Chapter 3

Oscillatory convection under gravity modulation
3.1 Weak nonlinear oscillatory convection in a viscoelastic fluid layer under gravity modulation

3.1.1 Introduction

Kim et al. (2003) have performed a weakly nonlinear analysis of Darcy flow for stationary and oscillatory mode of convection. They found that elasticity parameters are destabilizing factor and for a certain parameter range the overstability is a preferred mode. The literature says clearly that numerous data is available for stationary nonlinear convection but lack in oscillatory nonlinear convection. Based on Kim et al. (2003) problem in this section we have performed a weakly nonlinear oscillatory convection in a horizontal fluid layer under gravity modulation using complex non autonomous Ginzburg–Landau amplitude equation, and in the process quantify the heat transport.

3.1.2 Governing Equations

An infinitely extended horizontal layer of viscoelastic fluid, confined between two stress-free boundaries at $z = 0$ and $z = d$, is considered. The stress-free boundaries are maintained at constant temperature, and the fluid layer is heated from below. The hydrodynamic equations are simplified by assuming Oberbeck–Boussinesq approximation. The constitutive equations for non-Newtonian viscoelastic fluid model with the relaxation time $\lambda_1$ and retardation time $\lambda_2$ may be represented as (Rajib and Layek 2012):

\[
\nabla \cdot \vec{q} = 0, \quad (3.1.1)
\]

\[
\left( \lambda_1 \frac{\partial}{\partial t} + 1 \right) \left( \frac{\partial \vec{q}}{\partial t} + (\vec{q} \cdot \nabla) \vec{q} - \frac{1}{\rho_0} \nabla P + \frac{\rho}{\rho_0} \vec{g} \right) - \nu \left( \lambda_2 \frac{\partial}{\partial t} + 1 \right) \nabla^2 \vec{q} = 0, \quad (3.1.2)
\]

\[
\frac{\partial T}{\partial t} + (\vec{q} \cdot \nabla) T = \kappa_T \nabla^2 T, \quad (3.1.3)
\]

\[
\rho = \rho_0 [1 - \alpha_T (T - T_0)], \quad (3.1.4)
\]

where the physical variables have their usual meanings, and are given in Nomenclature. The externally imposed thermal boundary conditions and gravitational fields are given
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by

\[
T = T_0 + \Delta T, \quad \text{at } z = 0
\]

\[
= T_0, \quad \text{at } z = d, \quad (3.1.5)
\]

\[
\vec{g} = g_0 [1 + \chi^2 \delta \cos(\Omega t)] \hat{k}, \quad (3.1.6)
\]

where \( \Delta T \) is the temperature difference across the fluid layer, \( \chi \) is the smallness of amplitude of modulation, \( \delta, \Omega \) are amplitude and frequency of gravity modulation.

### 3.1.3 Basic state

The basic state is assumed to be quiescent, and the quantities in this state are given by

\[
\vec{q}_b = 0, p = p_b(z,t), \quad T = T_b(z,t), \quad \rho = \rho_b(z,t). \quad (3.1.7)
\]

Substituting Eq.(3.1.7) in Eqs.(3.1.1)-(3.1.4), we get the following relations, which helps us to define basic state pressure and temperature:

\[
\frac{\partial p_b}{\partial z} = -\rho_b g, \quad (3.1.8)
\]

\[
\kappa_T \frac{d^2 T_b}{dz^2} = 0, \quad (3.1.9)
\]

\[
\rho_b = \rho_0 \left[1 - \alpha_T (T_b - T_0)\right]. \quad (3.1.10)
\]

The solution of equation (3.1.9), subjected to the boundary conditions (3.1.5), is given by

\[
T_b = T_0 + \Delta T \left(1 - \frac{z}{d}\right). \quad (3.1.11)
\]

The finite amplitude perturbations on the basic state are superposed in the form:

\[
\vec{q} = \vec{q}_b + \vec{q}', \quad \rho = \rho_b + \rho', \quad p = p_b + p', \quad T = T_b + T'. \quad (3.1.12)
\]

We introduce the Eq.(3.1.12) and the basic state temperature field given by Eq.(3.1.11), and then use the stream function \( \psi \) as \( u' = \frac{\partial \psi}{\partial z}, w' = -\frac{\partial \psi}{\partial x} \), for two dimensional flow. The equations are then non-dimensionalized using the physical variables; \((x, y, z) = d(x^*, y^*, z^*), \quad t = \frac{d^2}{\kappa_T} t^*, \quad \psi = \kappa_T \psi^*, \quad T' = \Delta T T^*, \quad \lambda_1 = \frac{\kappa_T}{d}, \quad \lambda_2 = \frac{\kappa_T}{d^2}, \quad \text{and } \Omega = \frac{\kappa_T}{d^2} \Omega^*.\)
The resulting non-dimensionalized system of equations can be expressed as (dropping the asterisk)

\[
\left( \lambda_1 \frac{\partial}{\partial t} + 1 \right) \left( \frac{1}{Pr} \frac{\partial}{\partial t} \nabla^2 \psi - \frac{1}{Pr} \frac{\partial (\psi, \nabla^2 \psi)}{\partial (x,z)} + g_m Ra \frac{\partial T}{\partial x} \right) - \left( \lambda_2 \frac{\partial}{\partial t} + 1 \right) \nabla^4 \psi = 0,
\]

\[\text{(3.1.13)}\]

\[
\frac{\partial \psi}{\partial x} + \left( \frac{\partial}{\partial t} - \nabla^2 \right) T = \frac{\partial (\psi, T)}{\partial (x,z)},
\]

\[\text{(3.1.14)}\]

where \( g_m = (1 + \delta \cos(\Omega t)) \). The above system will be solved by considering stress free and isothermal boundary conditions as given below

\[
\psi = \frac{\partial^2 \psi}{\partial z^2} = T = 0 \quad \text{on} \quad z = 0 \quad z = 1.
\]

\[\text{(3.1.15)}\]

