Chapter 4

Oscillatory convection under thermal modulation
4.1 Weakly nonlinear oscillatory convection in a viscoelastic fluid saturating porous medium under temperature modulation

4.1.1 Introduction

Based on the following papers (where Bhadauria and Kiran (2013a) studied temperature modulation (Venezian 1969) in an anisotropic porous medium while performing a weakly nonlinear study for stationary mode of convection, Bhadauria and Kiran (2014a,b) studied gravity modulation in fluid and porous medium considering viscoelastic fluid, while performing a weakly nonlinear study for oscillatory mode of convection) in this section we study a weakly nonlinear thermal instability in a viscoelastic fluid saturated porous medium under temperature modulation, and quantify the heat transfer in terms of the amplitude of convection which is evaluated by complex Ginzburg–Landau equation.

4.1.2 Governing Equations

We considered an infinitely extended horizontal fluid saturated porous layer of depth ‘d’ as given in figure 4.1a. The porous layer is homogeneous and isotropic, and saturated with viscoelastic fluid. The porous medium is heated slowly from below. Using modified Darcy’s model (Alishaev 1975) and employing the Boussinesq approximation for this system, the governing equations of flow and temperature fields are expressed as

\[ \nabla \cdot \vec{q} = 0, \]

\[ \left( \frac{\lambda_1}{\partial t} + 1 \right) (-\nabla P + \rho \vec{g}) - \frac{\mu}{K} \left( \frac{\lambda_2}{\partial t} + 1 \right) \vec{q} = 0, \]

\[ \frac{\partial T}{\partial t} + (\vec{q} \cdot \nabla)T = \kappa_T \nabla^2 T, \]

\[ \rho = \rho_0[1 - \alpha_T(T - T_0)], \]

\[ \nabla \cdot \vec{q} = 0, \quad (4.1.1) \]

\[ \left( \frac{\lambda_1}{\partial t} + 1 \right) (-\nabla P + \rho \vec{g}) - \frac{\mu}{K} \left( \frac{\lambda_2}{\partial t} + 1 \right) \vec{q} = 0, \quad (4.1.2) \]

\[ \frac{\partial T}{\partial t} + (\vec{q} \cdot \nabla)T = \kappa_T \nabla^2 T, \quad (4.1.3) \]

\[ \rho = \rho_0[1 - \alpha_T(T - T_0)], \quad (4.1.4) \]
where the physical variables have their usual meanings as given in Nomenclature. The externally imposed thermal boundary conditions are considered as (Venezian 1969)

\[
T = T_0 + \frac{\Delta T}{2} [1 + \chi^2 \delta \cos(\Omega t)], \quad \text{at } z = 0
\]

\[
= T_0 - \frac{\Delta T}{2} [1 - \chi^2 \delta \cos (\Omega t + \theta)], \quad \text{at } z = d
\] (4.1.5)

where \( \Delta T \) is the temperature difference across the porous medium, \( \delta, \Omega \) are amplitude and frequency of temperature modulation, and \( \theta \) is the phase angle.

### 4.1.3 Basic state

The basic state is assumed to be quiescent and the quantities in this state are given

\[
\vec{q}_b = 0, p = p_b(z, t), \quad T = T_b(z, t), \quad \rho = \rho_b(z, t).
\] (4.1.6)

Substituting the Eq.(4.1.6) in Eqs.(4.1.1)-(4.1.4), we get the following relations which helps us to define basic state pressure and temperature:

\[
\frac{\partial p_b}{\partial z} = -\rho_b \vec{g},
\] (4.1.7)

\[
\frac{\partial T_b}{\partial t} = \kappa_T \frac{\partial^2 T_b}{\partial z^2},
\] (4.1.8)

\[
\rho_b = \rho_0 [1 - \alpha_T (T_b - T_0)].
\] (4.1.9)

The solution of equation (4.1.8), subjected to the boundary conditions (4.1.5), is given by

\[
T_b(z, t) = T_s(z) + \chi^2 \delta \text{Re}[T_1(z, t)],
\] (4.1.10)

where

\[
T_s(z) = T_0 + \frac{\Delta T}{2} \left(1 - \frac{2z}{d}\right),
\] (4.1.11)

\[
T_1(z, t) = \left\{ a_1(\zeta)e^{\frac{i\zeta^2}{\kappa_T}} + a_1(-\zeta)e^{-\frac{i\zeta^2}{\kappa_T}} \right\} e^{-i\Omega t},
\] (4.1.12)

and \( a_1(\zeta) = \frac{\Delta T}{2} \frac{(e^{i\theta} - e^{-i\theta})}{(e^{\zeta} - e^{-\zeta})} \) and \( \zeta^2 = \frac{-i\alpha_T \Omega}{\kappa_T} \). Here \( T_s(z) \) is the steady part, while \( T_1(z, t) \) is the oscillatory part of the basic state temperature field \( T_b(z, t) \). The finite amplitude perturbations on the basic state are superposed in the form:

\[
\vec{q} = \vec{q}_b + \vec{q}', \quad \rho = \rho_b + \rho', \quad p = p_b + p', \quad T = T_b + T'.
\] (4.1.13)
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We introduce the Eq.(4.1.13) and the basic state temperature field in Eqs.(4.1.1)−(4.1.4), and then using stream function and non-dimensionalized quantities as in chapter 3, the resulting non-dimensionalized system of equations are

$$
\left( \lambda_2 \frac{\partial}{\partial t} + 1 \right) \nabla^2 \psi + Ra_D \left( \lambda_1 \frac{\partial}{\partial t} + 1 \right) \frac{\partial T}{\partial x} = 0
$$

(4.1.14)

$$
- \frac{\partial T_b}{\partial z} \frac{\partial \psi}{\partial x} + \left( \frac{\partial}{\partial t} - \nabla^2 \right) T = \frac{\partial (\psi, T)}{\partial (x, z)}.
$$

(4.1.15)

The basic state solution which appears in Eq.(4.1.15), influences the stability problem through the factor $\frac{\partial T_b}{\partial z}$, which is given by

$$
\frac{\partial T_b}{\partial z} = -1 + \chi^2 \delta (f_2(z,t)),
$$

(4.1.16)

where

$$
f_2(z,t) = \text{Re} \left( f(z) e^{i\Omega t} \right)
$$

(4.1.17)

$$
f(z) = (A(\zeta)e^{\zeta z} + A(-\zeta)e^{-\zeta z}), \quad A(\zeta) = \frac{\zeta (e^{-i\theta} - e^{-\zeta})}{2 (e^{\zeta} - e^{-\zeta})} \quad \text{and} \quad \zeta = (1 - i)\sqrt{\frac{\Omega}{2}}.
$$

(4.1.18)

We write the nonlinear system of Eqs.(4.1.14)−(4.1.15), in the matrix form as given bellow

$$
\begin{bmatrix}
\left( \lambda_2 \frac{\partial}{\partial t} + 1 \right) \nabla^2 & Ra_D \left( \lambda_1 \frac{\partial}{\partial t} + 1 \right) \frac{\partial}{\partial x} \\
\frac{\partial}{\partial x} & \left( \frac{\partial}{\partial t} - \nabla^2 \right)
\end{bmatrix}
\begin{bmatrix}
\psi \\
T
\end{bmatrix}
= 
\begin{bmatrix}
-Ra_D \left( \lambda_1 \frac{\partial}{\partial t} + 1 \right) \frac{\partial T}{\partial x} \\
\frac{\partial (\psi, T)}{\partial (x, z)} + \chi^2 \delta f_2(z, t) \frac{\partial \psi}{\partial x}
\end{bmatrix}
$$

(4.1.19)

The above system will be solved by considering stress-free and isothermal boundary conditions as given in Eq.(2.2.26). In order to seek the solution of the above system Eq.(4.1.19) we introduce an asymptotic series or Poincaré expansion given in Eq.(2.3.1) for $Ra$, $\psi$ and $T$ in terms of a small perturbation parameter $\chi$ that show a deviation from the critical state of onset of convection.

