CHAPTER 7

AN OPTIMAL INVENTORY POLICY FOR ITEMS HAVING QUADRATIC TIME DEPENDENT DEMAND AND CONSTANT DETERIORATION RATE WITH TRADE CREDIT
7.1 Introduction

In most of the classical inventory models demand is considered as constant, but in most of the practical situation the demand changes with time, quantity etc. In this model, we have studied an order level inventory problem with demand rate represented by a continuous quadratic function of time and constant deterioration rate. The effect of permissible delay is also taken and incorporated in this model.

One of the challenging problems for the marketing researchers and practitioners is to study and analyze the inventory systems as the proper inventory systems not only reduce the costs, but also reduce stock-outs and improve customer satisfactions. Thus, proper inventory systems can improve the profitability and help in the survival of an organization. The researchers have considered various types of demands, viz. a linear time-dependent demand, stock dependent demand, price dependent demand etc. Time dependent demand indicates a uniform change in the demand rate of the product per unit time which is an unrealistic phenomenon and it rarely occurs in the market. On the other hand, exponentially time-varying demand indicates very rapid change in demand rate which is also unrealistic because the demand of any product can not undergo a rate which is so high as exponential demand is.

The concept of permissible delay is not new, even when currency was not in circulation, then also permissible delay was provided by suppliers to buyers. In general practice, suppliers are known to offer their customers a fixed period of time and do not charge any interest for this period. However, a higher interest is charged if the payment is not settled by the end of credit period. The permissible delay in payment produces three advantages to the supplier

⇒ It helps to attract new customer.
⇒ It helps in the bulk sale of goods.
It may be applied as an alternative to price discount because it does not provoke competitors to reduce their prices and thus introduce lasting price reductions.

In this chapter, an effort has been made to analyze an EOQ model for deteriorating item considering time-dependent quadratic demand rate and permissible delay in payment. Among the various time-varying demand in EOQ models, the more realistic demand approach is to consider a quadratic time dependent demand rate because it represents both accelerated and retarded growth in demand. The demand rate in this case is of the form, Here, $c = 0$ represents linear demand rate and $b = 0, c = 0$ represent the constant demand rate. Thus, it helps retailer to decide its optimal ordering quantity under the constraints of constant deterioration rate and quadratic time dependent demand.

### 7.2 Assumptions and Notations

The following are the assumptions:

(i) $a$ = initial demand rate.
(ii) $b$ = rate with which the demand rate increases.
(iii) $c$ = rate of change in the demand rate itself changes at a rate $c$.
(iv) $D(t) = a + bt + ct^2$, the demand rate.
(v) $A$ = ordering cost for items per order of inventory.
(vi) $C$ = unit cost of the item
(vii) $D_T$ = total demand during cycle period $T$
(viii) $i = $inventory carrying charge and expressed as % of the average inventory

(ix) $C_D = $cost of deterioration per cycle per unit time

(x) $C_H = $holding cost per cycle per unit time

(xi) $I(t) = $inventory at any time $t$

(xii) $\theta(t) = \theta$, constant rate of deterioration

(xiii) $M = $permissible delay time

(xiv) $P_T = $profit earned on the item at any time $T$

(xv) $I_T = $interest earned per cycle.

(xvi) $C_{VT} = $total variable cost per cycle.

(xvii) $C_T(P, T) = $variable cost per unit time.

(xviii) $s = $the time at which the inventory level reaches zero in the replacement cycle.

(xix) $I_P = $interest earned at time period $P$.

(xx) $I_C = $interest payable at constant rate.

7.3 Mathematical model

In the development of this model, we assume that the variable rate of demand is considered with the variable rate of deterioration. Depletion of the inventory occurs due to demand as well as due to deterioration which occurs only when there is inventory i.e. in the period $[0, T]$. So, the required differential equation is given by

$$\frac{dI(t)}{dt} + \theta(t)I(t) = -D(t) \quad 0 \leq t \leq T$$
Where, \( I(0) = I_0 \) and \( I(T) = 0 \). … (7.1)

Solving the above linear differential equation (7.1) with the boundary conditions and putting constant \( = I_0 \), we get:

\[
I(t) = e^{-\frac{bt^2}{2}} (I_0 - at - \frac{bt^2}{2} - \frac{ct^3}{3} - \frac{at^3}{6} - \frac{bt^4}{8} - \frac{ct^5}{10})
\] … (7.2)

It is obvious that at \( t = T \) i.e. at the end of the cycle period, \( I(T) = 0 \). So (7.2) gives

\[
I_0 = at + \frac{bT^2}{2} + \frac{cT^3}{3} + \frac{aT^3}{6} + \frac{bT^4}{8} + \frac{cT^5}{10}
\] … (7.3)