Introducing a small perturbation parameter \( \chi \) that show a deviation from the critical state of onset of convection, the variables for a weak nonlinear state may be expanded as power series of \( \chi \) as in Eq.(2.3.1). Here \( R_0 \) is the critical value of the critical Rayleigh number at which the onset of convection takes place in the absence of gravity modulation.

### 3.1.4 Bifurcation of periodic solution

In order to allow for anticipated frequency shift along the bifurcation solution, we introduce the fast time scale of time \( \tau \) and the slow time scale \( s \). Therefore, the scaling of time variable is such that \( \frac{\partial}{\partial t} = \frac{\partial}{\partial \tau} + \chi^2 \frac{\partial}{\partial s} \). In the first order problem the nonlinear term in energy equation will vanish therefore, the first order problem reduces to the linear stability problem for overstability.

**At the lowest order**, we have

\[
\begin{bmatrix}
\frac{1}{Pr} \left( \lambda_1 \frac{\partial}{\partial \tau} + 1 \right) \frac{\partial}{\partial \tau} \nabla^2 - \left( \lambda_2 \frac{\partial}{\partial \tau} + 1 \right) \nabla^4 & R_0 \left( \lambda_1 \frac{\partial}{\partial \tau} + 1 \right) \frac{\partial}{\partial x} \\
\frac{\partial}{\partial \tau} - \nabla^2 & \left( \frac{\partial}{\partial \tau} - \nabla^2 \right)
\end{bmatrix}
\begin{bmatrix}
\psi_1 \\
T_1
\end{bmatrix}
= \begin{bmatrix}
0 \\
0
\end{bmatrix}
\]

The solution of the lowest order system subject to the boundary conditions Eq.(3.1.15), is assumed to be

\[
\psi_1 = \left( \mathbb{B}(s) e^{i\omega \tau} + \mathbb{B}(s) e^{-i\omega \tau} \right) \sin ax \sin \pi z,
\]

\[\text{(3.1.16)}\]

\[
T_1 = \left( \mathbb{A}(s) e^{i\omega \tau} + \mathbb{A}(s) e^{-i\omega \tau} \right) \cos ax \sin \pi z.
\]

\[\text{(3.1.17)}\]
The undetermined amplitudes are functions of slow time scale and are related by the following relation:

\[ \mathfrak{B}(s) = -\frac{c + i\omega}{a} \mathfrak{A}(s), \]  

(3.1.18)

where \( c = a^2 + \pi^2 \). The values of the critical Rayleigh number and the corresponding wave number for stationary mode of convection are

\[ R_0 = \frac{c^3}{a^2}, \]  

(3.1.19)

\[ a = \frac{\pi}{\sqrt{2}}, \]  

(3.1.20)

which are classical results of Chandrasekhar (1961). We find critical Rayleigh number for oscillatory convection as:

\[ R_0 = \frac{c^3}{a^2} - \left( \left( \lambda_1 + \lambda_2 Pr \right)c + 1 \right) \frac{c\omega^2}{a^2 Pr}, \]  

(3.1.21)

which is same as obtained by Rajib et al. (2012). Here we calculate the corresponding critical wave number while minimizing critical Rayleigh number with respect to the square of wave number. The critical Rayleigh number and corresponding wave number does not depend on \((\lambda_1, \lambda_2)\) in stationary mode but in oscillatory mode. Also we see that the overstability can occur for a particular wave number \( a \) only, if the following inequality holds

\[ \lambda_1 > \lambda_2 + \frac{1 + Pr}{cPr}. \]  

(3.1.22)

In the second order, we get the following relations

\[ \psi_2 = 0, \]  

(3.1.23)

\[ \left( \frac{\partial}{\partial \tau} - \nabla^2 \right) T_2 = \frac{\partial \psi_1}{\partial x} \frac{\partial T_1}{\partial z} - \frac{\partial \psi_1}{\partial z} \frac{\partial T_1}{\partial x}. \]  

(3.1.24)

From the above relation, according to Kim et al. (2003), we can deduce that the velocity and temperature fields have the terms having frequency \( 2\omega \) and independent of past time scale. Thus, we write the second order temperature term as follows:

\[ T_2 = \{ T_{20} + T_{22}e^{2i\omega \tau} + T_{22}e^{-2i\omega \tau} \} \sin 2\pi z, \]  