4.1.4 **Bifurcation of periodic solution**

In order to allow for anticipated frequency shift along the bifurcation solution, we introduce the fast time scale of time $\tau$ and the slow time scale of $s$. Therefore, the scaling of time variable is such that $\frac{\partial}{\partial t} = \frac{\partial}{\partial \tau} + \chi^2 \frac{\partial}{\partial s}$. In the first order problem the nonlinear term in energy equation will be vanished therefore, the first order problem reduces to the linear

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stability problem for supercritical flow.

**At the lowest order**, we have

\[
\begin{bmatrix}
(\lambda_2 \frac{\partial}{\partial \tau} + 1) \nabla^2 & R_0 c (\lambda_1 \frac{\partial}{\partial \tau} + 1) \frac{\partial}{\partial x} \\
\frac{\partial}{\partial x} & (\frac{\partial}{\partial \tau} - \nabla^2)
\end{bmatrix}
\begin{bmatrix}
\psi_1 \\
T_1
\end{bmatrix}
= \begin{bmatrix}
0 \\
0
\end{bmatrix}
\] (4.1.20)

The solution of the lowest order system subject to the boundary conditions Eq. (2.2.26), is assumed to be

\[
\psi_1 = (A_1(s)e^{i\omega \tau} + A_1^{*}(s)e^{-i\omega \tau}) \cos ax \sin \pi z, \quad (4.1.21)
\]

\[
T_1 = (B_1(s)e^{i\omega \tau} + B_1^{*}(s)e^{-i\omega \tau}) \sin ax \sin \pi z. \quad (4.1.22)
\]

The undetermined amplitudes are functions of slow time scale and are related by the following relation:

\[
B_1(s) = -\frac{c + i\omega}{a} A_1(s) \quad (4.1.23)
\]

where \( c = a^2 + \pi^2 \). The values of the Darcy-Rayleigh number and the corresponding wave number for stationary mode of convection

\[
Ra_{D}^{st} = \frac{(\pi^2 + a^2)^2}{a^2} \quad (4.1.24)
\]

\[ a_c = \pi, \quad (4.1.25) \]

which are classical results of Horton and Rogers(1945) and Lapwood (1948). We find Darcy-Rayleigh number and corresponding critical wave number for oscillatory convection as given bellow:

\[
Ra_{D}^{osc} = \frac{(\lambda_2 \pi^4 + \pi^2 + 2a^2 \pi^2 \lambda_2 + a^2 + a^4 \lambda_2)}{\lambda_1 a^2} \quad (4.1.26)
\]

\[
a_c^2 = \sqrt{\pi^4 + \frac{\pi^2}{\lambda_2}}, \quad (4.1.27)
\]

which are same as obtained by Kim et al.(2003). The critical Darcy–Rayleigh number and corresponding wave number does not depend on \( (\lambda_1, \lambda_2) \) in stationary mode but in oscillatory mode. Also we see that the supercritical flow can occur for a particular wave number only if the following inequality holds

\[
\lambda_1 > \lambda_2 + \frac{1}{c}. \quad (4.1.28)
\]
The dimensionless frequency of the neutral oscillatory mode is
\[ \omega^2 = \frac{c(\lambda_1 - \lambda_2) - 1}{\lambda_2 \lambda_1}. \] (4.1.29)

**In the second order**, we get
\[
\frac{\partial(\psi_1, T_1)}{(x, z)} = \pi a^2 \left\{ A_1(s) B_1(s) e^{2i\omega \tau} + \overline{A}_1(s) \overline{B}_1(s) e^{-2i\omega \tau} + A_1(s) B_1(s) + \overline{A}_1(s) B_1(s) \right\} \sin 2\pi z. \] (4.1.30)

From the above relation, we can deduce that the velocity and temperature fields have the terms having frequency \(2\omega\) and independent of past time scale. Thus, we write the second order temperature term as follows:
\[
T_2 = \left\{ T_{20} + T_{22} e^{2i\omega \tau} + \overline{T}_{22} e^{-2i\omega \tau} \right\} \sin 2\pi z \] (4.1.31)

where \(T_{22}\) and \(T_{20}\) are temperature fields having the terms having the frequency \(2\omega\) and independent of fast time scale, respectively. The solutions of the second order problems are:
\[
T_{20} = \frac{a}{8\pi} \left\{ A_1(s) B_1(s) + \overline{A}_1(s) B_1(s) \right\}, \quad \psi_{20} = 0 \] (4.1.32)
and
\[
T_{22} = \frac{\pi a}{8\pi^2 + 4i\omega} A_1(s) B_1(s). \] (4.1.33)

The horizontally averaged Nusselt number, \(Nu(s)\), for the oscillatory mode of convection is given by using the expression of \(T_2\), given in Eq.(4.1.31), one can simplify Eq.(3.1.28) and obtain Nu as
\[
Nu(s) = 1 + \frac{c}{2} \left( \frac{2\pi^2 \sqrt{c^2 + \omega^2}}{\sqrt{64\pi^4 + 16\omega^2}} \right) |A_1(s)|^2. \] (4.1.34)

It is clear that the thermal modulation is effective at third order and affects \(Nu(s)\) through \(A_1(s)\) which is evaluated at third order.

**At the third order**, we have
\[
\begin{bmatrix} (\lambda_2 \frac{\partial}{\partial \tau} + 1) \nabla^2 & R_{0e} \left( \lambda_1 \frac{\partial}{\partial \tau} + 1 \right) \frac{\partial}{\partial x} \end{bmatrix} \begin{bmatrix} \psi_3 \\ T_3 \end{bmatrix} = \begin{bmatrix} R_{31} \\ R_{32} \end{bmatrix} \] (4.1.35)

where
\[
R_{31} = -\lambda_2 \frac{\partial}{\partial s}(\nabla^2 \psi_1) - R_{0e} \lambda_1 \frac{\partial}{\partial s} \left( \frac{\partial T_1}{\partial x} \right) - R_2 \left( \frac{\partial}{\partial \tau} + 1 \right) \left( \frac{\partial T_1}{\partial x} \right), \] (4.1.36)

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\[ R_{32} = \frac{\partial \psi_1}{\partial x} \frac{\partial T_2}{\partial z} + \frac{\partial f_2(z,s)}{\partial x} \frac{\partial \psi_1}{\partial s} - \frac{\partial T_1}{\partial s}. \]  

(4.1.37)

Substituting \( \psi_1, T_1 \) and \( T_2 \) into Eqs. (4.1.36-4.1.37), we obtain the expressions for \( R_{31} \) and \( R_{32} \) easily. Now under the stability condition for the existence of third order solution, these equations yield the following Landau equation that describes the temporal variation of the amplitude \( A_1(s) \) of the convection cell

\[
\frac{\partial A_1(s)}{\partial s} - \gamma_1^{-1} F(s) A_1(s) + \gamma_1^{-1} k |A_1(s)|^2 A_1(s) = 0, 
\]

