q(f_k))|) = f^*.$$ 

Then, $\widehat{\Theta}$ is a bounded linear operator such that $(\{Q(f_k)\}, \widehat{\Theta})$ is a retro Banach frame for $\mathcal{H}^*$ with respect to $\mathcal{Z}_Q$.

Remark 6.2.4. The condition (6.2.5) in Theorem 6.2.4 can not be relaxed. Indeed, let $T = O$ be the zero operator on $\mathcal{H}$.

Then

$$\|\xi(I + TT^*)^{-1}\| = 1, \quad \text{for all } \xi > 0.$$ 

But there exists no reconstruction operator $\widehat{\Theta}$ such that $(\{Q(f_k)\}, \widehat{\Theta})$ is a retro Banach frame for $\mathcal{H}^*$ with respect to some associated Banach space of scalar valued sequences.

Remark 6.2.5. A necessary and sufficient condition for the construction of retro Banach frames which are images of a given retro Banach frame under $Q \in \mathcal{B}(\mathcal{H}, \mathcal{H})$ is obtained by Theorem 6.2.3 and Theorem 6.2.4.

In the definition of a retro Banach frame, three Banach spaces are involved. One of the Banach space is the associated Banach space of scalar valued sequences with
respect to which a certain system admit retro Banach frame for the underlying space. Let \( \mathcal{F}_o \) be a retro Banach frame for \( \mathcal{H}^* \) with respect to an associated sequence space \( \mathcal{Z}_d \). It would be interesting to know whether the image of \( \mathcal{F}_o \) (under a bounded linear operator on \( \mathcal{H} \)) constitutes a retro Banach frame for \( \mathcal{H}^* \) with respect to \( \mathcal{Z}_d \). This is a deep problem in some sense. The following theorem provides necessary and sufficient conditions for the construction of retro Banach frames from bounded linear operators on the underlying space with respect to the same associated Banach space of scalar valued sequences.

**Theorem 6.2.5.** Let \( \mathcal{F} \equiv (\{f_n\}, \Theta) (\{f_n\} \subset \mathcal{H}, \Theta : \mathcal{Z}_d \to \mathcal{H}^*) \) be a retro Banach frame for \( \mathcal{H}^* \) with respect to an associated sequence space \( \mathcal{Z}_d \) and let \( \mathcal{Q} \in \mathcal{B}(\mathcal{H}, \mathcal{H}) \). Assume that \( \mathcal{W} \in \mathcal{B}(\mathcal{Z}_d, \mathcal{Z}_d) \) is such that for all \( f^* \in \mathcal{H}^* \), \( \mathcal{W} : \{f^*(f_n)\} \to \{f^*(\mathcal{Q}(f_n))\} \). Then, there exists a bounded linear operator \( \widehat{\Theta} \) such that \( (\{\mathcal{Q}(f_n)\}, \widehat{\Theta}) \) is a retro Banach frame for \( \mathcal{H}^* \) with respect to \( \mathcal{Z}_d \) if and only if

\[
\|\mathcal{W}(\{f^*(f_n)\})\|_{\mathcal{Z}_d} \geq c\|\mathcal{J}(\{f^*(\mathcal{Q}(f_n))\})\|_{\mathcal{Z}_d}, \text{ for all } f^* \in \mathcal{H}^*,
\]

(6.2.7)

where \( \mathcal{J} \in \mathcal{B}(\mathcal{Z}_d, \mathcal{Z}_d) \) is an operator such that for all \( f^* \in \mathcal{H}^* \), \( \mathcal{J} : \{f^*(\mathcal{Q}(f_n))\} \to \{f^*(f_n)\} \) and \( c \) is a positive constant.