(3.1.25)
where \( T_{22} \) and \( T_{20} \) are temperature fields having the terms having the frequency \( 2\omega \) and independent of fast time scale, respectively. The solutions of the second order problems are:

\[
T_{20} = \frac{a}{8\pi} \{ A(s) \bar{B}(s) + \bar{A}(s) B(s) \},
\]

(3.1.26)

and

\[
T_{22} = \frac{\pi a}{8\pi^2 + 4i\omega} \bar{A}(s) \bar{B}(s).
\]

(3.1.27)

The horizontally averaged Nusselt number, \( \text{Nu}(\tau) \), for the oscillatory mode of convection is given by:

\[
\text{Nu}(s) = 1 - \chi^2 \left( \frac{\partial T_2}{\partial z} \right)_{z=0}
\]

(3.1.28)

Using the expression of \( T_2 \), given in Eq.(3.1.25), we simplify Eq.(3.1.28) as

\[
\text{Nu}(s) = 1 + \left( \frac{c}{2} + 2\pi^2 \frac{\sqrt{c^2 + \omega^2}}{\sqrt{64\pi^4 + 16\omega^2}} \right) |A(s)|^2.
\]

(3.1.29)

It is clear that the gravity modulation is effective at third order, and affects \( \text{Nu}(s) \) through \( \bar{A}(s) \), which is evaluated at third order.

**At the third order**, we have

\[
\begin{bmatrix}
\frac{1}{\rho r} \left( \lambda_1 \frac{\partial}{\partial \tau} + 1 \right) \frac{\partial}{\partial x} \nabla^2 - \left( \lambda_2 \frac{\partial}{\partial \tau} + 1 \right) \nabla^4 & R_0 \left( \lambda_1 \frac{\partial}{\partial \tau} + 1 \right) \frac{\partial}{\partial x} \\
\frac{\partial}{\partial x} & \left( \frac{\partial}{\partial \tau} - \nabla^2 \right)
\end{bmatrix}
\begin{bmatrix}
\psi_3 \\
T_3
\end{bmatrix}
= \begin{bmatrix}
R_{31} \\
R_{32}
\end{bmatrix}
\]

where the expressions for \( R_{31} \) and \( R_{32} \) are given in the appendix. Now under the stability condition for the existence of third order solution, we obtain the following Landau equation that describes the temporal variation of the amplitude \( \bar{A}(s) \) of the convection cell

\[
\frac{\partial \bar{A}(s)}{\partial s} - \gamma_1^{-1} F(s) \bar{A}(s) + \gamma_1^{-1} k |\bar{A}(s)|^2 \bar{A}(s) = 0,
\]

(3.1.30)

where the coefficients \( \gamma_1, F(s) \) and \( k \) are given in the appendix. Writing \( \bar{A}(s) \) in the phase-amplitude form, we get

\[
ar{A}(s) = |\bar{A}(s)| e^{i\phi}.
\]

(3.1.31)

Now substituting the expression Eq.(3.1.31) in Eq.(3.1.30), we get the following equations for the amplitude \( |\bar{A}(s)| \):

\[
\frac{\partial |\bar{A}(s)|^2}{\partial s} - 2p_r |\bar{A}(s)|^2 + 2l_r |\bar{A}(s)|^4 = 0,
\]

(3.1.32)
\[
\frac{\partial (ph(A(s)))}{\partial s} = p_i - l_i |A(s)|^2,
\]
(3.1.33)

where \( \gamma^{-1}_1 F(s) = p_r + i p_i, \gamma^{-1}_1 k = l_r + i l_i \) and \( ph(.) \) represents the phase shift.

### 3.1.5 Results and discussion

The bifurcation of a convective layer of a viscoelastic fluid has been analysed by means of weakly nonlinear theory under gravity modulation. The amplitude equations for the bifurcations are also obtained. In order to illustrate the effects of relaxational parameters \( \lambda_1, \lambda_2 \), the frequency \( \Omega \) and the amplitude \( \delta \) of modulation on heat transport, we plot the curves of Nusselt number versus time \( s \). It is observed that the relation Eq. (3.1.22) leads to an interesting result; that for a viscoelastic fluid layer heated underneath; the oscillatory type of instability is possible only when the relaxation parameter \( \lambda_1 \) is greater than the retardation parameter \( \lambda_2 \). Also, it is clear from the relation Eq. (3.1.21) that the oscillatory convection depends on both relaxation and retardation times.

The results corresponding to the gravity modulation has been depicted in figures 3.1-3.4, where we have plotted Nu with respect to the slow time \( s \). It is found that the value of Nu starts with 1, thus showing the conduction state initially that is heat transfer across the fluid layer is taking place through conduction when \( s \) is small. The values of Nu increases for intermediate values of \( s \) thus showing that convection is in progress and finally when \( s \) is very large, the oscillatory state is achieved. The effect of the Prandtl number is important, because many practical available viscoelastic fluids have large Prandtl numbers. It is quite interesting to note that when the Prandtl number is small, the critical value of the Rayleigh number decreases significantly for increasing Prandtl number so that the Prandtl number has a tendency to destabilize the system, compatible with results obtained by Tan et al. (2007), Kim et al. (2003). In figure 3.1a, as \( Pr \) increases there is an increment in heat transfer compatible with the results obtained by Bhadauria et al. (2013c) for considering low viscous fluids.

