CHAPTER 1

INTRODUCTION

1.1 INTRODUCTION TO GRAPH THEORY

The birth of Graph Theory began with Euler’s most famous work on the Königsberg Bridge Problem [29] in 1736. In the city of Königsberg there were seven bridges, on each bank of the Pregel river connecting two islands and portions of the city. The picture of Königsberg city is shown in Figure 1.1.

Figure 1.1 Königsberg Bridges

The residents tried to walk in such a way so as to cross each bridge exactly once and return to the starting point. This problem has been restated in graph-theoretic terms by representing the four regions of land by vertices and the seven bridges connecting the land masses by edges. This yields a
multigraph as shown in Figure 1.2. In this multigraph, a closed path is required which includes each edge exactly once.

![Figure 1.2 Graph of Königsberg Bridges](image1)

Configurations (diagrams) of points or vertices and connections or edges, like the one in the Königsberg Bridge Problem, occur in diverse applications. They may represent physical networks such as electrical circuits,

![Figure 1.3 Four-Colored India Map](image2)
roadways or country maps. A four-colored India map is shown in Figure 1.3. Formally such configurations were modelled by combinatorial structures called graphs. In general, a graph consists of points called vertices and connections called edges. In the recent years, Graph Theory has established itself as an important mathematical tool for formulating problems in a wide variety of disciplines, ranging from Operations Research and Chemistry to Genetics and Linguistics, and from Electrical Engineering and Geography to Sociology and Architecture. Once a problem has been formulated in graph-theoretical language, the concepts of Graph Theory can be used in analyzing the problem. Problem formulation aids in better understanding and also gives an insight as to whether the problem can be completely solved or why it is hard.

A few problems of society that have been modelled using Graph Theory can be classified under any one of the following types: Existence Problems, Construction Problems, Enumeration Problems and Optimization Problems. The famous Graph Theory problems such as Königsberg Bridge Problem, the Knight’s Tour Problem, the Four Color Problem, the Utilities Problem and the Queen’s Problem are considered to be Existence Problems. Fleury’s Algorithm, Tarry’s Algorithm, Kruskal and Prim’s Algorithms fall under Construction Problems. Graph Labeling Problems can be classified under Enumeration Problems. The Shortest Path Problem, the Scheduling Problem, the Minimum Connector Problem and the Travelling Salesman Problem can be considered as Optimization Problems. Hence, Graph Theory not only serves as a mathematical tool to formulate and solve real life problems, it has also emerged as a worthwhile mathematical discipline in its own right.

The main focus of this thesis is Graph Labeling Problems. This chapter comprises basic definitions and concepts in Graph Theory, an introduction to graph labeling, literature survey, applications of graph labeling and shell graphs and an overview of the thesis.
1.2 BASIC DEFINITIONS AND CONCEPTS

A few basic definitions and concepts that are required for the present work are provided from Doughlas B. West [28] and Bondy, Murty [16].

A graph $G$ is a triple consisting of a nonempty vertex set $V(G)$, an edge set $E(G)$, and a relation that associates with each edge two vertices called its end points. The cardinality of $V(G)$ is the number of vertices in $G$ called the order of $G$ and the cardinality of $E(G)$ is the number of edges in $G$ called the size of $G$.

A loop is an edge whose end points are equal. Multiple edges are edges having the same pair of end points. A simple graph is a graph having no loops or multiple edges. When $u$ and $v$ are the endpoints of an edge they are adjacent and are neighbours. A graph is finite if its vertex set and edge set are finite. If $v$ is an end point of an edge, then $v$ and $e$ are incident. The degree of a vertex $v$ is the number of edges incident with $v$ (loops at $v$ being counted twice). A vertex of degree one is called a pendant vertex or a leaf. An edge incident with a pendant vertex is called a pendant edge. A subgraph of a graph $G$ is a graph $H$ such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$ and the assignment of endpoints to edges in $H$ is the same as in $G$.

A walk of a graph $G$ is an alternating sequence of vertices and edges $v_0, e_1, v_1, e_2, v_2, \ldots, e_n, v_n$ beginning and ending with vertices such that each edge $e_i$ is incident with $v_{i-1}$ and $v_i$. The walk starting with $v_0$ as initial vertex and ending with $v_n$ as the terminal vertex is called the $v_0$-$v_n$ walk. The vertices other than $v_0$ and $v_n$ are termed as the internal vertices of the walk. A walk is called a trail if all its edges are distinct. A trail is called a path if all its vertices are distinct. A path on $n$ vertices is denoted by $P_n$. A $v_0$-$v_n$ walk is called closed if $v_0 = v_n$. A closed walk $v_0, e_1, v_1, e_2, \ldots, e_n, v_n = v_0$ ($n \geq 3$) in which the vertices $v_0, v_1, \ldots, v_n$ are distinct, is called a cycle. The $n$-cycle $C_n$ denotes a cycle on $n$
vertices. The length of a walk, trail, path or cycle is the number of edges present in it. A graph which has no cycles is called an acyclic graph.

A graph $G$ is connected if it has a $u$-$v$ path whenever $u$, $v \in V(G)$. Otherwise $G$ is disconnected. If $G$ has a $u$-$v$ path then $u$ is connected to $v$. A maximal connected subgraph of $G$ is a subgraph that is connected and is not contained in any other connected subgraph of $G$. The components of a graph $G$ are its maximal connected subgraphs.