**Proof.** Suppose first that \( \mathcal{F}_Q \equiv (\{\mathcal{Q}(f_n)\}, \widehat{\Theta}) \) is a retro Banach frame for \( \mathcal{H}^* \) with respect to \( \mathcal{Z}_d \). Let \( A_Q \) and \( B_Q \) be a choice of retro frame bounds for \( \mathcal{F}_Q \) and let \( \mathcal{P} : \mathcal{H}^* \to \mathcal{Z}_d \) be the analysis operator associated with \( \mathcal{F} \) which is given by

\[
\mathcal{P} : f^* \to \{f^*(f_n)\}, f^* \in \mathcal{H}^*.
\]

Choose \( \mathcal{J} = \mathcal{P}\widehat{\Theta} \) and \( c = \frac{A_Q}{\|\mathcal{P}\|} > 0 \).

Then

\[
\|\mathcal{W}(\{f^*(f_n)\})\|_{\mathcal{Z}_d} = \|\{f^*(\mathcal{Q}(f_n))\}\|_{\mathcal{Z}_d}
\]
\[ A_0 \| f^* \| \geq A_0 \| f^* \| \]
\[ \geq c \| \{ f^*(f_n) \} \|_{z_d} \]
\[ = c \| J(\{ f^*(Q(f_n)) \}) \|_{Z_d}, \text{ for all } f^* \in \mathcal{H}^*. \]

For the reverse part, suppose that (6.2.7) is satisfied.

We compute
\[
\| \{ f^*(Q(f_n)) \} \|_{Z_d} = \| \mathcal{W}(\{ f^*(f_n) \}) \|_{Z_d}
\leq \| \mathcal{W} \| \| \{ f^*(f_n) \} \|_{Z_d}
\leq \| \mathcal{W} \| \| P \| \| f^* \|, \text{ for all } f^* \in \mathcal{H}^*, \tag{6.2.8}
\]

By using (6.2.7), we have
\[
cA \| f^* \| \leq c \| \{ f^*(f_n) \} \|_{Z_d} \quad (\text{where } A \text{ is lower retro frame bound of } \mathcal{F})
\]
\[ = c \| J(\{ f^*(Q(f_n)) \}) \|_{Z_d}
\leq \| \mathcal{W}(\{ f^*(f_n) \}) \|_{Z_d} \quad (= \| \{ f^*(Q(f_n)) \} \|_{Z_d}) \tag{6.2.9}
\]

Set \( a_0 = cA \) and \( b_0 = \| \mathcal{W} \| \| P \| \). Then, by using (6.2.8) and (6.2.9), we have
\[
a_0 \| f^* \| \leq \| \{ f^*(Q(f_n)) \} \|_{Z_d} \leq b_0 \| f^* \|, \text{ for each } f^* \in \mathcal{H}^*.
\]

Choose \( \hat{\Theta} = \Theta J \). Then, \( \hat{\Theta} \in \mathcal{B}(Z_d, \mathcal{H}^*) \) is such that
\[
\hat{\Theta}(\{ f^*(Q(f_n)) \}) = f^*, \text{ for all } f^* \in \mathcal{H}^*.
\]

Hence \( \{ Q(f_n) \}, \hat{\Theta} \) is a retro Banach frame for \( \mathcal{H}^* \) with respect to \( Z_d \) and with one of the choice of retro frame bounds \( a_0, b_0 \).
6.3 Compact Operators Generated by Retro Banach Frames

We begin this section with the following counterexample.

"There exists a compact linear operator on a Banach space such that its image on a given retro Banach frame need not be a retro Banach frame for the underlying space."

This is given in the following example.

**Example 6.3.1.** Let $\mathcal{F} \equiv \{f_k\}, \Theta$ be a retro Banach frame for $\mathcal{X}^*$ and let $T$ be a compact linear operator on $\mathcal{X}$. Then, in general, there exists no reconstruction operator $\Theta_0^*$ such that $(\{T(f_k)\}, \Theta_0^*)$ is retro Banach frame for $\mathcal{X}^*$. Indeed, let $\mathcal{X} = L^2(\Omega, \mu)$ be the discrete signal space and let $\{\chi_k\} \subset \mathcal{X}$ be an orthonormal basis for $\mathcal{X}$. Then, there exists a reconstruction operator $\Theta$ such that $\mathcal{F} \equiv \{\chi_k\}, \Theta$ is an exact retro Banach frame for $\mathcal{X}^*$.

