Chapter 5

Radial Continuity of Metric Projections

This chapter deals with radial continuities of metric projections which are generalizations of lower semi-continuity and upper semi-continuity. Metric Projection $P_V$ is said to be lower semi-continuous (upper semi-continuous) at $x$ if for each open set $W$ with $P_V(x) \cap W \neq \emptyset$ ($P_V(x) \subset W$) there exists a neighbourhood $U$ of $x$ such that $P_V(y) \cap W \neq \emptyset$ ($P_V(y) \subset W$) for all $y \in U$. Upper semi-continuity and lower semi-continuity of the metric projections have played a key role in discussing the structure of the set $V$ and the geometry of the space (see Brosowski and Deutsch(1974b) and references cited therein). Brosowski and Deutsch(1972,1974b) introduced in normed linear spaces some simple and more general radial continuity criteria (called Outer Radial Lower, Inner Radial Lower and Outer Radial Upper), which require that the restriction of the metric projection to certain prescribed line segments be lower semi-continuous (lsc) or upper semi-continuous (usc). It turned out (see Brosowski and Deutsch(1972,1974b)) that these criteria, which are much weaker than lsc or usc, were strong enough to generalize a number of known results and weak enough so that many of the known theorems have valid converses (which they did not have under the strong hypotheses of lsc or usc).

In this chapter, we extend these concepts and also the concepts of Inner Radial Upper (IRU) continuity introduced in normed linear spaces by Panda(1974) to convex metric spaces and prove some results including those of Brosowski and Deutsch in such spaces. The results of this chapter are proved in Narang and Tejpal(communicated). Analogous results for farthest point mapping have been proved in Narang et al.(2006).
The first section deals with Outer Radial Lower continuity of metric projections. Second section is concerned with Inner Radial Lower continuity. In the third section, we give some results related to Outer Radial Upper continuity, while the fourth section is related to Inner Radial Upper continuity. The fifth section of this chapter deals with sets having both IRL and ORU continuous metric projections.

The proofs of some of the lemmas and subsequent theorems given in this chapter follow the line of arguments presented in Brosowski and Deutsch(1974b) and Mhaskar and Pai(2000) with suitable modifications.

5.1 Outer Radial Lower Continuity of Metric Projections

First generalization of lower semi-continuity is the following:

**Definition 5.1.1.** Let $V$ be a non-empty subset of an M-space $(X, d)$ and $x_0 \in X$. We say that $P_V$ is Outer Radial Lower (ORL)-continuous at $x_0$ if for each $v_0 \in P_V(x_0)$ and each open set $W$ for which $P_V(x_0) \cap W \neq \emptyset$ there exists a neighbourhood $U$ of $x_0$ such that $P_V(x) \cap W \neq \emptyset$ for all $x \in U \cap G_1(v_0, x_0, -)$. $P_V$ is called ORL-continuous if it is ORL-continuous at each point.

The following lemma, which was proved for normed linear spaces by Brosowski and Deutsch(1974b)- Lemma 2.2 and holds for M-spaces will be used in proving our first theorem:

**Lemma 5.1.1.** Let $V$ be a non-empty subset of an M-space $(X, d)$ and $x_0 \in X$ then the following are equivalent:

(i) $P_V$ is ORL-continuous at $x_0$.

(ii) for each $v_0, v_1 \in P_V(x_0)$ and each $\epsilon > 0$ there exists $\delta > 0$ such that $P_V(x) \cap B(v_1, \epsilon) \neq \emptyset$ for all $x \in \{z \in G_1(v_0, x_0, -) : d(z, x_0) < \delta\}$.

(iii) for each $v_0, v_1 \in P_V(x_0)$ and every sequence $x_n$ in $G_1(v_0, x_0, -)$ with $x_n \to x_0$ there exists $v_n \in P_V(x_n)$ such that $v_n \to v_1$.  

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Proof. $(i) \Rightarrow (ii)$. Let $v_0, v_1 \in P_V(x_0)$ and $\epsilon > 0$ be given. Consider $B(v_1, \epsilon)$. Clearly $P_V(x_0) \cap B(v_1, \epsilon) \neq \emptyset$. By ORL-continuity of $P_V$, there exists a neighbourhood $U$ of $x_0$ such that $P_V(x) \cap B(v_1, \epsilon) \neq \emptyset$ for every $x \in U \cap G_1(v_0, x_0, -)$. Since $U$ is an open set, there exists some $\delta' > 0$ such that $B(x_0, \delta') \subset U$. Let $\delta = \frac{\delta'}{d(x_0, v_0)}$. Therefore, $P_V(x) \cap B(v_1, \epsilon) \neq \emptyset$ for every $x \in \{z \in G_1(v_0, x_0, -) : d(z, x_0) < \delta\}$.