(4.1.38)

where

\[ \gamma_1 = \left[ \frac{a^2 R_{00} \lambda_1}{c(1 + i\omega \lambda_2)} - \frac{\lambda_2(c + i\omega)}{(1 + i\omega \lambda_2)} - 1 \right], \]

\[ F(s) = \left[ \frac{a^2 R_{00}(1 + i\omega \lambda_1)}{c(1 + i\omega \lambda_2)} - 2\delta I_1(c + i\omega) \right], \]

\[ k = -\pi^2 \left( \frac{2c^2 + 2ci\omega}{8\pi^2} + \frac{c^2 + \omega^2}{8\pi^2 + 4i\omega} \right) \]

and

\[ I_1 = \int_0^1 f_2(z,s) \sin^2(\pi z) \, dz. \]

Writing \( A_1(s) = |A_1(s)|e^{i\phi} \)

(4.1.39)

Now substituting the expression Eq. (4.1.39) in Eq. (4.1.38), we get the following equation for the amplitude \( |A_1(s)| \):

\[
\frac{\partial |A_1(s)|^2}{\partial s} = 2p_r |A_1(s)|^2 - 2l_r |A_1(s)|^4 \]  

(4.1.40)

\[
\frac{\partial (p\phi(A_1(s)))}{\partial s} = p_i - l_i |A_1(s)|^2 \]

(4.1.41)

where \( \gamma_1^{-1} F(s) = p_r + ip_i \), \( \gamma_1^{-1} k = l_r + il_i \) and \( p\phi(.) \) represents the phase shift. One can observe here from Eq. (4.1.38) for the case \( l_r > 0 \) and \( Ra^{osc}_D > Ra_c \) i.e. \( p_r > 0 \), the solution gives as \( \bar{A} \sim A_0 e^{p_r s} \) as \( s \to -\infty \), and \( \bar{A} \to 0 \) is unstable solution, and a new stable solution develops, \( \bar{A} = \sqrt{\frac{p_i}{l_i}} \) as \( s \to \infty \), whatever be the value of \( A_0 \).

This is called supercritical pitch fork bifurcation, the base system being linearly unstable for \( Ra^{osc}_D > Ra_c \) but settling down as a new laminar flow. The steady state amplitude exists when \( Ra_c \) takes positive values. Supercritical pitch fork bifurcation diagram has been explained in the figure 4.1b. We have calculated the mean value of Nusselt number (MNu) for better understanding the effect of temperature modulation on heat transport, a representative time interval that allows a clear comprehension of the modulation effect needs to be chosen. The interval \((0, 2\pi)\) seemed an appropriate interval to calculate MNu.

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Figure 4.1: a. Physical configuration of the problem: b. Supercritical pitch fork bifurcation diagram
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The time-averaged Nusselt number $\text{MNu}$ is defined as

$$\text{MNu} = \frac{1}{2\pi} \int_0^{2\pi} \text{Nu} \, ds.$$  

(4.1.42)

The amplitude $A_1(s)$ is obtained numerically and hence $\text{MNu}$ is also to be numerically evaluated. An interesting observation that can be observed in $I_1$, which determines whether the modulation amplifies or diminishes the amplitude of convection. A discussion of the results now follows culminating in a listing of conclusions.

### 4.1.5 Results and Discussion

In this section we made an attempt to investigate oscillatory mode of convection in a viscoelastic fluid saturated porous medium under temperature modulation. We derive the non autonomous complex Ginzburg–Landau equation to evaluate an amplitude of convection under solvability condition. We consider a weakly nonlinear theory to study heat transport in the porous medium. It quite interest that the oscillatory mode of convection is possible only when the values of $\lambda_1, \lambda_2$ are consider as given in the Eq.(4.1.28). To exhibit supercritical flow in the system it is important to take $\lambda_1$ more than $\lambda_2$. For small amplitude and lower values of frequency of modulation the maximum the heat transfer hence we consider $\delta$ around 0.1 and $\Omega$ at 2.0. In order to find out the effect of temperature modulation on the system we consider the following three types of temperature profiles at the boundaries of the problem:

1. In-phase modulation (IPM)($\theta = 0$).

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2. Out-phase modulation (OPM) ($\theta = \pi$).

3. Only Lower boundary modulated (LBMO) ($\theta = -i\infty$),
   which means that the modulation effect will not be considered in upper boundary but only in lower boundary.

In order to illustrate the effects of relaxation parameters $\lambda_1, \lambda_2$, the frequency $\Omega$ and the amplitude $\delta$ of modulation on heat transport, we obtain Nu numerically as a function of slow time scale $s$ and depicted the curves of the Nusselt number versus time $s$. It is clear from the relation Eq. (4.1.26) that the oscillatory convection depends on both stress relaxation and strain retardation times. The stability curves corresponding to the oscillatory critical Rayleigh number $R_{0c}$ has been presented in figure 4.2. In figure 4.2a it is found that the effect of increasing the value of stress relaxation parameter $\lambda_1$ is to reduce $R_{0c}$ hence it has destabilizing effect on the system. However, opposite effect is found for strain retardation parameter $\lambda_2$ as it is increases $R_{0c}$ increase hence it delays the onset of convection and enhance the stability of the system shown in figure 4.2b.

The results corresponding to the temperature modulation on the system shown in

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figures 4.3-4.5, where we have plotted Nu versus \( s \). From these figures it is observed that Nu starts with 1 thus showing the conduction state initially, which means heat transfer across the porous medium is taking place through conduction when \( s \) is small. The value of Nu starts increase for intermediate values of \( s \) thus showing the convection is in progress and finally when \( s \) is very large, the steady state is achieved. The results has been presented in figure 4.3 for in phase modulation case. The results of this case is just as unmodulated system. Where the modulation effect will not be taking place. The effect of an increment in the value of stress relaxation parameter \( \lambda_1 \) given in figure 4.3a is to enhance the value of Nu on increasing \( \lambda_1 \). For lower values of slow time \( s \) there is an enhancement in Nu, further values of time \( s \), Nu achieve steady state. The opposite effect is achieved for strain retardation parameter \( \lambda_2 \), i.e an increment in \( \lambda_2 \) is to suppress the heat transfer in the system given in figure 4.3b. There is no effect is obtained for an amplitude and frequency of modulation given in figure 4.3c. Hence in phase modulation is of negligible effect on heat transfer.