Figure 3.1b, shows the effect of viscoelastic parameter \( \lambda_1 \) on the oscillatory convection. For fixed value of other parameters, the critical Rayleigh number for the onset of oscillatory convection decreases with an increase in the value of \( \lambda_1 \), indicating that the

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Figure 3.1: Effect of different values of system parameters on Nu
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Effect of increasing viscoelastic parameter is to advance the onset of oscillatory convection. Thus, it is confirmed that the elastic behavior of the non-Newtonian fluids leads to the oscillatory motions, hence heat transfer increases. Further, the effect of retardation parameter $\lambda_2$ is found to stabilize the system as the heat transfer decreases on increasing $\lambda_2$, given in figure 3.1c. The effects of frequency $\Omega$ and the amplitude of modulation $\delta$ on heat transport is given in figures 3.1d-e, respectively. In figure 3.1d, one can see that an increment in amplitude of modulation increases the magnitude of Nu, thus enhances the heat transfer and advancing the onset of convection. An opposite effect is obtained in the case of frequency of modulation $\Omega$ as given in the figure 3.1e. Hence, we found that the effect of gravity modulation decreases as the frequency of modulation increases, and finally when $\Omega$ is very large, the effect of modulation disappears altogether, thus confirming the results of Venezian (1969) and Yang (1997). In figure 3.1f, we compare the results of oscillatory and stationary instabilities. It is found that heat transfer is more in oscillatory mode of convection than in stationary mode. It can be observed that ($Nu^{st} < Nu^{osc}$) for the same wave number. This implies that oscillatory instability sets in before the stationary instability. Similar results has also been obtained by Rajib and Layek (2012), Kim et al. (2003). In figures 3.2a-b, we plot the amplitude of convection $A(s)$ versus time $s$, it is found that amplitude enhances the heat transfer as $\delta$ increases but opposite in case of frequency $\Omega$, thus confirming the results obtained by Bhadauria et al. (2012, 2013). and Siddheshwar et al. (2012a,b).

In figures 3.3-3.4, the stream lines and the corresponding isotherms are depicted for gravity modulation, respectively at $s = 0.0, 0.3, 0.6, 0.8, 1.0, 2.0$ for $\lambda_1 = 0.4, \lambda_2 = 0.1, \delta = 0.8, 0.3$.
Figure 3.3: Streamlines at (a)s=0(b)s=0.3(c)s=0.6(d)s=0.8(e)s=1(f)s=2
0.1 Ω = 2.0 and χ = 0.5. From the figures, we found that initially when the time is small the magnitude of streamlines is also small given in figures 3.3a-b, and isotherms are straight that is the system is in conduction state, figures 3.4a-b. However, as time increases, the magnitude of streamlines increases and the isotherms loses their evenness, thus showing that the convection is taking place in the system. Convection becomes faster on further increasing the value of time s. However, the system achieves the study state beyond s = 0.16 as there is no change in the streamlines and isotherms, figures 3.3-3.4d-f.

3.1.6 Conclusions

We have analyzed the effect of gravity modulation on overstability of Bénard convection by performing a weakly nonlinear stability analysis resulting in the complex Ginzburg–Landau amplitude equation. The following conclusions are made:

1. Effect of relaxation time λ₁ is to advance the onset of convection and hence enhance the heat transport.

2. Effect of retardation time λ₂ is to delay the onset of convection and hence decrease the heat transport.

3. It is important that for oscillatory convection the relaxation time of fluid must dominant over retardation time.

4. The critical Rayleigh number depends on λ₁, λ₂ for oscillatory mode of convection, but for stationary case it is independent.

5. An increment in the amplitude of modulation δ is to advance the convection and hence heat transfer.

6. The frequency of modulation, Ω is to decrease the heat transfer.
Figure 3.4: Isotherms at (a)\(s=0\)(b)\(s=0.3\)(c)\(s=0.6\)(d)\(s=0.8\)(e)\(s=1\)(f)\(s=2\)
Appendix

The dimensionless frequency of the neutral oscillatory mode is

$$\omega^2 = \frac{cPr(\lambda_1 - \lambda_2) - 1(1 + Pr)}{\lambda_1(\lambda_1 + \lambda_2Pr)}.$$

The expressions given in Eq. (3.1.30) are

$$R_{31} = \lambda_2 \frac{\partial}{\partial s} (\nabla^4 \psi_1) - R_0 \lambda_1 \frac{\partial}{\partial s} \left( \frac{\partial T_1}{\partial x} \right) - (R_2 + R_0 \delta \cos(\Omega s)) \left( \lambda_1 \frac{\partial}{\partial t} + 1 \right) \left( \frac{\partial T_1}{\partial x} \right)$$

$$- \frac{1}{Pr} \left( \lambda_1 \frac{\partial}{\partial t} + 1 \right) \frac{\partial}{\partial s} (\nabla^2 \psi_1) - \frac{1}{Pr} \lambda_1 \frac{\partial}{\partial s} \left( \frac{\partial}{\partial \tau} \nabla^2 \psi_1 \right),$$

$$R_{32} = \frac{\partial \psi_1}{\partial x} \frac{\partial T_2}{\partial z} - \frac{\partial T_1}{\partial s}.$$