So substituting the value of \( I_0 \) from the equation (7.3) in the equation (7.2), expanding the exponential power and multiplying and neglecting the higher power of \( \theta \), we get

\[
= -at - \frac{bt^2}{2} - \frac{ct^3}{3} + at + \frac{bT^2}{2} + \frac{cT^3}{3} + \frac{aT^3}{6} + \frac{bT^4}{8} + \frac{cT^5}{10} - \frac{at^2 T \theta}{2} - \frac{bT^2 t^2 \theta}{4} + \frac{aT \theta}{6} - \frac{cT^3 t^2 \theta}{6} - \frac{bT^4 \theta}{8} + \frac{cT^5 \theta}{10}
\]

\[0 \leq t \leq T \] … (7.4)

The total demand during cycle period \( T \) is given by \( \int_0^T D(t) \, dt \). Thus it can easily seen that the amount of items deteriorates during one cycle is given by

\[
D_T = I_0 - \int_0^T D(t) \, dt
\]

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So the total cost of deterioration is given by putting the value of \( D_T \) from the equation (7.5), \( C_D \) is computed.

\[
C_D = C D_T
\]

\[
C_D = C \left( \frac{\alpha T^3 \theta}{6} + \frac{b T^4 \theta}{8} + \frac{c T^5 \theta}{10} \right)
\]

... (7.6)

The holding cost of the inventory is given by

\[
C_H = i C \int_0^T I(t) dt
\]

Putting the value of \( I(t) \) from the equation number (7.4) in the above equation, we have

\[
= i C \left\{ \frac{\alpha T^2}{2} + \frac{b T^3}{3} + \frac{c T^4}{4} + \frac{\alpha T^4 \theta}{12} + \frac{b T^5 \theta}{15} + \frac{c T^6 \theta}{18} \right\}
\]

... (7.7)

At any time \( t \) the cost of the unpaid inventory is the cost of the current inventory at that time, minus profit of amount sold during time \( M \), minus the interest earned from
the sales revenue during time $M$. The extra amount that can be paid can be determined by profit on the amount sold after the permissible delay time $M$. Therefore, the interest payable per cycle for the inventory not being sold after due date is given by

$$P_T = I_p \int_M^P \left( C(t) - (s - C) \int_0^M D(t) dt - s l_e \int_0^M D(t) dt \right) dt - (s$$

$$- C I_p \int_0^{P-M} D(t) dt$$

$$= - \frac{2}{3} b C I_p M^3 + \frac{3}{2} b C I_p M^2 P - \frac{2}{3} b C I_p M^3 P - b C I_p M P^2 + \frac{3}{2} b C I_p M^2 P^2$$

$$+ \frac{1}{6} b C I_p P^3 - c C I_p M P^3 + \frac{1}{6} c C I_p P^3 + \frac{1}{2} a I_p M^2 s$$

$$+ \frac{5}{6} b I_p M^3 s + \frac{1}{2} a I_c I_p M^3 P s + \frac{1}{12} c I_p M^4 s + \frac{1}{3} b I_c I_p M^4 s$$

$$+ \frac{1}{4} c I_c I_p M^4 s - \frac{3}{2} b I_p M^2 P s - \frac{1}{2} a I_c I_p M^2 P s + \frac{2}{3} b I_p M^3 P s$$

$$- \frac{1}{3} b I_c I_p M^3 P s - \frac{1}{4} c I_c I_p M^4 P s - \frac{1}{2} a I_p P^2 s + b I_p M^2 P s$$

$$- \frac{3}{2} c I_p M^2 P^2 s - \frac{1}{3} b I_p P^3 s + c I_p M P^3 s - \frac{1}{4} c I_p P^4 s - \frac{1}{4} c I_p P^4 s$$

$$- a C I_p M T + a C I_p P T - \frac{1}{2} b C I_p M T^2 + \frac{1}{2} b C I_p P T^2$$

$$- \frac{1}{3} c C I_p M T^3 + \frac{1}{3} c C I_p P T^3 - \frac{1}{12} a C I_p M^3 \theta - \frac{1}{40} b C I_p M^5 \theta$$

$$- \frac{1}{90} c C I_p M^6 \theta + \frac{1}{12} a C I_p P^4 \theta + \frac{1}{40} b C I_p P^5 \theta + \frac{1}{90} c C I_p P^6 \theta$$

$$+ \frac{1}{6} a C I_p M^3 T \theta - \frac{1}{6} a C I_p P^3 T \theta + \frac{1}{12} b C I_p M^3 T^2 \theta$$