Define $T_o : \mathcal{X} \to \mathcal{X}$ by

$$T_o(f = \{\xi_1, \xi_2, \xi_3, \xi_4, \ldots\}) = \left\{0, \xi_1, \frac{\xi_2}{2}, \frac{\xi_3}{3}, \frac{\xi_4}{4}, \ldots\right\}, \quad f = \{\xi_j\} \in \mathcal{X}.$$

First we show that $T_o \in \mathcal{K}(\mathcal{X}, \mathcal{X})$.

For this define $T_n : \mathcal{X} \to \mathcal{X}$ by

$$T_n(f = \{\xi_1, \xi_2, \xi_3, \xi_4, \ldots\}) = \left\{0, \xi_1, \frac{\xi_2}{2}, \frac{\xi_3}{3}, \frac{\xi_4}{4}, \ldots, \frac{\xi_n}{n}, 0, 0, 0, \ldots\right\}, \quad f = \{\xi_j\} \in \mathcal{X} \ (n \in \mathbb{N}).$$

Then, each $T_n$ is an operator of finite rank, and is compact.

We compute

$$\|(T_o - T_n)(f)\|^2 = \sum_{j=n+1}^{\infty} \left|\frac{\xi_j}{j}\right|^2$$
\[
\frac{1}{(n+1)^2} \sum_{j=n+1}^{\infty} |\xi_j|^2
\leq \frac{\|f\|^2}{(n+1)^2}.
\]

This gives

\[
\|T_o - T_n\| \leq \frac{1}{(n+1)^2}.
\]

Therefore

\[
T_n \to T_o.
\]

Hence \(T_o \in \mathcal{K}(\mathcal{X}, \mathcal{X})\).

Furthermore, there exists no reconstruction operator \(\Theta_0^\times\) such that \(\{T_o(\chi_k)\}, (\Theta_0^\times)\) is retro Banach frame for \(\mathcal{X}^*\) with respect to any \(\mathcal{Z}_d\). Indeed, let \((\gamma_0, \delta_0)\) be a choice of retro frame bounds for \((\{T_o(\chi_k)\}, (\Theta_0^\times))\).

Then

\[
\gamma_0\|f^*\| \leq \|f^*(T_o(\chi_k))\|_{\mathcal{Z}_d} \leq \delta_0\|f^*\|, \text{ for all } f^* \in \mathcal{X}^*.
\] (6.3.1)

Choose \(f_0^* = \chi_1 \in \mathcal{X}^*\). Then, we have \(f_0^*(T_o(\chi_k)) = 0\), for all \(k \in \mathbb{N}\). Therefore, by using retro frame inequality (6.3.1), we obtain \(f_0^* = 0\), a contradiction. Thus, the image of a retro Banach frame (even exact) under a compact linear operator on \(\mathcal{X}\) need not be a retro Banach frame for the underlying space.

We now discuss the construction of a compact linear operator from a given retro Banach frame which can generate a retro Banach frame for the underlying space. Let \((\{f_k\}, \Theta)\) be an exact retro Banach frame for \(\mathcal{X}^*\). Then, \(f_j \notin [f_k]_{k \neq j}\), for all \(j \in \mathbb{N}\). Therefore, by the Hahn-Banach Theorem, we can find a sequence \(\{f_k^*\} \subset \mathcal{X}^*\) such
that $f_j^*(f_m) = \delta_{j,m}$, for all $j, m \in \mathbb{N}$. We call the sequence $\{f_k^*\} \subset \mathcal{X}^*$ an admissible system of $\mathcal{F}$.

Define $T : \mathcal{X} \rightarrow \mathcal{X}$ by

$$
T(f) = \sum_{i=1}^{\infty} \frac{f_i^*(f) f_i}{\|f_i\| \|f_i^*\|}, f \in \mathcal{X}.
$$

(6.3.2)

Then, one can verify that $T \in \mathcal{K}(\mathcal{X}, \mathcal{X})$. Indeed, the operator $T$ can be approximated by a sequence of finite rank operators in operator norm topology. This is summarized in the following lemma.