The coefficients given in Eq. (3.1.30) are

$$\gamma_1 = \left[ 1 - a \Delta_1 R_0 \lambda_1 + \frac{c^2 \Delta_1 \lambda_2 (c + i\omega)}{a} + \frac{c \Delta_1 (c + i\omega)(1 + 2i\omega \lambda_1)}{aPr} \right],$$

$$F(s) = \left[ a \Delta_1 R_2 (1 + i\omega \lambda_1)(1 + \delta \cos(\Omega s)) \right],$$

$$k = \left( \frac{c^2 + i\omega}{4} + \frac{\pi^2 (c^2 + \omega^2)}{8\pi^2 + 4i\omega} \right)$$ and

$$\Delta_1 = \frac{aPr}{i\omega c Pr (1 + i\omega \lambda_1) + (1 + i\omega \lambda_2)c^2}.$$
Chapter-3.2: Weak nonlinear oscillatory convection in porous layer

3.2 Weak nonlinear oscillatory convection in a viscoelastic fluid saturated porous medium under gravity modulation

3.2.1 Introduction

Based on Kim et al. (2003) and Bhadauria and Kiran (2014a) in this section we have performed a weakly nonlinear thermal instability in a viscoelastic fluid saturated porous medium under gravity modulation, and quantify the Nusselt number in terms of the amplitude of convection by solving the complex Ginzburg–Landau equation. Finally, till now no experimental work has been found in the literature in support of this viscoelastic model for flow in porous media.

3.2.2 Problem Formulation

An infinitely extended horizontal fluid saturated porous medium of depth ‘d’ has been considered. The porous medium is homogeneous, isotropic and saturated with viscoelastic fluid. The porous medium is heated slowly from below, the configuration of the problem is given in figure 3.5a. Using modified Darcy’s model (Alishaev 1975) and employing the Boussinesq approximation for this system, the governing equations of flow and temperature fields are expressed as:

\[ \nabla \cdot \vec{q} = 0, \]  
\[ \left( \frac{1}{\lambda_1} \frac{\partial}{\partial t} + 1 \right) (-\nabla P + \rho \vec{g}) - \frac{\mu}{K} \left( \frac{1}{\lambda_2} \frac{\partial}{\partial t} + 1 \right) \vec{q} = 0, \]  
\[ \frac{\partial T}{\partial t} + (\vec{q} \cdot \nabla) T = \kappa T \nabla^2 T, \]  
\[ \rho = \rho_0 [1 - \alpha_T (T - T_0)], \]

where the physical variables have their usual meanings and are given in Nomenclature. The externally imposed gravitational field and the thermal boundary conditions are given.

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by

\[ \vec{g} = g_0 [1 + \chi^2 \delta \cos(\Omega t)] \hat{k}, \]  
\[ T = T_0 + \Delta T, \quad \text{at } z = 0 \]
\[ = T_0, \quad \text{at } z = d, \]  

where \( \Delta T \) is the temperature difference across the porous medium, \( \chi \) is the smallness of amplitude of modulation \( \delta \), \( \Omega \) are amplitude and frequency of gravity modulation.

### 3.2.3 Basic state

The basic state is assumed to be quiescent, and the quantities in this state are given by

\[ \vec{q}_b = 0, \quad p = p_b(z, t), \quad T = T_b(z, t), \quad \rho = \rho_b(z, t). \]  

Substituting the Eq. (3.2.7) in Eqs. (3.2.1) – (3.2.4), we get the following relations, which helps us to define basic state pressure and temperature:

\[ \frac{\partial p_b}{\partial z} = -\rho_b g, \]  
\[ \kappa_T \frac{d^2 T_b}{dz^2} = 0, \]  
\[ \rho_b = \rho_0 [1 - \alpha_T (T_b - T_0)]. \]  

The solution of equation (3.2.9), subjected to the boundary conditions (3.2.6), is given by

\[ T_b = T_0 + \Delta T \left( 1 - \frac{z}{d} \right). \]  

The finite amplitude perturbations on the basic state are superposed in the form:

\[ \vec{q} = \vec{q}_b + q', \quad \rho = \rho_b + \rho', \quad p = p_b + p', \quad T = T_b + T'. \]  

We introduce the Eq. (3.2.12) and the basic state temperature field given by Eq. (3.2.11) in Eqs. (3.2.1) – (3.2.4), and then use the stream function \( \psi \) and non-dimensionalized factors as given chapter 3.1 we obtain the following non-dimensionalized system as

\[ \left( \lambda_2 \frac{\partial}{\partial t} + 1 \right) \nabla^2 \psi + Ra(1 + \chi^2 \cos(\Omega t)) \left( \lambda_1 \frac{\partial}{\partial t} + 1 \right) \frac{\partial T}{\partial x} = 0 \]  

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\[
\frac{\partial \psi}{\partial x} + \left( \frac{\partial}{\partial t} - \nabla^2 \right) T = \frac{\partial (\psi, T)}{\partial (x, z)}.
\]  
(3.2.14)

The above system will be solved by considering stress free and isothermal boundary conditions as given in Eq.(2.2.26). Introduce a small perturbation parameter \( \chi \) that show deviation from the critical point of onset of convection, then the variables for a weak nonlinear state may be expanded as power series of \( \chi \) as in Eq.(2.3.1). Here \( R_0 \) is the critical value of the Darcy–Rayleigh number at which the onset of convection takes place in the absence of gravity modulation.