$$- \frac{1}{12} b C I_p P^3 T^2 \theta - \frac{1}{12} a C I_p M T^3 \theta + \frac{1}{18} c C I_p M^3 T^3 \theta$$

$$+ \frac{1}{6} a C I_p P T^3 \theta - \frac{1}{18} c C I_p P^3 T^3 \theta + \frac{1}{8} b C I_p M^4 T \theta$$

$$+ \frac{1}{8} b C I_p P T^4 \theta - \frac{1}{10} c C I_p M^5 T \theta + \frac{1}{10} c C I_p P T^5 \theta$$

$$... (7.8)$$
Interest earned per cycle, $I_T$ is the interest earned during the positive inventory is given by

$$ I_T = sI_c \left\{ \int_0^M D(t)tdt + \int_0^{T-P} D(t)tdt \right\} $$

$$ = \frac{1}{2}aI_cM^2S + \frac{1}{3}bI_cM^3S + \frac{1}{4}cI_cM^4S + \frac{1}{2}aI_cP^2S + \frac{1}{3}bI_cP^3S + \frac{1}{4}cI_cP^4S - aI_cPS \quad (7.9) $$

The total variable cost is the sum of ordering cost, carrying cost, deterioration cost and the interest payable cost minus the interest earned and is given by substituting the respective values in the above equation number (7.10) from (7.6), (7.7), (7.8) and (7.9), we can find the value of $C_{VT}$.

$$ C_{VT} = A + C_D + C_H + P_T - I_T \quad (7.10) $$

And hence the average cost per unit time is given by

$$ C_T(P, T) = \frac{C_{VT}}{T} $$
\[ \frac{\partial C_{r}(P, T)}{\partial T} = 0 \]
\[ \Phi = \frac{A}{T^2} \left[ \frac{3}{2} b c I_p M^2 P - \frac{1}{6} c c I_p M^3 P^3 + \frac{b c I_p M^2 P}{2 T^2} + \frac{3 c C I_p M^2 P^2}{3 T^2} - \frac{b C I_p P^3}{2 T^2} - \frac{b c I_p M^3}{6 T^2} - \frac{b C I_p P^4}{b I_c M^3 S} \right. \\
+ \frac{c C I_p M P S^3}{c c I_p P^4} + \frac{2 b c I_p M^3}{3 T^2} + \frac{a (I_c - I_p) M^2 S}{2 T^2} - \frac{3 c I_p M^4 S}{6 T^2} - \frac{b c I_p M^4 S}{b I_c M^3 S} \right.
\\
- \frac{5 b c I_p M^3 S}{6 T^2} - \frac{a I_c I_p M^3 S}{6 T^2} + \frac{3 c I_p M^4 S}{4 T^2} - \frac{12 T^2}{3 T^2} - \frac{1}{2} a c I_c I_p M^3 P S - \frac{b I_c P S}{3 T^2} - \frac{3 c I_c I_p M^3 P S}{2 c I_p M^3 P S} \right. \\
+ \frac{2 T^2}{4 T^2} - \frac{a I_c I_p M^4 P S}{2 T^2} + \frac{1}{2} a c I_c I_p M^4 P S - \frac{b I_c I_p M^4 P S}{2 T^2} - \frac{3 c I_c I_p M^4 P S}{4 T^2} + \frac{2 T^2}{3 T^2} - \frac{3 c I_c I_p M^4 P S}{4 T^2} + \frac{c (I_c + I_p) P^3 S}{3 T^2} + \frac{c (I_c + I_p) P^4 S}{4 T^2} \\
- \frac{1}{2} a c - \frac{1}{2} b c I_p (M - P) T - \frac{1}{2} b c I_c S + b I_c P S - \frac{3}{2} c I_c P^2 S \\
+ \frac{2}{3} b c I_T - \frac{2}{3} c c I_p M T + \frac{2}{3} c C I_p P T - \frac{2}{3} b I_c T + \frac{3}{2} c I_p S T \\
+ \frac{1}{4} c C i T^2 - \frac{1}{4} c I_c S T^2 \\
+ \theta \left\{ \frac{(a C I_p (M^3 - P^3)}{12 T^2} + \frac{b c I_p (M^3 - P^3)}{40 T^2} + \frac{c c I_p (M^6 - P^6)}{90 T^2} - \frac{1}{12} b C I_p (M^3 - P^3) + \frac{1}{3} a C T - \frac{a C I_p M T}{3 T} + \frac{1}{3} a C I_p P T - \frac{a C I_p M T}{5 c C I_p (M^3 - P^3)} \\
+ \frac{3 b C T^2}{8} + \frac{1}{4} a C i T^2 - \frac{3}{8} b C I_p (M - P) T^2 + \frac{2}{5} c C T^3 \\
+ \frac{b C i T^3}{15} - \frac{7}{2} c C I_p (M - P) T^3 + \frac{5}{18} c C i T^3 \right\} = 0 \]