For out of phase modulation (OPM) same results obtained for the case of \( \lambda_1, \lambda_2 \) given in figures 4.4a-b. Here one can notice that effect of modulation reflect on Nu, initially Nu start with 1 at conduction state and increases further in time shows sinusoidal behaviour. The effects of frequency \( \Omega \) and the amplitude \( \delta \) of modulation on heat transport is clearly visible in the case OPM given in figures 4.4c-d. Figure 4.4c show an effect of amplitude of modulation on heat transport as \( \delta \) increases heat transfer enhances in the system. For frequency of modulation \( \Omega \) reduces heat transfer as it is increases given in figure 4.4d. Which means the effect of temperature modulation decreases as the frequency of modulation increases, and finally when \( \Omega \) is very large, the effect of modulation disappears altogether, thus confirming the results of Venezian (1969), Bhadauria et al.(2010,2013c,d) and Siddheshwar et al.(2012a,b). Here we are not presenting results for LBMO case as they are similar to OPM case. On comparing the results in figure 4.5a, it is found that \( \text{Nu}_{\text{LBMO}} \) is lower than \( \text{Nu}_{\text{OPM}} \) but higher than \( \text{Nu}_{\text{IPM}} \) as given below:

\[
\text{Nu}_{\text{IPM}} < \text{Nu}_{\text{LBMO}} < \text{Nu}_{\text{OPM}}
\]

This is also clear due to the fact that in phase modulation of the boundary temperature

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Figure 4.4: Variation of Nu with respect to s OPM case

Figure 4.5: Comparison
does not substantially modify the temperature gradient across the porous medium, therefore not much effect on heat transfer. However, in cases of OPM and LBMO, the effect of temperature modulation on heat transfer is quite visible [figures 4.4a-d, figure 4.5a], and is oscillatory in nature. Here we can find certain frequencies where the value of Nu is high thus destabilizing the system and low thus stabilizing the system. In figure 4.5b we compare the results of oscillatory and stationary mode of convection for OPM case. It is found that heat transfer is more in oscillatory mode of convection than in stationary mode. It can be observed that \( \text{Nu}_{st} < \text{Nu}_{osc} \) for the same wave number. This implies that oscillatory instabilities can set in before stationary mode.

A better way of presenting our results according to Siddheshwar et al. (2013) the effect of modulation on mean Nusselt number depends on both the phase difference \( \theta \) and frequency \( \Omega \) of modulation than only on the choice of the small amplitude modulation. We present the effect of \( \Omega \) in the figure 4.6 and the effect of \( \theta \) in figure 4.7. From the figures 4.6-4.7 it is evident that for a given frequency of modulation there is a range of \( \theta \) in which MNu increases with increasing \( \theta \) and another range in which MNu decreases.

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Thus, one can conclude that, the combination of choices of $\Omega$ and $\theta$ can be made depending on the demands on heat transport in an application situation. Heat transfer can be regulated (enhanced or reduced) with the external mechanism of temperature modulation effectively. Our results are compatible with results of Malashetty et al. (2002). We also can observe our results in figures 4.6-4.7 are the results which are similar to Siddheshwar et al. (2013) for the Newtonian fluid case. It is clear that for temperature modulation the boundary temperatures should not be in in-phase modulation (synchronized), where the effect of modulation is negligible on heat transport.

4.1.6 Conclusions

We have analyzed the effect of temperature modulation on supercritical flow of Bénard–Darcy convection by performing a weakly nonlinear stability analysis resulting in the complex Ginzburg–Landau amplitude equation. The following conclusions are made by the previous analysis.

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1. Effect of IPM is negligible on heat transport in the system.

2. In the case of IPM, the effect of $\delta$ and $\Omega$ are also found to be negligible on heat transport.

3. Effect of $\lambda_1$ is to enhance the heat transport for all three types of modulations.

4. Effect of $\lambda_2$ is to decrease the heat transport for all three types of modulations.

5. In the case of IPM, Nu increase steadily for intermediate value of time $s$ and ultimately becomes constant when $s$ is large.

6. In the case of OPM and LBMO, Nu shows an oscillatory nature.

7. The results upon MNu follows as in the case of Nu.
Chapter-4.2: **Double diffusive oscillatory convection**

4.2 Heat and mass transfer for oscillatory convection in a binary viscoelastic fluid layer subjected to temperature modulation at the boundaries

4.2.1 Introduction

Thermohaline convection is an important fluid dynamics phenomenon that involves motions driven by two different density gradients diffusing at different rates. In two component fluid convection, the buoyancy force is affected not only by the difference of temperatures, but also by the difference of concentration of the fluid. The best example of double diffusive convection can be seen in oceanography, under ground water and lakes, atmospheric pollution, modeling of solar ponds, electrochemistry, chemical processes, laboratory experiments, magma chambers and sparks (Huppert and Sparks 1984), Fernando and Brandt (1994) formation of microstructure during the cooling of molten metals, migration of moisture through air contained in fibrous insulations, fluid flows around shrouded heat dissipation fins, grain storage system, the dispersion of contaminants through water saturated soil, solidification of binary mixtures, crystal P growth, and the underground disposal of nuclear wastes. In the case of non-Newtonian fluids; in particular viscoelastic fluids are important with a lot of industrial applications. The convection in viscoelastic fluids are important in chemical processing industries. The understanding of convective motion and its behaviour is important for controlling many industrial processes e.g. geothermal reservoirs, enhanced oil recovery, filtration, polymer filament package and composite impregnations. Malashetty and Swamy (2010) investigated linear and weak nonlinear thermal instability of double diffusive convection in a viscoelastic fluid. The onset criterion for both stationary and oscillatory convection is derived while using finite amplitude analysis analytically. The truncated representation of Fourier series method is used to find the heat and mass transfers for weak nonlinear theory. Using Runge–Kutta method they solved finite amplitude equations and quantified heat and mass transfer in terms of the Nusselt and Sherwood number. In this
section the double diffusive oscillatory thermal convection in a viscoelastic fluid layer has been investigated and found much more effective results than in the case of stationary convection.

4.2.2 Mathematical Formulation

We consider an infinitely extended horizontal layer of incompressible binary viscoelastic fluid mixture, confined between two free free boundaries at \( z = 0 \) and \( z = d \), and subjected to a time periodic temperature at the boundaries. A cartesian frame of reference is chosen with the origin at the lower boundary and the \( z \) axis vertically upward. The fluid layer is heated from below. The hydrodynamic equations are simplified by assuming Oberbeck–Boussinesq approximation. The disturbances are varied in the vertical direction. The viscoelastic fluid of the Oldroyd type is used to model the momentum equation.

The constitutive equations are given bellow

\[
\nabla \cdot \vec{q} = 0, \quad (4.2.1)
\]
\[
\left( \bar{\lambda}_1 \frac{\partial}{\partial t} + 1 \right) \left( \frac{\partial \vec{q}}{\partial t} + (\vec{q} \cdot \nabla)\vec{q} + \frac{1}{\rho_0} \nabla p - \frac{\rho}{\rho_0} \vec{g} \right) - \nu \left( \bar{\lambda}_2 \frac{\partial}{\partial t} + 1 \right) \nabla^2 \vec{q} = 0, \quad (4.2.2)
\]
\[
\frac{\partial T}{\partial t} + (\vec{q} \cdot \nabla) T = \kappa_T \nabla^2 T, \quad (4.2.3)
\]
\[
\frac{\partial S}{\partial t} + (\vec{q} \cdot \nabla) S = \kappa_S \nabla^2 S, \quad (4.2.4)
\]
\[
\rho = \rho_0 \left[ 1 - \alpha_T (T - T_0) + \beta_S (S - S_0) \right], \quad (4.2.5)
\]

where the physical variables have their usual meanings and are given in Nomenclature.