**Lemma 6.3.2.** Let $\mathcal{F} \equiv (\{f_k\}, \Theta)$ be an exact retro Banach frame for $\mathcal{X}^*$ with admissible system $\{f_k^*\} \subset \mathcal{X}^*$. Then, $T : f \rightarrow \sum_{i=1}^{\infty} \frac{f_i^*(f) f_i}{\|f_i\| \|f_i^*\|}$ defines a compact linear operator on $\mathcal{X}$.

Now a natural question arises: what is the importance of the compact linear operator $T$ given in Lemma 6.3.2? The compact linear operator given in Lemma 6.3.2 always generates a retro Banach frame for the underlying space. This is given in the following theorem.

**Theorem 6.3.3.** Let $\mathcal{F} \equiv (\{f_k\}, \Theta)$ be an exact retro Banach frame for a Banach space $\mathcal{X}^*$ with admissible system $\{f_k^*\} \subset \mathcal{X}^*$. Then, there exists a compact linear operator $T : \mathcal{X} \rightarrow \mathcal{X}$ and a reconstruction operator $\Theta^* \in \mathcal{B}(\mathcal{Z}_\infty, \mathcal{X}^*)$ such that $(\{T(f_k), \Theta^*\})$ is a retro Banach frame for $\mathcal{X}^*$, where $\mathcal{Z}_\infty$ an associated Banach space of scalar valued sequences.

**Proof.** The existence of a compact linear operator $T \in \mathcal{B}(\mathcal{X}, \mathcal{X})$ associated with $\mathcal{F}$ is guaranteed by Lemma 6.3.2. More precisely, $T$ is given by the equation (6.3.2).
For the second part since $\mathcal{F}$ is a retro Banach frame for $\mathcal{X}^*$, we can find positive constants $A_0, B_0$ such that

$$A_0\|f^*\| \leq \|\{f^*(f_k)\}\|_{Z_d} \leq B_0\|f^*\|, \text{ for each } f^* \in \mathcal{X}^*. \tag{6.3.3}$$

Assume that there exists no reconstruction operator $\Theta^\times$ such that $\{T(f_k), \Theta^\times\}$ is a retro Banach frame for $\mathcal{X}^*$ with respect to any associated Banach space of scalar valued sequences. Then, by the Hahn-Banach Theorem there is a non-zero functional $f_0^*$ such that

$$f_0^*(Tf_n) = 0, \text{ for all } n \in \mathbb{N}.$$ 

Therefore

$$0 = f_0^*(Tf_n), \text{ for all } n \in \mathbb{N}$$

$$= f_0^*\left(\mathbb{N} \sum_{i=1}^{\infty} \frac{f_i^*(f_n)f_i}{i^2\|f_i\|\|f_i^*\|}\right), \text{ for all } n \in \mathbb{N}$$

$$= f_0^*\left(\frac{f_n}{n^2\|f_n\|\|f_n^*\|}\right), \text{ for all } n \in \mathbb{N}. \tag{6.3.4}$$

By using (6.3.4), we obtain $f_0^*(f_n) = 0$, for all $n \in \mathbb{N}$. Therefore, by lower retro frame inequality in (6.3.3), we have $f_0^* = 0$, a contradiction. Hence we can find a reconstruction operator $\Theta^\times \in \mathcal{B}(\mathcal{Z}_\infty, \mathcal{X}^*)$ such that $\{T(f_k), \Theta^\times\}$ is retro Banach frame for $\mathcal{X}^*$, where $\mathcal{Z}_\infty$ an associated Banach space of scalar valued sequences.