$(ii) \Rightarrow (iii)$. Let $v_0, v_1 \in P_V(x_0)$ and $x_n$ be a sequence in $G(v_0, x_0, -)$ with $x_n \to x_0$. For $\epsilon = 1$, there exists $\delta_1 > 0$ such that $P_V(x) \cap B(v_1, 1) \neq \emptyset$ for every $x \in V_{\delta_1}$ where $V_{\delta} = \{z \in G_1(v_0, x_0, -) : d(z, x_0) < \delta\}$. Choose $n_1$ such that $x_n \in V_{\delta_1}$ for every $n \geq n_1$. Choose $y_1 \in P_V(x_{n_1}) \cap B(v_1, 1)$. For $\epsilon = \frac{1}{2}$, there exists a $\delta_2$, $0 < \delta_2 < \delta_1$, such that $P_V(x) \cap B(v_1, \frac{1}{2}) \neq \emptyset$ for every $x \in V_{\delta_2}$. Let $n_2 > n_1$ be such that $x_n \in V_{\delta_2}$ for every $n \geq n_2$ and choose $y_2 \in P_V(x_{n_2}) \cap B(v_1, \frac{1}{2})$. Continuing in this way, we obtain a sequence of integers $\langle n_k \rangle$, a decreasing sequence of positive numbers $\langle \delta_k \rangle$, and a sequence $\langle y_k \rangle$ such that $\delta_k \to 0$, $x_n \in V_{\delta_k}$ for every $n \geq n_k$, and $y_k \in P_V(x_{n_k}) \cap B(v_1, \frac{1}{k})$. We define a sequence $\langle v_n \rangle$ by taking $v_n \subset P(x_n)$ for $n = 1, \ldots, n_1 - 1$, $v_{n_n} = y_k$ for every $k$ and $v_n \in P_V(x_n) \cap B(v_1, \frac{1}{k})$ for $n_k < n < n_{k+1}$. Then $v_n \in P_V(x_n)$ for every $n$ and $v_n \to v_1$.

$(iii) \Rightarrow (i)$. Suppose on contrary that there exists $v_0 \in P_V(x_0)$ and an open set $W$ with $P_V(x_0) \cap W \neq \emptyset$ such that for every neighbourhood $U$ of $x_0$ there exists an $x \in U \cap G_1(v_0, x_0, -)$ such that $P_V(x) \cap W = \emptyset$. Choose $v_1 \in P_V(x_0) \cap W$. Then for every $n$, there exists $x_n \in G_1(v_0, x_0, -)$ with $d(x_n, x_0) < \frac{1}{n}$ such that $P_V(x_n) \cap W = \emptyset$. Then $x_n \to x_0$, but if $v_n \in P_V(x_n)$ then $v_n \notin W$ so $v_n \not \to v_1$.

**Theorem 5.1.1.** For a non-empty subset $V$ of an $M$-space $(X, d)$, consider the following:

(i) $V$ is a sun,

(ii) The metric projection $P_V$ is ORL-continuous,

(iii) for each $x \in X$, every local best approximation is a global best approximation,

(iv) $V$ is a moon,

then $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv)$.
Proof. \((i) \Rightarrow (ii)\) Suppose \(V\) is a sun. Let \(x_0 \in X\), \(v_0, v_1 \in P_V(x_0)\) and \(\epsilon > 0\) be given. In view of Lemma 5.1.1, it is sufficient to show that \(P_V(z) \cap B(v_1, \epsilon) \neq \emptyset\) for all \(z \in G_1(v_0, x_0, -).\) As \(V\) is a sun, \(v_0 \in P_V(z)\) for each \(z \in G_1(v_0, x_0, -)\) i.e. for each \(z\) satisfying \(d(z, v_0) = d(z, x_0) + d(x_0, v_0).\) Consider

\[
d(z, v_1) \leq d(z, x_0) + d(x_0, v_1)
= d(z, v_0) - d(x_0, v_0) + d(x_0, v_1)
= d(z, v_0) - d(x_0, V) + d(x_0, V)
= d(z, v_0)
\leq d(z, v_1) \text{ as } v_0 \in P_V(z).
\]

This gives \(d(z, v_0) = d(z, v_1)\) and so \(v_1 \in P_V(z)\) and hence \(P_V(z) \cap B(v_1, \epsilon) \neq \emptyset.\)

\((ii) \Rightarrow (iii)\) This part has been proved in Theorem 3.2.5.

\((iii) \Rightarrow (iv)\) This part has been proved in Theorem 3.2.4. \hfill \Box

Note 5.1.1. In normed linear spaces, Theorem 5.1.1 was proved by Brosowski and Deutsch (1974b) - Theorem 2.3 (see also Mhashkar and Pai(2000) - Theorem 7, p.468). In general, none of the implications in Theorem 5.1.1 is reversible (see Brosowski and Deutsch (1974b)).