A tree is a connected acyclic graph. A forest is a graph in which every component is a tree. A binary tree is defined as a tree in which there is exactly one vertex of degree two, called root and each of the remaining vertices is of degree one or three. A complete binary tree is defined as a tree where each vertex has either 2 or 0 children.

A complete graph is a simple graph whose vertices are pairwise adjacent. The complete graph with $n$ vertices is denoted by $K_n$. A clique in a graph is a set of pairwise adjacent vertices. An independent set or stable set in a graph is a set of pairwise nonadjacent vertices. The complement $\tilde{G}$ of a simple graph is the simple graph with vertex set $V(G)$ defined by $uv \in E(\tilde{G})$ if and only if $uv \notin E(G)$.

A graph $G$ is bipartite if $V(G)$ is the union of two disjoint independent sets called partite sets of $G$. A graph is $k$-partite if $V(G)$ can be expressed as the union of $k$ independent sets. A complete bipartite graph or biclique is a simple bipartite graph such that two vertices are adjacent if and only if they are in different partite sets. Biclique is denoted as $K_{r,s}$, where $r$ and $s$ are the sizes of the partite sets. An isomorphism from a simple graph $G$ to a simple graph $H$ is a bijection $f : V(G) \to V(H)$ such that $uv \in E(G)$ if and only if $f(u)f(v) \in E(H)$. $G$ is isomorphic to $H$, written as $G \cong H$ if there is an isomorphism from $G$ to $H$. 
A graph is *self-complementary* if it is isomorphic to its complement. A *decomposition* of a graph is a list of subgraphs such that each edge appears in exactly one subgraph in the list. The *union* of graphs $G_1$, $G_2$, $G_3$, \ldots, $G_k$, written as $G_1 \cup G_2 \cup G_3$, \ldots, $\cup G_k$, is the graph with vertex set $\bigcup_{i=1}^{k} V(G_i)$ and edge set $\bigcup_{i=1}^{k} E(G_i)$. Two graphs $G_1$ and $G_2$ are *disjoint* if there is no vertex common in $G_1$ and $G_2$. The graphs $G_1$ and $G_2$ are *edge-disjoint* if there is no edge in common. The *disjoint union* of the graphs $G_1$ and $G_2$ is denoted by $G_1 + G_2$. The *join* $G_1 \vee G_2$ of disjoint graphs is the graph obtained from $G_1 + G_2$ by joining each vertex of $G_1$ to each vertex of $G_2$. An edge $e$ is said to be *subdivided* when it is deleted and replaced by a path of length two connecting its end vertices, the internal vertex of this path being a new vertex.

### 1.3 GRAPH LABELING PROBLEMS AND LITERATURE SURVEY

Graph labeling is perceived to be one of the fastest growing areas of research in Graph Theory for the past five decades. An assignment of nonnegative integers to the vertices or edges, or both, subject to certain conditions is termed as a *graph labeling*. The origin of graph labeling, traces back to the introduction of graceful labeling by Rosa [88] in 1967. In the intervening years, variations of graceful labelings such as $k$-graceful labeling, almost graceful labeling, odd graceful labeling, edge-odd graceful labeling, one modulo three graceful labeling have also been introduced. Other labelings of interest include the harmonious labeling, cordial labeling, prime labeling, vertex prime labeling and $\rho$-labeling. For a good and an exhaustive survey on different graph labeling methods, applications and open problems, one may refer to the dynamic survey by Gallian [32]. The different types of graph labeling methods are presented here and the literature survey pertaining to them is also showcased in this section.
1.3.1 Graceful Labeling

The Ringel-Kotzig Conjecture [87], (the complete graph $K_{2n+1}$ can be cyclically decomposed into $(2n + 1)$ copies of a given tree with $n$ edges) led to the creation of a hierarchical series of ‘valuations’ of a graph by Rosa. One of the valuations termed as the $\beta$-labeling, introduced by Rosa [88] in 1967, served as a tool to attack the Ringel-Kotzig’s Conjecture. Golomb [36] later renamed this $\beta$-valuation as graceful labeling and this is the most widely used term in recent years.

**Definition 1.1** [88] A *graceful labeling* of a graph $G$ with $q$ edges is an injection $f: V(G) \to \{0, 1, 2, \ldots, q\}$ with the property that the resulting edge labels are also distinct, where an edge incident with vertices $u$ and $v$ is assigned the label $|f(u) - f(v)|$. A graph which admits a graceful labeling is called a *graceful graph*. An example of a graceful graph is given in Figure 1.4.

![Figure 1.4 A graceful labeling of $C_4$](image)

The following interesting theorem was proved by Rosa in his classical paper.

**Theorem 1.1** [88] If a tree $T$ with $n$ edges has a graceful labeling, then $K_{2n+1}$ has a decomposition into $(2n + 1)$ copies of $T$. 
This led to one of the most difficult conjectures in graph theory, popularly called Ringel-Kotzig-Rosa Conjecture in 1964 or Graceful Tree Conjecture [70], which states that ‘All trees are graceful’. The graceful tree conjecture was a starting point for this venture to find whether various kinds of graphs were graceful. In such an attempt, many graphs are shown to be graceful.