Remark 6.3.1. Let $\mathcal{Y}$ be a closed subspace of $\mathcal{X}$ and let $\mathcal{F} \equiv \{f_k, \Theta\}$ be an exact retro Banach frame for the Banach space $\mathcal{X}^*$. Then, we can find a sequence $\{g_j\} \subset \mathcal{Y}$, a compact operator $T_{\mathcal{Y}} \in \mathcal{B}(\mathcal{X}, \mathcal{X})$ and a reconstruction operator $\Theta^\times_{\mathcal{Y}}$ such that $\mathcal{F}_{\mathcal{Y}} \equiv \{T_{\mathcal{Y}}(g_j), \Theta^\times_{\mathcal{Y}}\}$ is a retro Banach frame for $\mathcal{Y}^*$. 
6.4 Schauder Frames associated with the Boundary Value Problems

In this section we discuss Schauder frames obtained from eigenfunctions associated with a given boundary value problem (BVP). There are various numerical and computational methods for obtaining eigenfunctions associated with a BVP [138], we consider Schauder frames of eigenfunctions. Recall that the Paley-Wiener Theorem says that if a given system is “sufficiently close” to a given system, then both have same behavior. For example, if a sequence \( \{f_n\} \subset \mathcal{H} \) is sufficiently close to an orthonormal basis \( \{\chi_n\} \subset \mathcal{H} \), then \( \{f_n\} \) also forms an orthonormal basis for the underlying space. By using the Paley-Wiener Theorem we can obtain Schauder frame for certain spaces, which need not be linearly independent. A compact linear operator on the Banach space \( \mathcal{X} \) is obtained provided two Schauder frames for \( \mathcal{X} \) are sufficiently closed (see Lemma 6.4.1).

Let \( \mathcal{X} = L^2(a,b) \). For a set of \( n \) boundary conditions, Consider a (BVP) given by

\[
(\ast) \quad \text{BVP} \quad \equiv \quad \nabla(f) = \lambda f, \quad \Lambda(f) = 0,
\]

where \( \nabla(\bullet) = (\bullet)^n + \Phi_1(\xi)(\bullet)^{n-1} + \cdots + \Phi_n(\xi)(\bullet) \) denotes the linear differential operator with \( \Phi_j \in C^{n-k}[a,b] \), and the equation \( \Lambda(f) = 0 \) denotes the set of \( n \) boundary conditions given by

\[
\Lambda_j(\Phi) = \Sigma_{k=1}^n [\alpha_j,k\Phi^{k-1}(a) + \beta_j,k\Phi^{k-1}(b)] = 0, \quad 1 \leq j \leq n.
\]

The BVP \( (\ast) \) admits a system \( \{\Phi_n(\xi)\} \) and \( \{\Psi_n(\xi)\} \) consisting of eigenfunctions associated with the BVP \( (\ast) \) (see [138] at page 66) such that

\[
\Phi_n(\xi) = A_n \left[ \cos \frac{2\pi n \xi}{b-a} + O\left(\frac{1}{n}\right) \right]
\]
and

\[ \Psi_m(\xi) = B_m \left[ \sin\frac{2\pi m \xi}{b-a} + O\left(\frac{1}{m}\right) \right], \]

where \( n, m \in N \cup \{0\} \).

Note that

\[ \left\| \Phi_n(\xi) - A_n \cos \frac{2\pi nt}{b-a} \right\|^2 < 1 \]

and

\[ \left\| \Psi_m(\xi) - B_m \sin \frac{2\pi mt}{b-a} \right\|^2 < 1. \]

Choose \( f_k = \{\Phi_n\} \bigcup \{\Psi_m\}, \ m, n \in \mathbb{N} \). Then, by the Paley and Wiener theorem in [139, pp. 208], there exists a sequence \( \{f_n^*\} \in \mathcal{X}^* \) such that \( \mathcal{G}_1 \equiv (\{f_k\}, \{f_k^*\}) \) is a Schauder frame for \( \mathcal{X} \). We say that \( \mathcal{G}_1 \) is a Schauder frame associated with the BVP (\( \ast \)). Let \( \mathcal{G}_2 = (\{g_k\}, \{g_k^*\}) \) be any Schauder frame for \( \mathcal{X} \) (of course, need not be linearly independent). Then, under certain closeness determined by the systems \( \mathcal{G}_1 \) and \( \mathcal{G}_2 \), we can find a compact linear operator on the underlying space. The following theorem provides sufficient condition for the existence of the said compact linear operator on \( \mathcal{X} \).