### 3.2.4 Bifurcation of periodic solution

In order to allow for anticipated frequency shift along the bifurcation solution, we introduce the fast time scale of time \( \tau \) and the slow time scale of \( s \). Therefore, the scaling of time variable is such that \( \frac{\partial}{\partial \tau} = \frac{\partial}{\partial t} + \chi^2 \frac{\partial}{\partial s} \). In the first order problem the nonlinear term in energy equation will vanish therefore, the first order problem reduces to the linear stability problem for overstability.

**At the lowest order**, we have

\[
\begin{bmatrix}
(\lambda_2 \frac{\partial}{\partial \tau} + 1) \nabla^2 & R_0 (\lambda_1 \frac{\partial}{\partial \tau} + 1) \frac{\partial}{\partial x} \\
\frac{\partial}{\partial x} & (\frac{\partial}{\partial \tau} - \nabla^2)
\end{bmatrix}
\begin{bmatrix}
\psi_1 \\
T_1
\end{bmatrix}
= \begin{bmatrix}
0 \\
0
\end{bmatrix}
\]  
(3.2.15)

The solution of the lowest order system subject to the boundary conditions Eq.(2.2.26), is assumed to be

\[
\psi_1 = (\mathbb{A}(s)e^{i\omega \tau} + \overline{\mathbb{A}}(s)e^{-i\omega \tau}) \sin ax \sin \pi z,
\]  
(3.2.16)

\[
T_1 = (\mathbb{B}(s)e^{i\omega \tau} + \overline{\mathbb{B}}(s)e^{-i\omega \tau}) \cos ax \sin \pi z.
\]  
(3.2.17)

The undetermined amplitudes are functions of slow time scale and are related by the following relation:

\[
\mathbb{B}(s) = -\frac{a}{c + i\omega} \mathbb{A}(s)
\]  
(3.2.18)

where \( c = a^2 + \pi^2 \). The values of the Darcy–Rayleigh number and the corresponding wave number for stationary mode of convection

\[
R_0 = \frac{c^2}{a^2}
\]  
(3.2.19)
which are classical results of Horton and Rogers (1945), and Lapwood (1948). We find Darcy–Rayleigh number and corresponding critical wave number for oscillatory convection as given below:

\[ R_0 = \frac{(\lambda_2 \pi^4 + \pi^2 + 2a^2 \pi^2 \lambda_2 + a^2 + a^4 \lambda_2)}{\lambda_1 a^2} \]  

\[ a_c^2 = \frac{\pi^4 + \frac{\pi^2}{\lambda_2}}{\lambda_2}, \]  

which are same as obtained by Kim et al. (2003). The critical Darcy Rayleigh number and corresponding wave number does not depend on \((\lambda_1, \lambda_2)\) in stationary mode but in oscillatory mode. Also we see that the overstability can occur for a particular wave number only, if the following inequality holds

\[ \lambda_1 > \lambda_2 + \frac{1}{c}. \]  

The dimensionless frequency of the neutral oscillatory mode is

\[ \omega^2 = \frac{c(\lambda_1 - \lambda_2) - 1}{\lambda_2 \lambda_1}. \]  

**In the second order**, we get

\[ \frac{\partial(\psi_1, T_1)}{(x, z)} = \frac{\pi a}{2} \left\{ A(s)\overline{B}(s)e^{2i\omega \tau} + A(s)\overline{B}(s)e^{-2i\omega \tau} + A(s)\overline{B}(s) + \overline{A}(s)\overline{B}(s) \right\} \sin 2\pi z. \]  

From the above relation, we can deduce that the velocity and temperature fields have the terms having frequency \(2\omega\) and independent of past time scale. Thus, we write the second order temperature term as follows:

\[ T_2 = \left\{ T_{20} + T_{22}e^{2i\omega \tau} + \overline{T}_{22}e^{-2i\omega \tau} \right\} \sin 2\pi z \]  

where \(T_{22}\) and \(T_{20}\) are temperature fields having the terms having the frequency \(2\omega\) and independent of fast time scale, respectively. The solutions of the second order problems are:

\[ T_{20} = \frac{a}{8\pi} \left\{ A(s)\overline{B}(s) + \overline{A}(s)\overline{B}(s) \right\}, \quad \psi_{20} = 0 \]
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and

\[ T_{22} = \frac{\pi a}{8\pi^2 + 4i\omega} A(s) B(s). \]  

(3.2.28)

The horizontally averaged Nusselt number, \( Nu(s) \), for the oscillatory mode of convection is given by using the expression of \( T_{22} \), given in Eq. (3.2.26), one can simplify Eq. (3.1.28) as

\[ Nu(s) = 1 + \left( \frac{ca_c^2}{2(c^2 + \omega^2)} + \frac{4\pi^2 a_c^2}{\sqrt{64\pi^4 + 16\omega^2(c^2 + \omega^2)}} \right) |A(s)|^2. \]  

(3.2.29)

It is clear that the thermal modulation is effective at third order and affects \( Nu(s) \) through \( A(s) \) which is evaluated at third order.