A **caterpillar** is a tree in which the removal of all the one degree vertices results in a path. Rosa [88] proved that all paths and caterpillars are graceful.

A **lobster graph** is a tree in which the removal of all one degree vertices results in a caterpillar. Bermond [9] conjectured that all lobsters are graceful. This conjecture is still open. Special classes of lobsters are shown to be graceful by Ng [85], Wang *et al.* [102], Sethuraman and Jeba Jesintha [91] and Mishra *et al.* [81, 82]

Hrnčiar and Haviar [42] have proved that all trees of diameter five are graceful. Jesintha and Sethuraman [64] have generated a random structure of graceful trees from graceful stars with $n$ edges using the transfer technique introduced by Hrnčiar and Haviar.

A **banana tree** is a tree obtained by connecting a vertex $v$ to one leaf of each of any number of stars, where $v$ is not in any of the stars. Chen, Liu and Yeh [21] conjectured that all banana trees are graceful. Bhat-Nayak and Deshmukh [10] have proved that certain classes of banana trees are graceful. Murugan and Arumugam [84] have given an algorithm to find graceful numbering of a special class of banana trees. In 2009, Sethuraman and Jesintha [92, 93] have completely proved this conjecture - all banana trees are graceful.

The graceful labeling of cycle-related graphs has also been a major focus of attention. Rosa [88] has showed that the $n$-cycle $C_n$, is graceful if and only if $n \equiv 0$ or 3 (mod 4). The **wheel graph** $W_n$ is the graph obtained by adding
an edge between \( K_1 \) and every vertex of the cycle \( C_n \). Frucht [31] has proved that the wheel graph \( W_n = K_1 + C_n \) is graceful.

The Helm graph \( H_n \) is the graph obtained from the wheel \( W_n \) by adding a pendant edge at each vertex on the cycle of the wheel. Helms are proved to be graceful by Ayel and Favaron [5]. A closed helm is the graph obtained from the helm graph by adding an edge between each pendant vertex in the helm.

Web graphs [68] are obtained by adding pendant edges at each vertex in the outer cycle of the closed helm. The web graph, denoted as \( W(2, n) \) was proved to be graceful by Kang et.al. [66]. Yang [66] denoted \( W(t, n) \) as the generalised web graph with \( t \) \( n \)-cycles and proved that \( W(3, n) \) and \( W(4, n) \) are graceful. Abhyankar [1] has shown that \( W(t, 5) \), for \( 5 \leq t \leq 13 \) is graceful.

Cycle with a chord was conjectured to be graceful by Bodendiek et.al. [15]. This conjecture was proved by Delorme et.al. [25]. Koh and Yap [69] defined a cycle with a \( P_k \)-chord as a cycle with the path \( P_k \) joining two non consecutive vertices of the cycle and have proved that such graphs are graceful when \( k = 3 \). Koh and Yap [69] had also conjectured that all cycles with a \( P_k \)-chord are graceful. This conjecture was proved for \( k \geq 4 \) by Punnim and Pabhapote [86]. Sethuraman and Elumalai [90] have defined and conjectured that the cycle \( C_n \) with parallel \( P_k \)-chords is graceful for all even \( k \) and have proved its gracefulness for \( k = 3, 4, 6, 8, 10 \).

Although numerous research on the gracefulness of graphs have been carried out, the characterization of graceful graphs is one of the most difficult problems in Graph Theory [36, 23]. In fact, the graceful labeling problem is rather a well known example of the problems in NP [41]. However, Golomb [36] has proved the following necessary condition for a graph to be graceful.

**Theorem 1.2** [36] Let \( G \) be a graceful graph with \( n \) vertices and \( e \) edges. Let the vertices be partitioned into two sets \( E \) and \( O \) having respectively the vertices
with even and odd labels. Then the number of edges connecting vertices in $E$ with vertices in $O$ is exactly $\frac{e+1}{2}$.

Rosa [88] has identified three reasons why a graph fails to be graceful. Also, a variety of sufficient conditions in graceful labeling have been proved over the past five decades.

1.3.2 $k$-Graceful Labeling


**Definition 1.2** A graph with $q$ edges is *k-graceful* if there is an injection $f : V(G) \rightarrow \{0, 1, 2, \ldots, (q + k - 1)\}$ such that the set of edge labels induced by the absolute value of the difference of the labels of adjacent vertices is $\{k, (k + 1), \ldots, (q + k -1)\}$. The following Figure 1.5 illustrates a $k$-graceful labeling in a wheel graph when $k = 2$.