In an MS-space, all the conditions of the above theorem are equivalent. In particular, we have

Corollary 5.1.1. Let \(X\) be an MS-space and \(V \subset X.\) Then \(V\) is a sun if and only if \(P_V\) is ORL-continuous.

5.2 Inner Radial Lower Continuity of Metric Projections

A second generalization of lower semi-continuity is:

Definition 5.2.1. Let \(V\) be a non-empty subset of a convex metric space \((X, d)\) and \(x_0 \in X.\) \(P_V\) is said to be Inner Radial Lower (IRL)-continuous at \(x_0\) if for each \(v_0 \in P_V(x_0)\) and each open set \(W\) with \(P_V(x_0) \cap W \neq \emptyset,\) there exists a neighbourhood \(U\) of \(x_0\) such
that $P_V(x) \cap W \neq \emptyset$ for every $x \in U \cap \{W(x_0, v_0, \lambda) : 0 \leq \lambda \leq 1\}$. $P_V$ is called IRL-continuous if it is IRL-continuous at each point of $X$.

Each lsc metric projection is IRL-continuous but not conversely (see Brosowski and Deutsch(1974b)).

The following lemma, which was proved for normed linear spaces in Brosowski and Deutsch(1974b) and holds for convex metric spaces, will be used in our next theorem:

**Lemma 5.2.1.** Let $V$ be a non-empty subset of a convex metric space $(X, d)$ and $x_0 \in X$ then the following are equivalent:

(i) $P_V$ is IRL-continuous at $x_0$.

(ii) for each $v_0, v_1 \in P_V(x_0)$, and each $\epsilon > 0$ there exists a $\delta > 0$ such that $P_V(x) \cap B(v_1, \epsilon) \neq \emptyset$ for all $x \in \{W(x_0, v_0, \lambda) : 1 - \delta \leq \lambda \leq 1\}$,

(iii) for each $v_0, v_1 \in P_V(x_0)$, and every sequence $\langle x_n \rangle$ in $\{W(x_0, v_0, \lambda) : 0 \leq \lambda \leq 1\}$ with $x_n \to x_0$, $d(v_1, P_V(x_n)) \to 0$ (i.e. there exists $v_n \in P_V(x_n)$ such that $v_n \to v_1$).

**Proof.** (i) $\Rightarrow$ (ii). Let $v_0, v_1 \in P_V(x_0)$ and $\epsilon > 0$ be given. Consider $B(v_1, \epsilon)$. Clearly $P_V(x_0) \cap B(v_1, \epsilon) \neq \emptyset$. By IRL continuity of $P_V$, there exists a neighbourhood $U$ of $x_0$ such that $P_V(x) \cap B(v_1, \epsilon) \neq \emptyset$ for every $x \in U \cap \{W(x_0, v_0, \lambda) : 0 \leq \lambda \leq 1\}$. Since $U$ is an open set, there exists some $\delta' > 0$ such that $B(x_0, \delta') \subset U$. Let $\delta = \frac{\delta'}{d(x_0, v_0)}$. Therefore, $P_V(x) \cap B(v_1, \epsilon) \neq \emptyset$ for every $x \in \{W(x_0, v_0, \lambda) : 1 - \delta \leq \lambda \leq 1\}$.

(ii) $\Rightarrow$ (iii). Suppose on contrary that there exists $v_0, v_1 \in P_V(x_0)$ and a sequence $x_n \in \{W(x_0, v_0, \lambda) : 0 \leq \lambda \leq 1\}$ with $x_n \to x_0$ but $d(v_1, P_V(x_n)) \geq \epsilon > 0$ for every $n$. Choose $\delta > 0$ such that $P_V(x) \cap B(v_1, \epsilon) \neq \emptyset$ for every $x \in \{W(x_0, v_0, \lambda) : 1 - \delta \leq \lambda \leq 1\}$. Then for $n$ sufficiently large, $x_n \in \{W(x_0, v_0, \lambda) : 1 - \delta \leq \lambda \leq 1\}$ and so $d(v_1, P_V(x_n)) < \epsilon$, which is a contradiction.