**Lemma 6.4.1.** Let \( \mathcal{G}_1 = (\{f_n\}, \{f_n^*\}) \) be a Schauder frame for \( \mathcal{X} \) associated with the BVP (\( \ast \)) and let \( \mathcal{G}_2 = (\{g_n\}, \{g_n^*\}) \) be another Schauder frame for \( \mathcal{X} \). If

\[ \sum_{n=1}^{\infty} \|g_n^* - f_n^*\| \|f_n\| < \infty, \quad (6.4.1) \]

then \( \tilde{\Theta} : f \to \sum_{n=1}^{\infty} (g_n^* - f_n^*)(f) f_n \) defines a compact linear operator on \( \mathcal{X} \).
Proof. By using (6.4.1), we have
\[ \left\| \sum_{k=p}^{p+n} (g_k^* - f_k^*)(f) f_k \right\| \leq \left( \sum_{k=p}^{p+n} \|g_k^* - f_k^*\| \right) \left\| f \right\| \]
\[ \leq \left( \sum_{k=p}^{p+n} \|g_k^* - f_k^*\| \right) \left\| f \right\| \]
\[ \to 0, \text{ as } n, p \to \infty. \]

Therefore, \( \tilde{\Theta} : \mathcal{X} \to \mathcal{X} \) given by \( \tilde{\Theta}(f) = \sum_{k=1}^{\infty} (g_k^* - f_k^*) f_k \) is a well defined linear operator. Moreover, \( \tilde{\Theta} \) is bounded.

Indeed
\[ \|\tilde{\Theta}\| \leq \sum_{k=1}^{\infty} \|g_k^* - f_k^*\| \left\| f_k \right\| < \infty. \]

Hence \( \tilde{\Theta} \) is bounded.

Choose \( \Theta_n(f) = \sum_{k=1}^{n} (g_k^* - f_k^*) f_k \), \( n \in \mathbb{N} \). Then, each \( \Theta_n \) is an operator of finite rank, and is compact.

We compute
\[ \|\tilde{\Theta} - \Theta_n\| = \left\| \sum_{k=n+1}^{\infty} (g_k^* - f_k^*) f_k \right\| \]
\[ \leq \sum_{k=n+1}^{\infty} \|g_k^* - f_k^*\| \left\| f_k \right\| \]
\[ \to 0, \text{ as } n \to \infty. \]

Hence \( \tilde{\Theta} : f \to \sum_{k=1}^{\infty} (g_k^* - f_k^*) f_k \) defines a compact linear operator on \( \mathcal{X} \).

Remark 6.4.1. The result given in Lemma 6.4.1 is known as Compact Approximation associated with the Boundary Value Problem (\( \star \)). This can be extended to a Banach space which admits a Schauder frame.
To conclude the chapter, we discuss a linear block of the Schauder frame for $\mathcal{X}$ with respect to a given sequence in $\mathcal{X}$.

**Definition 6.4.1.** Let $\mathcal{F} \equiv (\{ f_k \}, \{ f^*_k \})$ be a Schauder frame for $\mathcal{X}$ and let $\{ g_k \} \subset \mathcal{X}$. A **linear block** of $\mathcal{F}$ with respect to $\{ g_k \}$ is a sequence of the form

$$\{ f_k + \sum_{i=k+1}^{\infty} f^*_i (g_k) f_i \}_{k=1}^{\infty},$$

where the series converges in $\mathcal{X}$ for each $k \in \mathbb{N}$.