![Figure 1.5 A 2-graceful wheel graph](image)

Obviously, a 1-graceful graph is graceful. Graphs which are $k$-graceful for all $k$ are called *arbitrarily graceful*. Various kinds of graphs are shown to be $k$-graceful. Maheo, Thuillier [79] and Slater [98] have proved that the cycles $C_n$ are $k$-graceful if and only if either when $n \equiv 0$ or 1 (mod 4) with $k$ even and
1. Introduction

$k \leq \frac{n-1}{2}$ or when $n \equiv 3 \pmod{4}$ with $k$ odd and $k \leq \frac{n^2-1}{2}$. Maheo and Thuillier [79] have also shown that the wheel $W_n$ is $k$-graceful when $n$ is odd and conjectured that $W_n$ is $k$-graceful when $n$ is even except when $n = 6$ or $n = 8$. This conjecture was proved by Liang et al. [76]. The complete bipartite graph $K_{m,n}$ is proved to be $k$-graceful by Liang and Liu [77] and also by Li, Li, and Yan [75]. Yao et al. [107] have shown that a tree of order $p$ with maximum degree at least $\frac{p}{2}$ is $k$-graceful, for some $k$.

1.3.3 Almost Graceful Labeling

As a means of attacking graph decomposition problems, Rosa [88] invented another analogue of graceful labeling called $\rho^-$-labeling, by permitting the vertices of a graph with $q$ edges to assume labels from the set $\{0, 1, \ldots, (q+1)\}$, while the edge labels induced by the absolute value of the difference of the vertex labels are $\{1, 2, \ldots, (q - 1), q\}$ or $\{1, 2, \ldots, (q - 1), (q + 1)\}$. Frucht [31] used the term nearly graceful labeling instead of $\rho^-$-labeling. Moulton [83] introduced a slightly stronger concept called almost graceful labeling by permitting the vertices to take labels from the set $\{0, 1, \ldots, (q - 1), (q + 1)\}$ while the edge labels are $\{1, 2, \ldots, (q - 1), q\}$ or $\{1, 2, \ldots, (q - 1), (q + 1)\}$.

**Definition 1.3** An almost graceful labeling of a graph $G$ with $q$ edges and vertex set $V$ is an injection $f : V(G) \rightarrow \{0, 1, 2, \ldots, (q - 1), (q + 1)\}$ with the property that the resulting edge labels are also distinct, where the edge labels induced by the absolute value of the difference of the vertex labels are either $\{1, 2, 3, \ldots, (q - 1), q\}$ or $\{1, 2, \ldots, (q - 1), (q + 1)\}$. An example for an almost graceful labeling in a ladder graph is given in Figure 1.6.
Seoud and Elsa 

9 5 n

5, K m, K 1,m,n, P n + K 3(n ≥ 3), K 5 ∪ K 1,n, K 6 ∪ K 1,n and ladders.

Figure 1.6 An almost graceful ladder

1.3.4 Odd Graceful Labeling

The odd graceful labeling was introduced by Gnanajothi [35] in 1991. Gnanajothi proved that the class of odd graceful graphs lies between the class of graphs with \( \alpha \)-labelings and the class of bipartite graphs by showing that every graph with an \( \alpha \)-labeling has an odd graceful labeling and every graph with an odd cycle is not odd graceful. Note that, Rosa [88] defined a \( \alpha \)-labeling as a graceful labeling with the additional property that there exists an integer \( k \) so that for each edge \( xy \) either \( f(x) ≤ k < f(y) \) or \( f(y) ≤ k < f(x) \).

Definition 1.4 A graph \( G \) with \( q \) edges is said to be odd graceful if there is an injection \( f : V(G) \to \{0, 1, 2, \ldots, (2q - 1)\} \) such that, when each edge \( xy \) is assigned the label \( |f(x) - f(y)| \), the resulting edge labels are \{1, 3, 5, 7, \ldots, (2q - 1)\}. Figure 1.7 illustrates this labeling in a rooted tree of height two.
Gnanajothi [35] proved that the following graphs are odd graceful: paths $P_n$, even cycles $C_n$, complete bipartite graphs $K_{m,n}$, combs $P_n \odot K_1$ (graphs obtained by joining a single pendant edge to each vertex of $P_n$), books, the disjoint union of copies of $C_4$, crowns $C_n \odot K_1$ (graphs obtained by joining a single pendant edge to each vertex of $C_n$), if and only if $n$ is even. Gnanajothi [35] conjectured that all trees are odd graceful and proved the conjecture for all trees with order upto 10.

Barrientos [6] has proved that trees of order up to 12 are odd graceful. Barrientos [6] has also proved that the following graphs are odd graceful: every forest whose components are caterpillars, every tree with diameter at most five and all disjoint unions of caterpillars. Barrientos conjectured that every bipartite graph is odd graceful. Seoul, Diab and Elsahawi [94] have shown that a connected complete $r$-partite graph is odd graceful if and only if $r = 2$ and that the join of any two connected graphs is not odd graceful. Sekar [89] has shown that the following graphs are odd graceful: $C_m \odot P_n$ (the graph obtained by identifying an end point of $P_n$ with every vertex of $C_m$ where $n \geq 3$ and $m$ is even), graphs obtained from even cycles by identifying a vertex of the cycle with the end point of a star, lobsters and banana trees.
1.3.5 Edge-Odd Graceful Labeling

In 2009, Solairaju and Chitra [100] introduced a new type of labeling known as the *edge-odd graceful labeling*.

**Definition 1.5** A graph $G$ with $q$ edges is said to be *edge-odd graceful* if there is a bijection $f : E(G) \rightarrow \{1, 3, 5, \ldots, (2q - 1)\}$ such that, when each vertex is assigned the sum of all edge labels incident to it mod $2q$, the resulting vertex labels are distinct.