The externally imposed, thermal and solutal boundary conditions are given by

\[
T = T_0 + \frac{\Delta T}{2} [1 + \chi^2 \delta \cos(\Omega t)] \quad \text{at } z = 0
\]
\[
T = T_0 - \frac{\Delta T}{2} [1 - \chi^2 \delta \cos(\Omega t + \theta)] \quad \text{at } z = d \quad (4.2.6)
\]
\[
S = S_0 + \Delta S \quad \text{at } z = 0
\]
\[
S = S_0 \quad \text{at } z = d, \quad (4.2.7)
\]

where \( \Delta T, \Delta S \) are the temperature, concentration difference across the fluid layer, \( \chi \) is the smallness of amplitude of modulation, \( \delta, \Omega \) are amplitude and frequency of thermal

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modulation. The basic state is assumed to be quiescent, and the quantities in this state are given by

\[ \vec{q}_b = 0, p = p_b(z,t), \ T = T_b(z,t), \ \rho = \rho_b(z,t) \ S = S_b(z). \quad (4.2.8) \]

Substituting the Eq. (4.2.8) in Eqs. (4.2.1)-(4.2.5), we get the following relations, which help us in defining the basic state pressure, temperature and solute:

\[ \frac{\partial p_b}{\partial z} = -\rho_b g, \quad (4.2.9) \]
\[ \frac{\partial T_b}{\partial t} = \kappa_T \frac{\partial^2 T_b}{\partial z^2}, \quad (4.2.10) \]
\[ \kappa_S \frac{d^2 S_b}{dz^2} = 0, \quad (4.2.11) \]
\[ \rho_b = \rho_0 \left[ 1 - \alpha_T (T_b - T_0) + \beta_S (S_b - S_0) \right]. \quad (4.2.12) \]

The solution of Eqs. (4.2.10)-(4.2.11), subjected to the boundary conditions given in Eqs. (4.2.6)-(4.2.7), are given by

\[ T_b(z,t) = T_s(z) + \chi^2 \delta \text{Re}[T_1(z,t)], \quad (4.2.13) \]
\[ S_b = S_0 + \Delta S \left( 1 - \frac{z}{d} \right). \quad (4.2.14) \]

The finite amplitude perturbations on the basic state are superposed in the form:

\[ \vec{q} = \vec{q}_b + \vec{q}', \ \rho = \rho_b + \rho', \ p = p_b + p', \ T = T_b + T'. \quad (4.2.15) \]

We introduce the perturbed quantities Eq. (4.2.15) in Eqs. (4.2.1)-(4.2.5) and then non-dimensionalizing, one can obtained the following equations:

\[ \left( \lambda_1 \frac{\partial}{\partial t} + 1 \right) \left( \frac{1}{Pr} \frac{\partial}{\partial t} \nabla^2 \psi + Ra \frac{\partial T}{\partial x} - Rs \frac{\partial S}{\partial x} - \frac{1}{Pr} \frac{\partial (\psi, \nabla^2 \psi)}{\partial (x,z)} \right) = \left( \lambda_2 \frac{\partial}{\partial t} + 1 \right) \nabla^4 \psi, \quad (4.2.16) \]
\[ - \frac{\partial T_b}{\partial z} \frac{\partial \psi}{\partial x} + \left( \frac{\partial}{\partial t} - \nabla^2 \right) T = \frac{\partial (\psi, T)}{\partial (x,z)}, \quad (4.2.17) \]
\[ \frac{\partial \psi}{\partial x} + \left( \frac{\partial}{\partial t} - \Gamma \nabla^2 \right) S = \frac{\partial (\psi, S)}{\partial (x,z)}, \quad (4.2.18) \]
where \( Ra = \frac{\alpha g \Delta T d^3}{\nu k T} \) is thermal Rayleigh number, \( Rs = \frac{\beta g \Delta S d^3}{\nu k S} \) is the solutal Rayleigh number. The basic state solution which appears in Eq. (4.2.17), influences the stability problem through the factor \( \frac{\partial T_b}{\partial z} \), which is given by

\[
\frac{\partial T_b}{\partial z} = -1 + \chi^2 \delta (f_2(z,t)) ,
\]

where

\[
f_2(z,t) = \text{Re} \left( f(z)e^{i(\Omega t)} \right),
\]

\[
f(z) = (A(\zeta)e^{\zeta z} + A(-\zeta)e^{-\zeta z}) ,
A(\zeta) = \frac{\zeta (e^{-i\theta} - e^{-\zeta})}{2 (e^\zeta - e^{-\zeta})} \text{ & } \zeta = (1 - i)\sqrt{\frac{\Omega}{2}} .
\]

The above system Eqs. (4.2.16)-(4.2.18) will be solved by considering stress free, isothermal and isosolutal boundaries. Hence the boundary conditions for perturbed variables are given by

\[
\psi = \frac{\partial^2 \psi}{\partial z^2} = S = T = 0 \text{ on } z = 0 \text{ } z = 1.
\]

### 4.2.3 Finite amplitude equation

Using an asymptotic expansion given in Eq. (2.3.1) for physical quantities (\( Ra, \psi, T, S \)), the above system Eqs. (4.2.16)-(4.2.18) is solved for every order of \( \chi \). In order to allow for anticipated frequency shift along the bifurcation solution, we introduce the fast time scale of time \( \tau \) and the slow time scale of \( s \). Therefore, the scaling of time variable is such that \( \frac{\partial}{\partial t} = \frac{\partial}{\partial \tau} + \lambda \frac{\partial}{\partial s} \). In the first order problem, the nonlinear term in energy equation will vanish, therefore, the first order problem reduces to the linear stability problem for overstability.

At the lowest order, we have

\[
\begin{bmatrix}
\frac{1}{Pr} \left( \lambda_1 \frac{\partial}{\partial \tau} + 1 \right) \frac{\partial}{\partial \tau} \nabla^2 - \left( \lambda_2 \frac{\partial}{\partial \tau} + 1 \right) \nabla^4 R_0 \left( \lambda_1 \frac{\partial}{\partial \tau} + 1 \right) \frac{\partial}{\partial \tau} - Rs \left( \lambda_1 \frac{\partial}{\partial \tau} + 1 \right) \frac{\partial}{\partial \tau} \\
\frac{\partial T_b}{\partial z} \frac{\partial}{\partial \tau} \\
\frac{\partial}{\partial \tau} \\
0 \\
0 \\
0 \\
0 \\
0 \end{bmatrix} \begin{bmatrix} \psi \\ T_1 \\ S_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\]

\[
(4.2.23)
\]
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The solution of the lowest order system subject to the boundary conditions Eq.(4.2.22), is assumed to be

\[
\psi_1 = (A(s)e^{i\omega \tau} + \overline{A}(s)e^{-i\omega \tau}) \sin ax \sin \pi z, \quad (4.2.24)
\]

\[
T_1 = (B(s)e^{i\omega \tau} + \overline{B}(s)e^{-i\omega \tau}) \cos ax \sin \pi z, \quad (4.2.25)
\]

\[
S_1 = (C(s)e^{i\omega \tau} + \overline{C}(s)e^{-i\omega \tau}) \cos ax \sin \pi z. \quad (4.2.26)
\]