Recall that a Schauder frame $\mathcal{F} \equiv (\{ f_k \}, \{ f^*_k \})$ for $\mathcal{X}$ provides a series representation of each vector in $\mathcal{X}$ over $f_k$. If each $f_k$ is perturbed by a nonzero vector in $\mathcal{X}$ (e.g. see (6.4.2)), then there is a question of the reconstruction of each vector in $\mathcal{X}$ as an infinite series in terms of the perturbed sequence $\{ f_k + (\bullet) \}$. The following theorem provides sufficient conditions for the existence of a sequence in $\mathcal{X}^*$ which constitute a Schauder frame for $\mathcal{X}$ with respect to the linear block of a given Schauder frame for $\mathcal{X}$.

**Theorem 6.4.2.** Assume that $\mathcal{F} \equiv (\{ f_k \}, \{ f^*_k \})$ is Schauder frame for a Banach space $\mathcal{X}$. Let $\{ g_k \} \subset \mathcal{X}$ be a sequence of vectors and let

$$\max \left\{ \limsup_{i \geq k+1} \frac{\| f^*_i (g_k) f_i \|}{\| f_k \|}, \sum_{i=1}^{k} \| f^*_i (f) f_i \| \right\} < 1 \quad (k \in \mathbb{N}).$$

Then, there exists $\{ g^*_k \} \subset \mathcal{X}^*$ such that $\{ f_k + \sum_{i=k+1}^{\infty} f^*_i (g_k) f_i \}_{k=1}^{\infty}, \{ g^*_k \}$ is a Schauder frame for $\mathcal{X}$.

**Proof.** For each $k \in \mathbb{N}$, choose

$$\varphi_k = f_k + \sum_{i \geq k+1} f^*_i (g_k) f_i.$$
By using (6.4.3) there exists a \( \lambda \) (\( 0 < \lambda < 1 \)) such that
\[
\| \varphi_k - f_k \| < \lambda \| f_k \|, \text{ for all } k \geq 1. \tag{6.4.4}
\]
Then, by using (6.4.4), we have
\[
\| f_i^*(f)(f_i - \varphi_i) \| \leq \sum_{i=1}^{n} |f_i^*(f)| \| (f_i - \varphi_i) \|
\leq \lambda \sum_{i=1}^{n} \| f_i^*(f) f_i \|, \ f \in \mathcal{X}. \tag{6.4.5}
\]
Define \( \Theta : \mathcal{X} \to \mathcal{X} \) by
\[
\Theta \left( f = \sum_{i=1}^{\infty} f_i^*(f) f_i \right) = \sum_{i=1}^{\infty} f_i^*(f) \varphi_i. \tag{6.4.6}
\]
Then, \( \Theta \) is a well defined bounded linear operator on \( \mathcal{X} \).

Now, for all \( f \in \mathcal{X} \), we have
\[
\|(I(f) - \Theta(f))\| = \| f - \Theta(f) \|
= \lim_{n \to \infty} \left\| \sum_{i=1}^{n} f_i^*(f) f_i - \sum_{i=1}^{n} f_i^*(f) \varphi_i \right\|
\leq \lim_{n \to \infty} \sum_{i=1}^{n} | f_i^*(f) | \| f_i - \varphi_i \|
\leq \lambda \lim_{n \to \infty} \sum_{i=1}^{n} \| f_i^*(f) f_i \|.
\]
By using (6.4.3), we have \( \| I - \Theta \| < 1 \), so \( \Theta \) is an invertible operator on \( \mathcal{X} \).

Choose \( g_i^* = (\Theta^{-1})^* f_i^* \), \( i \in \mathbb{N} \).

Then, for all \( f \in \mathcal{X} \), by using (6.4.6), we have
\[
\sum_{i=1}^{\infty} g_i^*(f) \varphi_i = \sum_{i=1}^{\infty} ((\Theta^{-1})^* f_i^*)(f) \varphi_i
= \sum_{i=1}^{\infty} f_i^*(\Theta^{-1} f) \varphi_i
\]
= \Theta (\Theta^{-1} f) = f.

Hence \((\{f_k + \sum_{i=k+1}^{\infty} f^*_i(g_k) f_i\}_{k=1}^{\infty}, \{g_k^*\})\) is a Schauder frame for \(X\).