(iii) $\Rightarrow$ (i). Suppose on contrary that there exists $v_0 \in P_V(x_0)$ and an open set $W$ with $P_V(x_0) \cap W \neq \emptyset$ such that for every neighbourhood $U$ of $x_0$ there exists an $x \in U \cap \{W(x_0, v_0, \lambda) : 0 \leq \lambda \leq 1\}$ such that $P_V(x) \cap W = \emptyset$. Choose $v_1 \in P_V(x_0) \cap W$. Then for every $n$ there exists $x_n = W(x_0, v_0, \lambda_n)$ with $1 - \frac{1}{n} < \lambda_n < 1$ such that
\[ P_V(x_i) \cap B(v_1, \epsilon) = \emptyset \text{ for } i = 0, 1, 2, \ldots \text{ Hence } x_n \to x_0 \text{ but } d(v_1, P_V(x_n)) \geq \epsilon \text{ for every } n, \text{ a contradiction.} \]

**Theorem 5.2.1.** Let \((X, d)\) be a convex metric space satisfying Property(I) and \(x \in X\). If \(P_V(x)\) is convex then \(P_V\) is IRL-continuous at \(x\).

**Proof.** If \(P_V(x) = \emptyset\), the result is trivial. Let \(v_0, v_1 \in P_V(x)\) and \(x_n \in \{W(x, v_0, \lambda) : 0 \leq \lambda \leq 1\}\) with \(x_n \to x\). Thus \(x_n = W(x, v_0, 1 - \epsilon_n)\) where \(0 \leq \epsilon_n \leq 1\) and \(\epsilon_n \to 0\). Also \(d(x_n, x) = \epsilon_n d(v_0, x)\), \(d(x, v_0) = (1 - \epsilon_n)d(x, v_0)\). Let \(v_n = W(v_1, v_0, 1 - \epsilon_n)\). Then \(v_n \in P_V(x)\) as \(P_V(x)\) is convex and \(v_n \to v_1\). Also

\[
d(x_n, v_n) = d(W(x, v_0, 1 - \epsilon_n), W(v_1, v_0, 1 - \epsilon_n)) \\
\leq (1 - \epsilon_n)d(x, v_1) \\
= (1 - \epsilon_n)d(x, v_0) \\
= d(x_n, v_0) \\
= d(x_n, V)
\]

as \(x_n \in \{W(x, v_0, \lambda) : 0 \leq \lambda \leq 1\}\) and \(v_0 \in P_V(x)\) imply \(v_0 \in P_V(x_n)\) (Proposition 2.2.3)). This gives \(v_n \in P_V(x_n)\) and hence \(P_V\) is IRL-continuous at \(x\) by the above lemma. \(\square\)

**Remark 5.2.1.** The converse of above theorem is not true (see Brosowski and Deutsch (1974b)). Taking \(X\) to be the plane \(\mathbb{R}^2\) with the maximum norm and letting \(V\) to be the two point set \(\{(1, 0), (1, \frac{1}{2})\}\), one sees that \(P_V(0) = V\) is not convex but \(P_V\) is IRL-continuous at \(0\).

**Remark 5.2.2.** Since for convex sets \(V\), \(P_V(x)\) is convex (Narang(1998)) and for Chebyshev sets \(V\), \(P_V(x)\) being singleton is convex, we have

**Corollary 5.2.1.** If \(V\) is a convex or a Chebyshev set in a convex metric space \((X, d)\) satisfying Property (I) then \(P_V\) is IRL-continuous.

Neither of the sufficient conditions of the corollary are necessary.
**Example 5.2.1.** (Brosowski and Deutsch(1974b)). Let $X$ denote the plane $\mathbb{R}^2$ with the maximum norm and

$$V = \{(x, y) : x \leq 0\} \cup \{(x, y) : x \leq y\}$$

$V$ is neither convex nor Chebyshev but $P_V$ is IRL-continuous (since $P_V(x)$ is convex for every $x$).

### 5.3 Outer Radial Upper Continuity of Metric Projections

A first generalization of upper semi-continuity is:

**Definition 5.3.1.** Let $V$ be a non-empty subset of an M-space $(X, d)$ and $x_0 \in X$. $P_V$ is said to be **Outer Radial Upper (ORU)-continuous** at $x_0$ if for each $v_0 \in P_V(x_0)$ and each open set $W \supset P_V(x_0)$ there exists a neighbourhood $U$ of $x_0$ such that $P_V(x) \subset W$ for all $x \in U \cap G_1(v_0, x_0, -)$. $P_V$ is called ORU-continuous if it is ORU-continuous at each point of $X$.

**Example 5.3.1.** (Brosowski and Deutsch(1974b)) Let $X = l_2$ and $V = X \setminus B(0, 1)$ - complement of unit ball, then $P_V$ is usc and hence ORU-continuous.