The undetermined amplitudes are functions of slow time scale, and are related by the following relation:

\[
\mathbb{B}(s) = -\frac{a}{c + i\omega}A(s), \quad (4.2.27)
\]

\[
\mathbb{C}(s) = -\frac{a}{\Gamma c + i\omega}A(s), \quad (4.2.28)
\]

where \( c = a^2 + \pi^2 \) is total wavenumber. The critical Rayleigh number for oscillatory case is given by

\[
R_0 = \frac{c}{a^2} \left[ c^2(1 + \omega^2\lambda_1\lambda_2) \frac{1}{1 + \lambda_1^2\omega^2} - \omega^2 \left( \frac{1}{Pr} + \frac{(\lambda_2 - \lambda_1)c}{1 + \lambda_1^2\omega^2} \right) \right] + \frac{Rs(\Gamma c^2 + \omega^2)}{\Gamma^2 c^2 + \omega^2}. \quad (4.2.29)
\]

Here we calculate the corresponding critical wave number by minimizing critical Rayleigh number with respect to the wave number. The growth rate \( \omega^2 \) can be obtained from:

\[
a_1\omega^4 + a_2\omega^2 + a_3 = 0. \quad (4.2.30)
\]

Observing closely on \( a_1, a_2, a_3 \) (are given in appendix) it reveals that the necessary condition for the occurrence of the oscillatory convection is that the following inequalities hold:

\[
\lambda_1 > \lambda_2 \& \Gamma < 1. \quad (4.2.31)
\]

Also, the value of critical Rayleigh number and the corresponding wavenumber for stationary mode of convection is given by

\[
R_0 = \frac{c^3}{a^2} + \frac{Rs}{\Gamma}, \quad (4.2.32)
\]

\[
a = \frac{\pi}{\sqrt{2}}. \quad (4.2.33)
\]
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The critical Rayleigh number and corresponding wave number does not depend on \((\lambda_1, \lambda_2)\) in stationary mode but in oscillatory mode.

**In the second order**, we get the following relations

\[
\psi_2 = 0, \quad (4.2.34)
\]

\[
\left( \frac{\partial}{\partial \tau} - \nabla^2 \right) T_2 = \frac{\partial (\psi_1, T_1)}{\partial (x, z)}, \quad (4.2.35)
\]

\[
\left( \frac{\partial}{\partial \tau} - \Gamma \nabla^2 \right) S_2 = \frac{\partial (\psi_1, S_1)}{\partial (x, z)}. \quad (4.2.36)
\]

From the above relation, according to Kim et al. (2003), we can deduce that the velocity, temperature and solutal fields have the terms having frequency \(2\omega\) and independent of fast time scale. Thus, we write the second order temperature, solutal terms as follows:

\[
T_2 = \{T_{20} + T_{22} e^{2i\omega \tau} + \bar{T}_{22} e^{-2i\omega \tau}\} \sin 2\pi z, \quad (4.2.37)
\]

\[
S_2 = \{S_{20} + S_{22} e^{2i\omega \tau} + \bar{S}_{22} e^{-2i\omega \tau}\} \sin 2\pi z, \quad (4.2.38)
\]

where \((T_{20}, T_{22})\) and \((S_{20}, S_{22})\) are temperature and solutal fields having the terms having the frequency \(2\omega\) and independent of fast time scale, respectively. The second order solutions can be defined using \(T_2, S_2\) in Eqs.\((4.2.35)-(4.2.36)\). The horizontally averaged Nusselt, Sherwood numbers, for the oscillatory mode of convection is given by:

\[
\text{Nu}(s) = 1 - \chi^2 \left( \frac{\partial T_2}{\partial z} \right)_{z=0}, \quad (4.2.39)
\]

\[
\text{Sh}(s) = 1 - \chi^2 \left( \frac{\partial S_2}{\partial z} \right)_{z=0}. \quad (4.2.40)
\]

**At the third order**, we have

\[
\begin{bmatrix}
\frac{1}{\Pr} (\lambda_1 \frac{\partial}{\partial \tau} + 1) \frac{\partial}{\partial \tau} \nabla^2 - (\lambda_2 \frac{\partial}{\partial \tau} + 1) \nabla^4 & R_0 (\lambda_1 \frac{\partial}{\partial \tau} + 1) \frac{\partial}{\partial x} & -Rs (\lambda_1 \frac{\partial}{\partial \tau} + 1) \frac{\partial}{\partial x} \\
-\frac{\partial \psi_3}{\partial x} \frac{\partial}{\partial x} & (\frac{\partial}{\partial \tau} - \nabla^2) & 0 \\
\frac{\partial}{\partial x} & 0 & (\frac{\partial}{\partial \tau} - \Gamma \nabla^2)
\end{bmatrix}
\]

\[
\begin{bmatrix}
\psi_3 \\
T_3 \\
S_3
\end{bmatrix}
= \begin{bmatrix}
R_{31} \\
R_{32} \\
R_{33}
\end{bmatrix}, \quad (4.2.41)
\]
where the expressions for $R_{31}, R_{32}$ and $R_{33}$ are given in the appendix. Now under the stability condition for the existence of third order solution, we obtain the following Landau equation that describes the temporal variation of the amplitude $\mathcal{A}(s)$ of the convection cell

$$\frac{\partial \mathcal{A}(s)}{\partial s} - \gamma_1^{-1} F(s) \mathcal{A}(s) + \gamma_1^{-1} k |\mathcal{A}(s)|^2 \mathcal{A}(s) = 0.$$  \hspace{1cm} (4.2.42)

where the coefficients $\gamma_1, F(s)$ and $k$ are given in the appendix. Writing $\mathcal{A}(s)$ in the phase-amplitude form, we get

$$\mathcal{A}(s) = |\mathcal{A}(s)| e^{i\theta}.$$  \hspace{1cm} (4.2.43)

Now substituting the expression Eq.(4.2.43) in Eq.(4.2.42), we get the following equations for the amplitude $|\mathcal{A}(s)|$:

$$\frac{\partial |\mathcal{A}(s)|^2}{\partial s} - 2p_r |\mathcal{A}(s)|^2 + 2l_r |\mathcal{A}(s)|^4 = 0,$$  \hspace{1cm} (4.2.44)

$$\frac{\partial (\text{ph} (\mathcal{A}(s))))}{\partial s} = p_i - l_i |\mathcal{A}(s)|^2,$$  \hspace{1cm} (4.2.45)

where $\gamma_1^{-1} F(s) = p_r + i p_i$, $\gamma_1^{-1} k = l_r + i l_i$ and $\text{ph}(.)$ represents the phase shift. We have calculated the mean value of Nusselt (MNu), Sherwood (MSh) numbers for better understanding the effect of thermal modulation on heat and mass transports, a representative time interval that allows a clear comprehension of the modulation effect needs to be chosen. The interval $(0, 2\pi)$ seemed an appropriate interval to calculate MNu and MSh. The time averaged Nusselt and Sherwood numbers are defined as

$$\text{MNu} = \frac{1}{2\pi} \int_0^{2\pi} \text{Nu} ds,$$  \hspace{1cm} (4.2.46)

$$\text{MSh} = \frac{1}{2\pi} \int_0^{2\pi} \text{Sh} ds.$$  \hspace{1cm} (4.2.47)

An interesting observation that can be seen in $I_1$, which determines whether the modulation amplifies or diminishes the amplitude of convection. A discussion of the results now follows culminating in a listing of conclusions.
4.3 Results and discussions