In Figure 1.8, an edge-odd graceful labeling is given for the comb graph.

![Figure 1.8 An edge-odd graceful comb](image)

Solairaju and Chitra [100] have proved the following graphs to be edge odd graceful: combs, the graph obtained by appending $(2n + 1)$ pendant edges to each endpoint of $P_2$ or $P_3$ and the graph obtained by subdividing each edge of the star $K_{1,2n}$. Singun [97] has shown that that following graphs have edge-odd graceful labelings: $W_{2n}$, $W_n \odot K_1$, $W_n \odot K_m$ when $n$ is odd, $m$ is even and $n$ divides $m$.

1.3.6 One Modulo Three Graceful Labeling

The *one modulo three graceful labeling* was introduced by Sekar [89] in 2002.
Definition 1.6 A graph with $q$ edges is called a **one modulo three graceful graph** if there is an injective function $f: V(G) \to \{0, 1, 3, 4, 6, 7, \ldots, (3q - 3), (3q - 2)\}$ such that the edge labels induced by labeling each edge $e = uv$ with $|f(u) - f(v)|$ is $\{1, 4, 7, \ldots, (3q - 2)\}$.

In Figure 1.9, a one modulo three graceful labeling is given to a caterpillar graph.

![Figure 1.9 A one modulo three graceful caterpillar](image)

Sekar [89] proved that the following graphs are one modulo three graceful graphs: $P_m$, cycles $C_n$ if and only if $n \equiv 0 \pmod{4}$, the complete bipartite graphs $K_{m,n}$, caterpillars, stars, lobsters, banana trees, rooted trees of height 2, ladders and crowns $C_n \odot K_1$ for $n$ even. Sekar conjectured that every one modulo three graceful is graceful.

1.3.7 Harmonious Labeling

The *harmonious labeling* was introduced by Graham and Sloane [38]. Harmonious graphs naturally arose in the study of modular versions of additive base problems stemming from error correcting codes.

Definition 1.7 A graph $G$ with $q$ edges is said to be **harmonious** if there is an injection from the vertices of $G$ to the group of integers modulo $q$ such that
when each edge $xy$ is assigned the label $[f(x) + f(y)] \pmod{q}$, the resulting edge labels are distinct. Note that, when the graph is a tree exactly one label may be used on two vertices. Figure 1.10 illustrates this labeling in a tree.

![Figure 1.10 A harmonious tree](image)

Graham and Sloane [38] proved that the wheel $W_n$ is harmonious and the cycle $C_n$ is harmonious if and only if $n \equiv 1$ or $3 \pmod{4}$. The helm graphs are shown to be harmonious by Gnanajothi [35]. The gear graphs (graph obtained from a wheel graph $W_n$ by adding a vertex between every pair of adjacent vertices of the $n$-cycle) are proved to be harmonious by Chen [20]. Liu [78] has shown that crown graphs are harmonious. Deb and Limaye [24] have defined shell graphs (a shell graph is a cycle $C_n$ with (n-3) chords sharing a common end point called the apex) and have conjectured that all multiple shells (a multiple shell is a collection of edge disjoint shells that have their apex in common) are harmonious, and have shown that the conjecture is true for the balanced double shells and balanced triple shells (A multiple shell is said to be balanced with width $w$ if every shell has order $w$ or every shell has order $(w+1)$ or $w$). Yang et.al. [106] proved the conjecture is true for balanced quadruple shells. In 2011, Xi Yue [105] has proved that balanced quintuple shells are harmonious. Gallian et.al. [33] have proved that all prisms $C_m \times P_2$ with a single vertex deleted or single edge deleted are harmonious. Sethuraman and Elumalai [90] have shown that any given set of graphs $G_1, G_2, \ldots, G_t$ can be embedded in a harmonious graph.
1.3.8 Cordial Labeling

*Cordial labeling*, a variation of both graceful and harmonious labeling was introduced by Cahit [18] in the year 1987.

**Definition 1.8** The *cordial labeling* of a graph is a function \( f : V(G) \rightarrow \{0, 1\} \) such that each edge \( xy \) is assigned \( f^*(xy) = \left| f(x) - f(y) \right| \) with the property that \( \left| v_f(0) - v_f(1) \right| \leq 1 \) and \( \left| e^*_f(0) - e^*_f(1) \right| \leq 1 \), where \( v_f(i) \) denotes the number of vertices with label \( i \), for \( i = 0, 1 \) and \( e^*_f(i) \) denotes the number of edges with label \( i \), for \( i = 0, 1 \).

Figure 1.11 is an example for a cordial labeling in a helm graph.

![Cordial labeling in a helm graph](image_url)

Cahit [18] has proved the following graphs to be cordial: Every tree is cordial, the wheel graph \( W_n \) is cordial when \( n \not\equiv 3 \ (\text{mod} \ 4) \), the friendship graph \( C_3^t \) (the graph obtained by joining \( t \) copies of cycle \( C_3 \) with a common vertex) is cordial if and only if \( t \not\equiv 2 \ (\text{mod} \ 4) \). Eventually different authors have proved various graphs to be cordial. Lee *et.al.* [73] proved the following graphs to be cordial: the cartesian product of an arbitrary number of paths, the cartesian...
product of two cycles if and only if at least one of them is even. Shee and Ho [99] have defined the path union of the graph $G$ as a graph obtained by adding an edge from the graph $G_i$ to the graph $G_{i+1}$ for $i = 1, 2, \ldots (n - 1)$ and have shown that the following graphs are cordial: path union of cycles, path union of three or more copies of $K_5$, path union of wheels, fans and trees.