**Example 5.3.2.** (Brosowski and Deutsch(1974b)) Let $X$ be the Euclidean plane and $V$ be defined as

$$V = \{(x, y) : x \geq 1\} \cup \{(x, y) : x \leq -1\}$$

$$\cup \{(x, y) : |x| < 1, y \geq \sqrt{1-x^2}\} \setminus \{(1, 0), (-1, 0)\}.$$

Taking $x_0 = (0, 0)$, we have

$$P_V(x_0) = \{(x, y) : x^2 + y^2 = 1, y > 0\}.$$

Then $P_V$ is not ORU-continuous at $(0, 0)$. 73
The following lemma, which for normed linear spaces is given in Brosowski and Deutsch(1974b) - Lemma 4.2 and holds in M-spaces, deals with ORU-continuity.

**Lemma 5.3.1.** Let $V$ be a non-empty subset of an M-space $(X, d)$ and $x_0 \in X$. Consider the following statements:

(i) $P_V$ is ORU-continuous at $x_0$.

(ii) For each $v_0 \in P_V(x_0)$ and each $\epsilon > 0$ there exists a $\delta > 0$ such that

$$\sup_{v \in P_V(x)} d(v, P_V(x_0)) < \epsilon$$

for every $x$ in $G_1(v_0, x_0, -)$ with $d(x, x_0) < \delta$.

(iii) For each $v_0 \in P_V(x_0)$ and each sequence $<x_n>$ in $G_1(v_0, x_0, -)$ with $x_n \to x_0$, $d(v, P_V(x_0)) \to 0$.

(iv) For each $v_0 \in P_V(x_0)$ and each sequence $<x_n>$ in $G_1(v_0, x_0, -)$ with $x_n \to x_0$ and each sequence $<v_n>$ with $v_n \in P_V(x_n)$, $d(v_n, P_V(x_0)) \to 0$.

(v) For each $v_0 \in P_V(x_0)$ and each sequence $<x_n>$ in $G_1(v_0, x_0, -)$ with $x_n \to x_0$ and each sequence $<v_n>$ with $v_n \in P_V(x_n)$ and $v_n \to v$, $v \in P_V(x_0)$.

Then (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v). Moreover, if $P_V(x_0)$ is compact then (iv) $\Rightarrow$ (i) and if $V$ is compact then (v) $\Rightarrow$ (i).

**Proof.** (i) $\Rightarrow$ (ii). Choose $v_0 \in P_V(x_0)$ and let $W = \cup \{ B(v, \frac{\delta}{2}) : v \in P_V(x_0) \} \supset P_V(x_0)$. Then by ORU-continuity of $P_V$, there exists a $\delta > 0$ such that $P_V(x) \subset W$ for every $x \in G_1(v_0, x_0, -)$ with $d(x, x_0) < \delta$. Let $x \in \{ z \in G_1(v_0, x_0, -) : d(z, x_0) < \delta \}$ and $v \in P_V(x)$. Then there exists $v' \in P_V(x_0)$ such that $d(v', v) < \frac{\delta}{2}$ and so $d(v, P_V(x_0)) < \frac{\delta}{2}$.

It follows that $\sup \{ d(v, P_V(x_0)) : v \in P_V(x) \} \leq \frac{\delta}{2} < \epsilon$.

(ii) $\Rightarrow$ (iii). Let $v_0 \in P_V(x_0)$ and $<x_n>$ in $G_1(v_0, x_0, -)$ with $x_n \to x_0$. Therefore, for $\delta > 0$, there exists a positive integer $N$ such that $d(x_n, x_0) < \delta$ for all $n \geq N$. If $n \geq N$, $x_n \in B(x_0, \delta) \cap G_1(v_0, x_0, -)$ and so by (ii) $\sup \{ d(v, P_V(x_0)) : v \in P_V(x) \} < \epsilon$ for all $n \geq N$, which implies that $\sup \{ d(v, P_V(x_0)) : v \in P_V(x) \} \to 0$. 

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(iii) ⇒ (iv). This part is straightforward.

(iv) ⇒ (v). Let \( v_0 \in P_V(x_0) \) and \( \langle x_n \rangle \) in \( G_1(v_0, x_0, -) \) with \( x_n \to x_0 \). Let \( \langle v_n \rangle \) be a sequence in \( P_V(x_n) \) with \( v_n \to v \). By (iv), \( d(v_n, P_V(x_0)) \to 0 \). Consider \( d(v, P_V(x_0)) \leq d(v, v_n) + d(v_n, P_V(x_0)) \to 0 \). Therefore, \( v \in \overline{P_V(x_0)} \).