Effect of temperature modulation on double diffusive thermal convection in a viscoelastic fluid layer has been analysed, for overstable mode, by means of a weakly non-linear theory. The amplitude equations for the bifurcations states are also obtained. In order to illustrate the effects of various parameters of the system on heat and mass transports, we plot the curves of Nusselt, Sherwood numbers versus slow time $s$. It is observed that the relation given in Eq.(4.2.31) leads to an interesting result; that for a viscoelastic fluid layer heated and salted underneath; the oscillatory type of instability is possible only when the relaxation parameter $\lambda_1$ is greater than the retardation parameter $\lambda_2$, and the diffusivity ratio less than 1. Also, it is clear from the relation Eq.(4.2.29) that the oscillatory convection depends on both relaxation and retardation times. It can be notified that the modulation effect enters into the system at the third order $O(\chi^3)$. Without loss of generality $R_2 = R_0$ is assumed in the calculations and this is done to keep the parameters to a minimum. The results that we obtain and present here are only for particular set of parameter values. We observe here that the stationary critical Rayleigh number Eq.(4.2.30) does not depend on $\lambda_1, \lambda_2$, therefore for stationary case, the study reduces to double diffusive convection in Newtonian fluid. In order to analyze the effect of modulated temperature field, we discuss the following three cases:

1. In-phase modulation (IPM) ($\theta = 0$).

2. Out-phase modulation (OPM) ($\theta = \pi$).

3. Only Lower boundary modulated (LBMO) ($\theta = -i\infty$),
   which means only lower boundary temperature will be modulated and the upper boundary will be at fixed constant temperature.

The Nusselt and Sherwood numbers have been obtained as functions of the system parameters and an amplitude of convection. The parameters of the system are $Rs, Pr, \lambda_1, \lambda_2, \Gamma, \delta$ and $\Omega$. To see the effect of each parameter on the system, we fix all other parameters and study individual parameter. The results corresponding to the

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Figure 4.8: Effect of various parameters on heat transport OPM case
thermal modulation have been depicted in figures 4.8-4.11. In figures 4.8-4.9 we have plotted $\nu$, $Sh$ with respect to the slow time $s$ and discussed heat and mass transports. It is found that the value of $\nu$ starts with 1, thus showing the conduction state initially that is heat transfer across the fluid layer is taking place through conduction when $s$ is small. The values of $\nu$, $Sh$ increases for intermediate values of $s$ thus showing that convection is in progress and finally when $s$ is very large, the oscillatory state is achieved. The effect of the Prandtl number is important, because many practical available viscoelastic fluids have large Prandtl numbers. It is quite interesting to note that when the Prandtl number increases, the critical value of the Rayleigh number decreases significantly so that the Prandtl number has a tendency to destabilize the system, compatible with results obtained by Kim et al. (2003), Tan et al. (2007) and Bhadauria et al. (2012, 2014).

One can observe from Eq. (4.2.42) for the case $l_r > 0$ and $Ra > Ra_c$ i.e. $p_r > 0$, the solution gives as $A \sim A_0 e^{p_r s}$ as $s \to -\infty$, and $A \to 0$ is unstable solution, and a new stable solution develops, $A = \sqrt{\frac{p_r}{l_r}}$ as $s \to \infty$, whatever be the value of $A_0$. This is called supercritical pitch fork bifurcation, the base system being linearly unstable for $Ra > Ra_c$ but settling down as a new laminar flow. The steady state amplitude exists when $Ra_c$ takes positive values. Supercritical pitch fork bifurcation diagram has been presented in the figure 4.1a for the case of OPM. Similar results are also obtained for IPM and LBMO but not included here. thus conforms the results of Bhadauria and Kiran (2014c).

First we discuss our results for out of phase modulation (OPM): The effect of solutal Rayleigh number $Rs$ in figures 4.8a and 4.9a is to increase $\nu$ and $Sh$ so heat and mass transfer, hence it has a destabilizing effect on the system. Though the presence of a stabilizing gradient of solute will prevent the onset of convection, the strong finite amplitude motions, which exist for large Rayleigh numbers, tend to mix the solute and redistribute it so that the interior layers of the fluid are more neutrally stratified. As a consequence, the inhibiting effect of the solutal gradient is greatly reduced, and hence fluid will convect more and more heat and mass when $Rs$ is increased. The effect of Prandtl number on heat and mass transfer is shown in figures 4.8b and 4.9b and found that $Pr$ has a destabilizing effect and advance the convection, hence increases heat and mass transfer. In
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Figure 4.9: Effect of various parameters on mass transport OPM case
figures 4.8c and 4.9c, we depict the effect of viscoelastic parameter $\lambda_1$ on the oscillatory convection. For fixed values of other parameters, the critical Rayleigh number for the onset of oscillatory convection decreases with an increase in the value of $\lambda_1$, indicating that the effect of increasing viscoelastic parameter is to advance the onset of oscillatory convection. Thus, it confirms that the elastic behavior of the non-Newtonian fluids leads to the oscillatory motions, hence heat and mass transfer increases. Further, the effect of retardation parameter $\lambda_2$ given in figures 4.8d and 4.9d, is found to stabilize the system as the heat and mass transfer decreases on increasing $\lambda_2$. The effect of diffusivity ratio $\Gamma$ is delay the onset of convection and hence heat and mass transfer given in figures 4.8e and 4.9e. The effects of frequency $\Omega$ and the amplitude of modulation, $\delta$ on heat and mass transport are given in the figures 4.8f and 4.9f and figures 4.8g and 4.9g, respectively. It is found that an increment in amplitude of modulation increases the magnitude of Nu, Sh, thus enhances heat and mass transfer, and thus advancing the onset of convection. An opposite effect is obtained in the case of frequency of modulation as $\Omega$ increases. Hence, we found that the effect of thermal modulation decreases as the frequency of modulation increases, and finally when $\Omega$ is very large, the effect of modulation disappears altogether, thus confirming the results of Venezian (1969), Bhadauria and Kiran (2013a, 2014c, g).

In the case of IPM, the parameters’ effects are same as in OPM, however, in—phase modulation (IPM) of the boundary temperature does not substantially modify the temperature gradient across the layer, therefore not much effect of modulation on heat transfer. Thus, the results in the case of IPM figure 4.10 are same as that are in unmodulated case, compatible with the results of Malashetty and Swamy (2010), Kumar and Bhadauria (2011) and Rajib and Layak (2012) for unmodulated case. The effect of individual parameter is same as in the case of OPM. The similar results can be obtained for mass transfer. Here we do not present the results corresponding to LBMO, as they are similar to the results obtained in case of OPM. However, on comparing the results of all three cases it is found that $\text{Nu/Sh}^{LBMO}$ is lower than $\text{Nu/Sh}^{OPM}$ but higher than $\text{Nu/Sh}^{IPM}$ as given below:

$$\text{Nu/Sh}^{IPM} < \text{Nu/Sh}^{LBMO} < \text{Nu/Sh}^{OPM}$$

Finally in figures 4.11a-h we have presented our results in terms of averaged Nusselt
Figure 4.10: Effect of various parameters on heat transport IPM case

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Figure 4.11: Effect of $\theta$ on MNu for different values of $\Omega$ and Pr (a-d): Effect of $\Omega$ on MNu for different values of $\theta$ and Pr (e-h).
number to see the effect of temperature modulation. In figures 4.11a-d we observe the
effect of phase angle $\theta$ and in figures 4.11e-h the effect of frequency of modulation. Instead
of choosing the frequency of small amplitude modulation, it is to be noted that there is
good combination of selecting the range of $\theta, \Omega$ in which the rate of heat transfer regulated
effectively. From the figures it is evident that for a given frequency of modulation there is
a range of $\theta$ in which (MNu) increases with increasing $\theta$ and another range in which (MNu)
decreases. Thus, one can conclude that, the combination of choices of $\Omega$ and $\theta$ can be
made depending on the applications on heat and mass transports. Heat and Mass transfer
can be regulated (enhanced or reduced) with the external mechanism of temperature
modulation. One can notice here that the effect of modulation is negligible when both the
boundary temperatures are synchronized, hence for temperature modulation the boundary
temperatures should not be synchronized. Only asynchronized temperature boundaries
are effective for temperature modulation for either enhancing or diminishing heat and
mass transfer. Our results are compatible with results which are similar to Siddheshwar
et al.(2013) and Bhadauria and Kiran (2014c). The similar results can be obtained for
the case of averaged Sherwood number MSh.