**At the third order**, we have

\[
\begin{bmatrix}
(\lambda_2 \frac{\partial}{\partial \tau} + 1) \nabla^2 & R_0 \left( \lambda_1 \frac{\partial}{\partial \tau} + 1 \right) \frac{\partial}{\partial x}
\end{bmatrix}
\begin{bmatrix}
\psi_3 \\
T_3
\end{bmatrix}
= \begin{bmatrix}
R_{31} \\
R_{32}
\end{bmatrix}
\]

(3.2.30)

where

\[
\begin{align*}
R_{31} &= -\lambda_2 \frac{\partial}{\partial s}(\nabla^2 \psi_1) - R_0 \lambda_1 \frac{\partial}{\partial s} \left( \frac{\partial T_1}{\partial x} \right) - (R_2 + R_0 \delta \cos(\Omega s)) \left( \lambda_1 \frac{\partial}{\partial \tau} + 1 \right) \left( \frac{\partial T_1}{\partial x} \right), \\
R_{32} &= \frac{\partial \psi_1}{\partial \tau} \frac{\partial T_2}{\partial x} \frac{\partial}{\partial z} - \frac{\partial T_1}{\partial s}.
\end{align*}
\]

(3.2.31-3.2.32)

Substituting the \( \psi_1, \ T_1 \) and \( T_2 \) into Eqs. (3.2.31)-(3.2.32), we obtain the expressions for \( R_{31} \) and \( R_{32} \) easily. Now under the stability condition for the existence of third order solution, these equations yield the following Landau equation that describes the temporal variation of the amplitude \( A(s) \) of the convection cell

\[
\frac{\partial A(s)}{\partial s} - \gamma_1^{-1} F(s) A(s) + \gamma_1^{-1} k |A(s)|^2 A(s) = 0
\]

(3.2.33)

where

\[
\gamma_1 = \left[ \lambda_2 c - \frac{a^2 R_0 \lambda_1}{(c + i\omega)} + \frac{a^2 R_0 (1 + i\omega \lambda_1)}{(c + i\omega)^2} \right],
\]

\[
F(s) = \left[ \frac{a^2 R_0 (1 + i\omega \lambda_1)}{(c + i\omega)} + \frac{4\pi^2 a^4 R_0 (c^2 + i\omega \lambda_1)}{(c + i\omega)^2} \right].
\]

Writing \( A(s) \) in the phase-amplitude form, we get

\[
A(s) = |A(s)| e^{i\phi}
\]

(3.2.34)

Now substituting the expression Eq.(3.2.34) in Eq.(3.2.33), we get the following equations for the amplitude \( |A(s)| \):

\[
\frac{\partial |A(s)|^2}{\partial s} = 2p_r |A(s)|^2 - 2l_r |A(s)|^4
\]

(3.2.35)

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\[
\frac{\partial (ph(A(s))))}{\partial s} = p_i - l_i|A(s)|^2 \tag{3.2.36}
\]
where \(\gamma^{-1}_1 F(s) = p_r + ip_i\), \(\gamma^{-1}_1 k = l_r + il_i\), and \(ph(.)\) represents the phase shift. One can observe from the Eq. \((3.2.33)\) for the case \(l_r > 0\) and \(Ra > Ra_c\), i.e. \(p_r > 0\), the solution gives as \(A \sim A_0 e^{p_r s}\) as \(s \to -\infty\), and \(A \to 0\) is unstable solution, and a new stable solution develops, \(A = \sqrt{\frac{p_r}{l_r}}\) as \(s \to \infty\), whatever be the value of \(A_0\). This is called supercritical pitch fork bifurcation, the base system being linearly unstable for \(Ra > Ra_c\) but settling down as a new laminar flow. The steady state amplitude exists when \(Ra_c\) takes positive values. Supercritical pitch fork bifurcation diagram has been shown in the figure 3.5b.

### 3.2.5 Results and discussion

In this work, we carried out a study of heat transport for oscillatory convection in an horizontal porous medium saturated with viscoelastic fluid under gravity modulation. In order to illustrate the effects of relaxational parameters \(\lambda_1, \lambda_2\), the frequency \(\Omega\) and the amplitude \(\delta\) of modulation on heat transport, we plot the curves of Nusselt number versus time \(s\). It is observed that the relation Eq. \((3.2.24)\) leads to an interesting result; that for a horizontal porous layer heated underneath; the oscillatory type of instability is possible only when the relaxation parameter \(\lambda_1\) is greater than the retardation parameter \(\lambda_2\). Also, it is clear from the relation Eq. \((3.2.21)\) that the oscillatory convection depends on both relaxation and retardation times. The marginal stability curves for the stationary and oscillatory modes are plotted in the figure 3.6. For comparison, the curves representing exchange of stabilities and overstability at the marginal state are drawn. The solid curve represents the Rayleigh number for oscillatory convection as a function of wave number while the broken curves represent the same for stationary convection. To illustrate the effects of relaxational parameter and the retardation parameter on the onset of convection, we plot the curves of the Rayleigh number versus the wave number. One can see in figures 3.6a-b that the marginal overstability curve deviates from the stationary Newtonian curve by showing a bifurcation point on the Newtonian curve and we observe that in this case the onset of convection is characterized by stationary convection. To study the effect
Figure 3.5: a. Physical configuration of the problem b. Supercritical pitch fork bifurcation diagram.
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Figure 3.6: Effect of $\lambda_1$ and $\lambda_2$ on $R_{osc}^{0}$ and $\omega^2$