Now assume that \( P_V(x_0) \) is compact. Suppose on contrary, that (iv) holds but (i) fails then there is an open set \( W \supset P_V(x_0) \) such that for every \( n \) there is an \( x_n \in \{ z \in G_1(v_0, x_0, -) : d(z, x_0) < 1 + \frac{1}{n} \} \) satisfying \( P_V(x_n) \setminus W \neq \emptyset \). Choose \( v_n \in P_V(x_n) \setminus W \). Then \( x_n \to x_0 \) so by (iv), \( d(v_n, P_V(x_0)) \to 0 \). Choose \( y_n \in P_V(x_0) \) such that \( d(v_n, y_n) \to 0 \). By passing to a subsequence we may assume \( y_n \to y_0, y_0 \in P_V(x_0) \). Hence \( v_n \to y_0 \) also. Since \( y_0 \in W \) is open, \( v_n \in W \) for \( n \) large. But this is a contradiction.

Now assume that \( V \) is compact. If (v) holds but (i) fails then a similar argument yields a contradiction. \( \square \)

Remark 5.3.1. In general, the implication (iv) ⇒ (i) and (v) ⇒ (i) are false (see Brosowski and Deutsch(1974b)).

The following theorem gives conditions under which metric projection is ORU continuous:

**Theorem 5.3.1.** If \( V \) is a sun in an M-space \( (X, d) \) such that \( P_V(x) \) is compact for each \( x \in X \) then \( P_V \) is ORU-continuous.

Before proving this theorem, we establish a lemma.

**Lemma 5.3.2.** If \( V \) is a sun in an M-space \( (X, d) \), \( v_0 \in P_V(x_0) \) and \( \langle x_n \rangle \) is a sequence in \( G_1(v_0, x_0, -) \) with \( x_n \to x_0 \) then \( P_V(x_0) = \cap_{n=1}^{\infty} P_V(x_n) \).

**Proof.** Let \( v \in P_V(x_0) \). Then for each \( n \),

\[
\begin{align*}
d(x_n, v) & \leq d(x_n, x_0) + d(x_0, v) \\
& = d(x_n, x_0) + d(x_0, v_0) \\
& = d(x_n, v_0) \\
& = d(x_n, V)
\end{align*}
\]

(5.3.1)
as \( v_0 \in P_V(x_0) \) and \( V \) is a sun, so \( v_0 \in P_V(x_n) \). Then (5.3.1) implies \( v \in P_V(x_n) \) for each \( n \) and hence \( P_V(x_0) \subset \bigcap_{n=1}^{\infty} P_V(x_n) \).

Conversely if \( v \in \bigcap_{n=1}^{\infty} P_V(x_n) \) then \( d(x_n, v) = d(x_n, V) \) for each \( n \). Consider

\[
d(x_0, v) \leq d(x_0, x_n) + d(x_n, v) = d(x_0, x_n) + d(x_n, V) \to d(x_0, V).
\]

Therefore \( v \in P_V(x_0) \) and hence \( P_V(x_0) = \bigcap_{n=1}^{\infty} P_V(x_n) \). \( \square \)

**Proof of Theorem 5.3.1.** Let \( x_0 \in X \) and \( v_0 \in P_V(x_0) \). Suppose \( x_n \) is a sequence in \( G_1(v_0, x_0, -) \) with \( x_n \to x_0 \). As \( V \) is a sun, \( v_0 \in P_V(x_n) \) for every \( n \). Let \( W \) be an open set with \( W \supset P_V(x_0) \). By Lemma 5.3.2, \( P_V(x_0) = \bigcap_{n=1}^{\infty} P_V(x_n) \).

We claim that \( P_V(x_n) \subset P_V(x_{n-1}) \). Let \( v \in P_V(x_n) \). Then \( d(x_n, v) = d(x_n, V) \).

Consider

\[
d(x_{n-1}, v) \leq d(x_{n-1}, x_n) + d(x_n, v) = d(x_{n-1}, x_n) + d(x_n, V) \text{ as } v \in P_V(x_n) = d(x_{n-1}, x_n) + d(x_n, v_0) = d(x_{n-1}, v_0) \text{ as } x_n \in G_1(v_0, x_0, -) \text{ for all } n = d(x_{n-1}, V) \text{ as } v_0 \in \bigcap_{n=1}^{\infty} P_V(x_n)
\]

This implies \( v \in P_V(x_{n-1}) \) and \( P_V(x_n) \subset P_V(x_{n-1}) \) for all \( n \). Thus \( P_V(x_n) \) is a decreasing sequence of compact sets. Let \( v_n \in P_V(x_n) \), \( n = 1, 2, \ldots \). Then \( v_n \in P_V(x_1) \) for all \( n \). Since \( P_V(x_1) \) is compact, \( v_n \) has a convergent subsequence \( v_{n_i} \to v \). Now we
claim that \( v \in P_V(x_0) = \bigcap_{n=1}^{\infty} P_V(x_n) \). Let \( n \) be fixed but arbitrary. Consider

\[
\begin{align*}
d(x_n, v) &= d(x_n, \lim v_{n_i}) \\
&= \lim_i d(x_n, v_{n_i}) \\
&\leq \lim_i [d(x_n, x_{n_i}) + d(x_{n_i}, v_0)] \\
&= \lim_i [d(x_n, x_{n_i}) + d(x_{n_i}, V)] \\
&= \lim_i [d(x_n, x_{n_i}) + d(x_{n_i}, v_0)] \\
&= d(x_n, x_0) + d(x_0, v_0) \\
&= d(x_n, v_0) \\
&= d(x_n, V).
\end{align*}
\]