1.3.9 Prime Labeling

The prime labeling was introduced by Tout, Dabboucy and Howalla [101].

Definition 1.9 A graph $G$ with vertex set $V$ is said to have a prime labeling if its vertices are labeled with distinct integers $1, 2, \ldots, |V|$ such that for each edge $xy$, the labels assigned to $x$ and $y$ are relatively prime. An example for a prime labeling is given in Figure 1.12.

![Figure 1.12 Prime labeling in $C_6$](image)

Haxell et al. [40] proved that all large trees are prime graphs. Among the classes of trees known to have prime labelings are paths, stars, caterpillars, complete binary trees, spiders (trees with one vertex of degree at least three and with all other vertices with degree at most two), olive trees (a rooted tree consisting of $k$ branches such that the $i^{th}$ branch is a path of length $i$), all trees of order up to 50 and banana trees. Deretsky et al. [26] have proved that cycle $C_n$ have prime labeling and the disjoint union of $C_{2k}$ and $C_n$ admit prime labeling.
Lee et al. [74] have proved that the wheels $W_n$ are prime graphs if and only if $n$ is even. Helms, flowers, stars, complete bipartite graphs $K_{2,n}$ and $K_{3,n}$ except for $n = 3$ or $n = 7$ are proved to be prime graphs by Seoud et al. [94]. Babujee and Jagadesh [7] proved that the following graphs have prime labelings: $P_1 \odot K_{1,n}$, the union of $K_{1,n}$ and the graph obtained from $K_{1,n}$ by appending a pendant edge to every pendant edge of $K_{1,n}$. Babujee and Vishnupriya [8] proved that $n$ copies of $P_2$, the graph $P_n \cup P_n \cup \cdots \cup P_n$ and bistars have prime labeling.

1.3.10 Vertex Prime Labeling

Vertex prime labeling has been introduced by Deretsky, Mitchem and Lee [26]. It is a dual of prime labeling.

**Definition 1.10** A graph with $q$ edges has a vertex prime labeling if its edges can be labeled with distinct integers $1, 2, \ldots, q$ such that for each vertex of degree at least 2, the greatest common divisor of the labels on its incident edges is 1. In Figure 1.13, a star graph with vertex prime labeling is shown.

![Figure 1.13 A star graph with vertex prime labeling](image)

Deretsky, Mitchem and Lee [26] have shown that the following graphs have vertex prime labeling: all connected graphs, forests, union of the cycles
1. Introduction

$C_{2k}$ and $C_n$, union of the cycles $C_{2m}$, $C_{2n}$ and $C_{2k+1}$, union of the cycles $C_{2m}$, $C_{2n}$, $C_{2r}$ and $C_k$, the graph $5C_{2m}$, the graph with exactly two components one of which is not an odd cycle. Deretsky et al. [26] have conjectured that a 2-regular graph has a vertex prime labeling if and only if it does not have two odd cycles. Borosh et al. [17] have proved the conjecture of Deretsky et al. [26] for the graph $G$ which is the disjoint union of at most seven cycles.

1.3.1 $\rho$-Labeling

Rosa [88] introduced another kind of labeling in 1967 which is known as a $\rho$-labeling.

**Definition 1.11** A $\rho$-labeling (or $\rho$-valuation) of a graph is an injection from the vertices of the graph with $q$ edges to the set $\{0, 1, \ldots, 2q\}$, where if the edge labels induced by the absolute value of the difference of the vertex labels are $a_1, a_2, \ldots, a_q$, then $a_i = i$ or $a_i = (2q + 1 - i)$. An example for this labeling is given in Figure 1.14.

![Figure 1.14 $\rho$-labeling in lobster graph](image)

Rosa [88] proved that a cyclic decomposition of the edge set of the complete graph $K_{2q+1}$ into subgraphs isomorphic to a given graph $G$ with $q$
edges exists if and only if $G$ has a $\rho$-labeling. Rosa [88] has also proved that every graph with at most 11 edges and all lobsters have $\rho$-labeling. Gannon and El-Zanati [34] have proved that for any odd $n \geq 7$, $rC_n$ admits $\rho$-labeling. Aguado et al. [3] gave a $\rho$-labeling for the union of the cycles $C_r$, $C_s$ and $C_t$. Caro et al. [19] provided a construction for the adjacency matrix for every graph that has a $\rho$-labeling. Donovan et al. [27] proved that $rC_m$ has a $\rho$-labeling when $r \leq 4$ and conjectured that every 2-regular graph has a $\rho$-labeling.

1.4 APPLICATIONS OF GRAPH LABELING AND SHELL GRAPHS

Although labeling of graphs is primarily a theoretical subject in the field of graph theory, it serves as models in a wide range of other areas of study such as Astronomy, Circuit Design, Coding Theory, Radar, X-ray Crystallography, Communication Network Addressing, Database Management and in Secret Sharing Schemes. See [4, 12, 13, 14, 71, 72].