of relaxation time of the fluid on the onset of overstability, from the figure 3.6a, it can be seen that the critical Rayleigh number decreases with the increasing value of the relaxation time $\lambda_1$ for fixed values of $\lambda_2$, indicating that the effect of increasing relaxation time is to destabilize the system. The effect of retardation time $\lambda_2$ on the onset of overstability is shown in the figure 3.6b, where we observe that viscoelastic fluids with higher value of retardation time exhibits overstability at higher Rayleigh number and the critical Rayleigh number increases with increasing retardation time. Thus the effect

Figure 3.7: Effect of $\lambda_1$ and $\lambda_2$ on Nu for fixed values of other parameters

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of increasing retardation time has a stabilizing effect on the system. The effect of time relaxation and time retardation parameter on the critical value of dimensionless frequency for marginally oscillatory modes is obtained from the relation Eq. (3.2.24). The results are given by plotting square of frequency against square of wave number in figure 3.6c and figure 3.6d, the critical value of the frequency increases with increasing relaxation time figure 3.6c but with decreasing retardation time figure 3.6d.

The results corresponding to the gravity modulation has been depicted in figures 3.7–3.9, where we have plotted Nu with respect to the slow time $s$. It is found that the value of Nu starts with 1 thus showing the conduction state initially that is heat transfer across the porous medium is taking place through conduction when $s$ is small. The values of Nu increases for intermediate values of $s$ thus showing that convection is in progress and finally when $s$ is very large, the oscillatory state is achieved. As in figure 3.7a, the effect of an increment in the value of relaxation parameter $\lambda_1$ is destabilizing as the value of Nu increases on increasing $\lambda_1$. Further, the effect of retardation parameter $\lambda_2$ is found to stabilize the system as the heat transfer decreases on increasing $\lambda_2$, given in figure 3.7b. The effects of frequency $\Omega$ and the amplitude of modulation $\delta$ on heat transport is given in Figure 3.8: Effect of $\delta$ and $\Omega$ on Nu: c.Comparison
figures 3.8a-b. In figure 3.8a, one can see that, an increment in amplitude of modulation increases the magnitude of Nu, thus enhances the heat transfer and advancing the onset of convection. An opposite effect is obtained in the case of frequency of modulation as Ω increases given in figure 3.8b. Hence we found that the effect of gravity modulation decreases as the frequency of modulation increases, and finally when Ω is very large, the effect of modulation disappears altogether, thus confirming the results of Venezian (1969). In figure 3.8c, we compare the results of oscillatory and stationary instabilities. It is found that heat transfer is more in oscillatory mode of convection than in stationary mode. It can be observed that (Nu_{stat} < Nu_{osc}). This implies that oscillatory instability sets in before the stationary instability. Similar results have also been obtained by Rajib and Layek (2012), Kim et al. (2003). In figures 3.9-3.10, the stream lines and the corresponding isotherms are depicted for gravity modulation, respectively at s = 0.0, 0.12, 0.14, 0.15, 0.16, 0.17 for λ_1 = 0.4, λ_2 = 0.1, δ = 0.1 Ω = 2.0 and χ = 0.5. From the figures we found that initially when the time is small the magnitude of streamlines is also small given in figures 3.9a-b, and isotherms are straight that is the system is in conduction state figures 3.10a-b. However, as time increases, the magnitude of streamlines increases and the isotherms loses their evenness. This shows that the convection is taking place in the system. Convection becomes faster on further increasing the value of time s. However, the system achieves the study state beyond s = 0.16 as there is no change in the streamlines and isotherms figures 3.9-3.10d-f.

3.2.6 Conclusions

We have analyzed the effect of gravity modulation on overstability of Bénard–Darcy convection by performing a weakly nonlinear stability analysis resulting in the complex Ginzburg–Landau amplitude equation. The following conclusions are made:

1. Effect of relaxation time λ_1 is to advance the onset of convection and hence enhance the heat transport.

2. Effect of retardation time λ_2 is to delay the onset of convection and hence decrease the heat transport.
Figure 3.9: Streamlines at (a) $s=0.0$ (b) $s=0.12$ (c) $s=0.14$ (d) $s=0.15$ (e) $s=0.16$ (f) $s=0.17$
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Figure 3.10: Isotherms at (a) s = 0.0 (b) s = 0.12 (c) s = 0.14 (d) s = 0.15 (e) s = 0.16 (f) s = 0.17
3. The oscillatory critical Rayleigh–Darcy number depends on $\lambda_1, \lambda_2$, but in stationary case it is independent.

4. An increment in the amplitude $\delta$ of modulation is to advance the convection and hence heat transfer.

5. The frequency $\Omega$ of modulation decreases the heat transfer as its value increases.

6. Supercritical pitch fork bifurcation exits for Eq. (3.2.33).