This gives \( v \in P_V(x_n) \) for all \( n \) i.e. \( v \in \bigcap_{n=1}^{\infty} P_V(x_n) = P_V(x_0) \). Thus \( v \in P_V(x_0) \in W \). As \( W \) is open, there exists \( N \) such that \( v_n \in W \) for all \( n \geq N \). Thus \( P_V(x_n) \subset W \) for all \( n \geq N \). Therefore for some \( \delta > 0 \), \( P_V(x) \subset W \) for all \( x \) in \( G_1(v_0, x_0, -) \cap B(x_0, \delta) \) and so there exists a neighbourhood \( U \) of \( x_0 \) such that \( P_V(x) \subset W \) for all \( x \in G_1(v_0, x_0, -) \). Therefore \( P_V \) is ORU-continuous at \( x_0 \) and hence \( P_V \) is ORU-continuous. \( \square \)

**Corollary 5.3.1.** If \( V \) is a Chebyshev sun in an M-space \((X, d)\) then \( P_V \) is ORU-continuous.

### 5.4 Inner Radial Upper Continuity of Metric Projections

A second generalization of upper semi-continuity is:

**Definition 5.4.1.** Let \( V \) be a non-empty subset of a convex metric space \((X, d)\) and \( x_0 \in X \). \( P_V \) is said to be Inner Radial Upper (IRU)-continuous at \( x_0 \) if for each \( v_0 \in P_V(x_0) \) and each open set \( W \supset P_V(x_0) \) there exists a neighborhood \( U \) of \( x_0 \) such that \( P_V(x) \subset W \) for all \( x \in U \cap W(v_0, x_0, \lambda) \). \( P_V \) is called IRU-continuous if it is IRU-continuous at each point.

**Example 5.4.1.** Let \( X = l_2 \) and \( V = X \setminus B(0, 1) \). Example 5.3.1 shows that \( P_V \) is u.s.c. and hence it is IRU-continuous.
Concerning IRU-continuity in convex metric space we have:

**Proposition 5.4.1.** Let $V \subset X$ and $x_0 \in X$. Consider the following statements:

(i) $P_V$ is IRU-continuous at $x_0$.

(ii) For each $v_0 \in P_V(x_0)$ and each $\epsilon > 0$ there exists a $\delta > 0$ such that

$$\sup_{v \in P_V(x)} d(v, P_V(x_0)) < \epsilon$$

for every $x$ in $\{W(v_0, x_0, \lambda) : \lambda \in [0, 1]\}$ with $d(x, x_0) < \delta$.

(iii) For each $v_0 \in P_V(x_0)$ and each sequence $\langle x_n \rangle$ in $W(v_0, x_0, \lambda)$ with $x_n \to x_0$,

$$\sup_{v \in P_V(x_n)} d(v, P_V(x_0)) \to 0.$$

(iv) For each $v_0 \in P_V(x_0)$ and each sequence $\langle x_n \rangle$ in $W(v_0, x_0, \lambda)$ with $x_n \to x_0$ and each sequence $\langle v_n \rangle$ with $v_n \in P_V(x_n)$, $d(v_n, P_V(x_0)) \to 0$.

(v) For each $v_0 \in P_V(x_0)$ and each sequence $\langle x_n \rangle$ in $W(v_0, x_0, \lambda)$ with $x_n \to x_0$ and each sequence $\langle v_n \rangle$ with $v_n \in P_V(x_n)$ and $v_n \to v$, $v \in \overline{P_V(x_0)}$.

Then (i) $\Rightarrow$ (ii) $\iff$ (iii) $\iff$ (iv) $\Rightarrow$ (v). Moreover if $P_V(x_0)$ is compact, (iv) $\Rightarrow$ (i) and the first four statements are equivalent. If $V$ is compact then (v) $\Rightarrow$ (i) and hence all the five statements are equivalent.

**Proof.** (i) $\Rightarrow$ (ii) Choose $v_0 \in P_V(x_0)$ and $W = \cup \{B(v, \frac{x}{2}) : v \in P_V(x_0)\} \supseteq P_V(x_0)$.

By the IRU-continuity, there exists a neighbourhood $B(x_0, \delta)$ of $x_0$ such that $P_V(x) \subset W$ for every $x$ in $B(x_0, \delta) \cap \{W(v_0, x_0, \lambda) : \lambda \in [0, 1]\}$.