\[ \frac{\partial C_T(P, T)}{\partial P} = 0 \]
\[ a c l_p + a l_c s - 2 b l_c P s + 3 c l_c P^2 s + \frac{3 b c l_p M^2}{2 T} - \frac{2 c C l_p M^3}{3 T} - \frac{b C l_p M^2}{T} + \frac{3 c C l_p M^2 P}{T} - \frac{3 b l_p M^2 S}{2 T} \]
\[ - \frac{a l_c l_p M^2 S}{2 T} + \frac{2 c l_p M^3 S}{2 T} - \frac{b l_c l_p M^3 S}{3 T} - \frac{c l_c l_p M^4 S}{4 T} \]
\[ + \frac{3 c l_p M P^2 S}{T} - \frac{c l_p P^3 S}{T} + \frac{b C l_p T + b l_c T - 3 c l_p P S T}{T} \]
\[ + \frac{1}{3} c C l_p + c l_c S T^2 \]
\[ + \theta \left( \left\{ \frac{1}{2} a c l_p P^2 + \frac{a C l_p P^3}{3 T} + \frac{b C l_p P^4}{8 T} + \frac{c C l_p P^5}{15 T} - \frac{b C l_p P^2 T}{4} \right\} \right) \]
\[ = 0 \]

... (7.12.2)

Solving the equation (7.12.1) and (7.12.2) for T and P. The values of T and P will be optimal i.e. T (=T*) and P (=P*) Hessian determinant is greater than zero

\[ H = \begin{vmatrix} \frac{\partial^2 C_T(P, T)}{\partial T^2} & \frac{\partial^2 C_T(P, T)}{\partial T \partial P} \\ \frac{\partial^2 C_T(P, T)}{\partial P \partial T} & \frac{\partial^2 C_T(P, T)}{\partial P^2} \end{vmatrix} \]

Corresponding minimum average cost: \( C_T^*(P^*, T^*) = C_T(P, T) (T^*, P^*) \)

### 7.4 Solution Procedure

The total average cost \( C_T(P, T) \) given above, is a function of the two variables T and P. The necessary condition for \( C_T(P, T) \) to be minimum is given by the equations
(7.12.1) and (7.12.2). These equations are highly non-linear in nature, which can be easily solved by Newton Method of higher variables when the values of the parameters are prescribed. The optimal solutions of the equations (7.12.1) and (7.12.2) are $T^*$ and $P^*$ provided these values satisfy the conditions $H > 0$. Subsisting these values in equation (7.12), the optimal average cost $C_T(P, T) (T^*, P^*)$ can be obtained.

### 7.5 Numerical examples

To illustrate the obtained result following are some examples

Considering, $A = 250$ units, $s = 46$, $I_p = 0.18$, $I_c = 0.15$, $a = 12$, $b = 7.3$, $c = 0.9$, $\theta(t) = 0.23$, $i = 0.26$ and $C = 1.72$ units

**Case I**, for $M = 0$,

$\Rightarrow T^* = 1.9$ year, and $P^* = 1.43$ year and $C_T(P, T)^* = 45.71$ units

**Case II**, for $M = 0.3$ year,

$\Rightarrow T^* = 2.05$ year, and $P^* = 1.7$ year and $C_T(P, T)^* = 27.88$ units

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7.6 Conclusion

In this chapter, some realistic features are considered. These features are likely to be associated with an inventory of consumer goods. The assumption of a quadratic time-dependent demand rate and permissible delay in payments are very realistic in the market. The deterioration rate is also assumed to be constant. From the numerical example, we observe that for $0 < a, b, c \leq 1$, the total average cost is minimum. If we put $c = 0$ in equation number (7.11), we get the total average cost as in the model of (Singh and Singh, 2009) with demand rate $D(t) = a + bt$. To illustrate the model two numerical examples are done, with different situations.

The above model can be converted into constant demand model by taking `$b=0, c=0$`, linear demand model by taking `$c=0$`, or for items having no deterioration by taking `$i(t)=0$`. This study can further be extended for items having various types of other demand patterns like periodic time dependent demand, stock dependent demand, exponential demand rate as well as the effect of inflation and time value of money also can be incorporated in this model to make it more realistic for the business environment. Thus, this kind of model will also help the retailers or buyers in deciding their optimal order quantity to have minimum inventory cost, payment time, and benefits of permissible delay in payments.