The graceful labeling which is the most fascinating and premier graph labeling method introduced by Rosa [88], has its application in Communication Networks [72], Coding theory [12], Network addressing [13] and in X-ray Crystallography.

In a communication network [43], having a fixed number of communication centres, say $(n + 1)$ centres, that are taken as the vertices of a graph and if those vertices are numbered or labeled with 0, 1, 2, \ldots, $n$ then the lines between any two centres could be labeled with the difference between the two communicating centre labels (that is, vertex labels). If the communication centre grid was laid out in a fashion conducive to a graceful graph, then it is possible to label the connections (edges) between centres such that each connection would have a distinct label. There are many advantages in creating a communication network analogous to a graceful graph. One advantage is that if
a link goes faulty, a simple algorithm could detect which two centres are no longer linked, since each connection is labeled with the difference between the two communication centres. Another advantage is that this network has the same properties as a graceful graph, such as cyclic decomposition, which makes it convenient to detect faults in the centres or the links.

Another application of graceful labeling is in the field of Coding Theory [43]. Graceful labeling along with semi-graceful labeling and quasi graceful labeling can be used to create a special kind of a ruler. This is done by assigning each vertex label of a graceful graph to the ruler called as the Golomb ruler [36]. The markings on the Golomb ruler by the vertex labels of a graceful graph are used to measure time and not the distance in Coding Theory. Golomb rulers are used in radar type codes, self orthogonal codes in algebraic coding and synch set codes in lasers with rotating disks.

The anticipated and the best use of graceful labeling is in Network Addressing [72]. In a wireless network, each station (vertex) is assigned a channel (positive integers that are labels) such that interference can be avoided. With the smaller distance between the stations, the interference become higher, so the difference in the channel assignment has to be greater. Thus, when this network is represented as a graceful graph, then the graceful labeling helps to detect which stations have more interference between them and which stations have less interference, as the graceful labeling yields distinct edge labels.

The graceful labeling or the odd graceful labeling can be applied in the research field of X-ray Crystallographic Analysis [11,30]. X-ray Crystallography is a tool used for identifying the atomic and molecular structure of a crystal, in which the crystalline atoms cause a beam of incident X-rays to diffract into many specific directions. By measuring the angles and intensities of these diffracted beams, a crystallographer can produce a three dimensional picture of the density of electrons within the crystal. From this
electron density, the mean positions of the atoms in the crystal can be determined, as well as their chemical bonds, their disorders and various other information can be obtained. As the position of an atom in crystal structures are determined by X-ray diffraction patterns, a series of mathematical calculations is used to produce a diffraction to the particular arrangement of atoms in that crystal. Measurements indicate the set of inter atomic distances in crystal lattices. Thus, the graceful labeling or the odd graceful labeling concept can be used to find the finite set of integers $R = \{0 = a_1 < a_2 < \ldots < a_n\}$ (vertices) that corresponds to the atom position, so that diffraction is equivalent to the distinct edge lengths which are the difference between two integers.

Edge labeling problems [37,104] originated due to the study of electrical networks in which the electrical terminals represent the vertices of the graph and the electrical resistances are the distances between the vertices or the vertex labels. These have applications in the field of Electrical Engineering. Labeling the edges of a graph is also a powerful method in solving several planar graph drawing problems successfully.

The harmonious labeling, introduced by Graham and Sloane [39], is closely related to problems in error-correcting codes[67] which has applications in compact discs, in computer memories and in photographs from spacecraft.

Cordial labeling finds its application in DNA code word design problems [80] and in noisy communication channels [67].

Prime labeling and vertex prime labeling have a practical use in the area of Cryptography [65], especially in public key cryptography algorithms that play a vital role in security applications.
1. Introduction

The applications of shell graphs are presented as follows.

In telecommunication networks, graph connectivity models of wireless networks refer to point-to-point connection of nodes and edges that are designed to represent wireless network topologies. Thus, a wireless communication network can often be represented as interconnection of stations (transmitter and receiver) and links that connect two stations. The assumption is that the network is subject to natural failure or enemy attack aimed at isolating stations from each other. The vulnerability parameter is one of important key issues in assessing the performance of point-to-point wireless networks [2]. The problem of determining the vulnerability and designing wireless network which are invulnerable to enemy attack are of paramount importance. The path network topology and the star network topology are more vulnerable to attacks and failure. Hence in order to build a more reliable network, we can utilize both configurations (path and star networks) to form one new network. Thus, the shell or the fan network model, which combines path and star networks, is invulnerable to attacks and failures. Shell graphs, \( C(n; n-3) \), were defined by Deb and Limaye [24] as the cycle \( C_n \) with \( (n - 3) \) chords sharing a common endpoint called the apex.

![Shell graph C(n, n - 3)](image)

**Figure 1.15 Shell graph \( C(n, n - 3) \)**

The layout pattern of the interconnections between computers in a network is called network topology. Shell graphs or shell structures have a
prominent importance in network topology. The Local Area Network (LAN) [72] uses the shell network topology. LANs are capable of transmitting data at very fast rates, much faster than data can be transmitted over a telephone line, but the distances are limited. Other characteristics of LAN are low error rate, good response time and sharing of hardware, software and data files.