4.4 Conclusions

We have analyzed the effect of thermal modulation on overstability of viscoelastic
double diffusive convection by performing a weakly nonlinear stability analysis resulting
in the complex Ginzburg–Landau amplitude equation. The following conclusions are
made by the previous analysis

1. Effect of IPM is negligible on heat and mass transport in the system.

2. In the case of IPM, the effect of $\delta$ and $\Omega$ are also found to be negligible on heat and
mass transport.

3. Effect of $Rs, Pr, \lambda_1$ is to enhance the heat and mass transport for all three types of
modulations.

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4. Effect of $\lambda_2, \Gamma$ is to decrease the heat and mass transport for all three types of modulations.

5. In the case of IPM, $\text{Nu}$ increase steadily for intermediate value of time $s$ and ultimately becomes constant when $s$ is large.

6. In the case of OPM and LBMO, $\text{Nu}$ shows an oscillatory nature at intermediate and large values of time $s$.

7. Supercritical pitch fork bifurcation exits for Eq. (4.2.42).

8. The effects of $\theta, \Omega$ observed in terms of $\text{MNu}$, similar results obtained for $\text{MSh}$.

Our results can be summarised as (in the case of OPM, LBMO)

1. $[\text{Nu/Sh}]_{Rs=60} < [\text{Nu/Sh}]_{Rs=80} < [\text{Nu/Sh}]_{Rs=100}$
2. $[\text{Nu/Sh}]_{Pr=5} < [\text{Nu/Sh}]_{Pr=7} < [\text{Nu/Sh}]_{Pr=10}$
3. $[\text{Nu/Sh}]_{\lambda_1=0.5} < [\text{Nu/Sh}]_{\lambda_1=0.6} < [\text{Nu/Sh}]_{\lambda_1=0.7}$
4. $[\text{Nu/Sh}]_{\lambda_2=0.09} < [\text{Nu/Sh}]_{\lambda_2=0.08} < [\text{Nu/Sh}]_{\lambda_2=0.07}$
5. $[\text{Nu/Sh}]_{\Gamma=0.4} < [\text{Nu/Sh}]_{\Gamma=0.3} < [\text{Nu/Sh}]_{\Gamma=0.2}$
6. $[\text{Nu/Sh}]_{\delta=0.1} < [\text{Nu/Sh}]_{\delta=0.3} < [\text{Nu/Sh}]_{\delta=0.5}$
7. $[\text{Nu/Sh}]_{\Omega=100} < [\text{Nu/Sh}]_{\Omega=50} < [\text{Nu/Sh}]_{\Omega=20} < [\text{Nu/Sh}]_{\Omega=10}$

**Appendix**

The coefficients in Eq. (4.2.30) are defined as:

\[
\begin{align*}
    a_1 &= (\lambda_1 + Pr\lambda_2)\lambda_1 c, \\
    a_2 &= c \left[ Pr(1 + \lambda_1 \lambda_2 \Gamma^2 c^2) + (1 + \lambda_1^2 \Gamma^2 c^2) + Pr(\lambda_2 - \lambda_1) c \right] - a^2 Pr(1 - \Gamma)\lambda_1^2 Rs, \\
    a_3 &= c^3 \Gamma c^2 (1 + Pr) + Pr\Gamma^2 c^4 (\lambda_2 - \lambda_1) - a^2 Pr(1 - \Gamma)Rs.
\end{align*}
\]
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The expressions given in Eqs.(4.2.39)-(4.2.40), one can simplify as

\[ \text{Nu}(s) = 1 + a^2 \left( \frac{c}{2(c^2 + \omega^2)} + \frac{\pi^2}{\sqrt{16\pi^4 + 4\omega^2}} \right) |\mathcal{A}(s)|^2, \]

\[ \text{Sh}(s) = 1 + a^2 \left( \frac{c}{2(\Gamma^2 c^2 + \omega^2)} + \frac{\pi^2}{\sqrt{16\Gamma^2\pi^4 + 4\omega^2}} \right) |\mathcal{A}(s)|^2. \]

The expressions given in Eq.(4.2.41) are

\[ R_{31} = \lambda_2 \frac{\partial}{\partial s} (\nabla^2 \psi_1) - R_0 \lambda_1 \frac{\partial}{\partial s} \left( \frac{\partial T_1}{\partial x} \right) - R_2 \left( \lambda_1 \frac{\partial}{\partial t} + 1 \right) \left( \frac{\partial T_1}{\partial x} \right) + R_s \lambda_1 \frac{\partial}{\partial s} \left( \frac{\partial S_1}{\partial x} \right), \]

\[ R_{32} = \frac{\partial \psi_1}{\partial x} \frac{\partial T_2}{\partial z} - \frac{\partial T_1}{\partial s} + \delta f \frac{\partial \psi}{\partial x}, \]

\[ R_{33} = \frac{\partial \psi_1}{\partial x} \frac{\partial S_2}{\partial z} - \frac{\partial S_1}{\partial s}. \]

The coefficients given in Eq.(4.2.42) are

\[ \gamma_1 = \left[ \frac{\lambda_2 c^2 + c(1 + 2i\omega \lambda_1)}{Pr} + a^2 \lambda_1 \left( \frac{Rs}{c + i\omega} - \frac{R_0}{c + i\omega} \right) + \frac{R_0 a^2(1 + i\omega \lambda_1)}{(c + i\omega)^2} - \frac{Rsa^2(1 + i\omega \lambda_1)}{(c + i\omega)^2} \right], \]

\[ F(s) = \left[ \frac{R_0 a^2(1 + i\omega \lambda_1)}{(c + i\omega)} - \frac{2R_0 a^2(1 + i\omega \lambda_1)}{(c + i\omega)^2} \delta I_1 \right], \text{where } I_1 = \int_0^1 f_2 \sin^2(\pi z)dz, \]

\[ k = \frac{a^4 cR_0(1 + i\omega \lambda_1)}{4(c^2 + \omega^2)(c + i\omega)} + \frac{a^4 \pi^2 R_0(1 + i\omega \lambda_1)}{(8\pi^2 + 4i\omega)(c + i\omega)^2} - \frac{a^4 \pi^2 R_0(1 + i\omega \lambda_1)}{4(\Gamma^2 c^2 + \omega^2)(c + i\omega)} - \frac{a^4 \pi^2 R_0(1 + i\omega \lambda_1)}{(8\pi^2 \Gamma + 4i\omega)(c + i\omega)^2}. \]