One of the attributes of an interconnection network is robustness [103]. In Computer Science, robustness is the ability of a computer system to cope with errors during execution and cope with erroneous input. In distributed systems, agents exchange information with each other locally. This exchange of information is possible through an interconnection network of agents that can be modelled by a graph structure. In fact, connectivity of the underlying graph structure is a necessary requirement for the consensus protocol to work. Moreover, the structure of the underlying network affects various properties of a system including convergence rates, connectivity of the network under edge (interconnection among agents) or vertex (agent) failures. A highly connected network is obviously less affected by an edge or vertex failure and is therefore, more robust to these deletions. Thus, the shell graph structure plays a key role when understanding the effects of edge or vertex failures and also in maintaining the structural robustness of the overall network. Labeling in a shell graph structure is used in routing problems [22] in Computer Science as shell graphs have fixed structures.

1.5 AN OVERVIEW OF THE THESIS

This thesis pertains to the study of graph labelings on various shell related graphs. The thesis comprises seven chapters on graceful labeling, variations of graceful labeling and other graph labelings.

In Chapter 1, an introduction to Graph Theory and graph labeling is given. The basic definitions and concepts in Graph Theory which are used in
the subsequent chapters of our thesis are stated. Various graph labelings such as graceful labeling, $k$-graceful labeling, odd graceful labeling, almost graceful labeling, one modulo three graceful labeling, edge-odd graceful labeling, harmonious labeling, cordial labeling, prime labeling, vertex prime labeling and $\rho$-labeling are defined and a literature survey on these labelings is presented. Finally, the applications of graph labeling and shell graphs are given.

In Chapter 2, the definitions of shell graph and multiple shell graph, introduced by Deb and Limaye [24], are presented. Based on the definition of the shell graph, we define new graphs called shell bow graph and the uniform shell bow graph. In this chapter, we have proved that all uniform shell bow graphs are graceful and edge-odd graceful. Towards the end of this chapter we have proved that all uniform shell bow graphs admit prime labeling and vertex prime labeling.


In Chapter 3, we have introduced a new graph called the shell butterfly graph. In this chapter, we prove that the shell butterfly graphs admit various kinds of labelings such as graceful labeling, almost graceful labeling, edge-odd graceful labeling, $k$-graceful labeling, and harmonious labeling. In the latter part of this chapter we introduce and define a new labeling called $\rho^*$-labeling. This $\rho^*$-labeling is a variation of the $\rho$-labeling introduced by Rosa [88]. We have proved that shell butterfly graphs admit $\rho^*$-labeling.

The results in this chapter have been published in the following journals: International Journal of Pure and Applied Mathematics 101(5) (2015), 949-956; Proceedings of the National Conference in Pure and
In Chapter 4, we define the subdivided shell graphs and the subdivided uniform shell bow graphs. In this chapter, we have proved that subdivided shell graphs and subdivided uniform shell bow graphs are odd graceful. We have also proved that subdivided shell graphs and subdivided uniform shell bow graphs admit one modulo three graceful labeling. Further, we have proved that subdivided uniform shell bow graphs admit $\rho^*$-labeling, cordial labeling, prime labeling and vertex prime labeling.

The contents of this chapter have been published in the following journals: Eduventure, 7(13) (2014), 103-105; International Research Journal of Mathematical Sciences 3(2) (2014), 645-647; Proceedings of International Conference on Viable Synergies in Mathematical and Natural Sciences, (2016), 174-178 and one result was presented at the National Symposium on Mathematics and Computer Applications, WCC, Chennai, (2014).

In Chapter 5, we define two more new graphs called shell flower graph and subdivided shell flower graph. In this chapter, we have proved that all shell flower graphs are graceful. We have also proved that subdivided shell flower graphs of the same order are odd graceful and cordial.

Chapter 6 deals with the disjoint union of two subdivided shell graphs of any order. We have proved that the disjoint union of two subdivided shell graphs is odd graceful, one modulo three graceful, $k$-graceful and admits $\rho^*$-labeling.

The results in this chapter have been published in the following journals: Annals of Pure and Applied Mathematics 8(2) (2014), 19-25; Proceedings of the National Conference on Mathematics and Computer Applications (2015), 261-266 and one result was presented at the International Workshop on Parallel Computing, VIT University, Chennai (2014).

In Chapter 7, we have defined a new graph $SSG(n)$. In this chapter, we have proved that, $SSG(n)$ is odd graceful, $k$-graceful and admits $\rho^*$-labeling when $n = 2$. We have also proved that, the graph $SSG(n)$ is odd graceful, one modulo three graceful, cordial and admits $\rho^*$-labeling when $n = 3$.

The contents of this chapter have been published in the Proceedings of the National Conference on Mathematics and Computer Applications, (2015), 261-266 and two results have been presented at the International Workshop on Parallel Computing, VIT University, Chennai, (2014). One result has been accepted for publication in the International Journal of Computer Aided Engineering and Technology (2016).

Throughout the thesis, a few open problems are posed and future research on shell related graphs with respect to different graph labelings is also discussed.