Let $x \in B(x_0, \delta) \cap \{W(v_0, x_0, \lambda) : \lambda \in [0, 1]\}$ and $v \in P_V(x)$ then there exists $v' \in P_V(x_0)$ such that $d(v, v') < \frac{\epsilon}{2}$ and so $d(v, P_V(x_0)) < \frac{\epsilon}{2}$ as $d(v, P_V(x_0)) = \inf_{w \in P_V(x_0)} d(v, w)$. It follows that $\sup_{v \in P_V(x)} d(v, P_V(x_0)) \leq \frac{\epsilon}{2} < \epsilon$.

(ii) $\Rightarrow$ (iii) Let $v_0 \in P_V(x_0)$ and $\langle x_n \rangle$ be a sequence in $W(v_0, x_0, \lambda)$ with $\langle x_n \rangle \to x_0$. Therefore, for $\delta > 0$ there exists a positive integer $N$ such that $d(x_n, x_0) < \delta$ for all $n \geq N$. If $n \geq N$, $x_n \in B(x_0, \delta) \cap W(v_0, x_0, \lambda)$ and so by the hypothesis,

$$\sup_{v \in P_V(x_n)} d(v, P_V(x_0)) < \epsilon$$

for all $n \geq N$, which implies $\sup_{v \in P_V(x_n)} d(v, P_V(x_0)) \to 0$.

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(iii) $\Rightarrow$ (ii) Suppose (ii) does not hold i.e. there exists $v_0 \in P_V(x_0)$ and $\varepsilon > 0$ such that for every $\delta > 0$, $\sup_{v \in P_V(x)} d(v, P_V(x_0)) \geq \varepsilon$ for all $x$ in $\{W(v_0, x_0, \lambda) : \lambda \in [0, 1]\}$ with $d(x, x_0) < \delta$. Let $\langle x_n \rangle$ be a sequence in $\{W(v_0, x_0, \lambda) : \lambda \in [0, 1]\}$ with $\langle x_n \rangle \to x_0$. Then $\sup_{v \in P_V(x_n)} d(v, P_V(x_0)) \geq \varepsilon$ for all $n$ and so $\sup_{v \in P_V(x_n)} d(v, P_V(x_0))$ does not converge to 0, a contradiction.

(iii) $\Rightarrow$ (iv) Let $v_0 \in P_V(x_0)$ and $\langle x_n \rangle$ be a sequence in $W(v_0, x_0, \lambda)$ with $x_n \to x_0$ and $v_n$ be a sequence in $P_V(x_n)$. By (iii), $\sup_{v \in P_V(x_n)} d(v, P_V(x_0)) \to 0$ and hence $d(v_n, P_V(x_0)) \to 0$.

(iv) $\Rightarrow$ (iii) Suppose on contrary i.e. for some $v_0 \in P_V(x_0)$ there exists a sequence $\langle x_n \rangle$ in $W(v_0, x_0, \lambda)$ with $x_n \to x_0$ such that $\sup_{v \in P_V(x_n)} d(v, P_V(x_0))$ does not converge to 0. Let $v_n$ be a sequence in $P_V(x_n)$. So we have $\sup_{v \in P_V(x_n)} d(v_n, P_V(x_0))$ does not converge to 0. This implies $d(v_n, P_V(x_0))$ does not converge to 0 for some $v_n \in P_V(x_n)$, a contradiction.

(iv) $\Rightarrow$ (v) Let $v_0 \in P_V(x_0)$, $\langle x_n \rangle$ be a sequence in $W(v_0, x_0, \lambda)$ with $x_n \to x_0$ and $v_n$ be a sequence in $P_V(x_n)$ with $v_n \to v$. By hypothesis, $d(v_n, P_V(x_0)) \to 0$ i.e. lim $d(v_n, P_V(x_0)) = 0$. This implies $d(v, P_V(x_0)) = 0 \Rightarrow v \in \overline{P_V(x_0)}$.

Suppose $P_V(x_0)$ is compact. To prove (iv) $\Rightarrow$ (i).

Suppose on contrary that there exists $v_0 \in P_V(x_0)$ and some open set $W \supset P_V(x_0)$ such that for every $n$ there is an $x_n \in \{W(v_0, x_0, \lambda) : 1 - \frac{1}{n} < \lambda \leq 1\}$ satisfying $P_V(x_n) \cap (X \setminus W) \neq \emptyset$. Choose $v_n \in P_V(x_n) \cap (X \setminus W)$. Then $x_n \to x_0$ and so $d(v_n, P_V(x_0)) \to 0$. Choose $\langle y_n \rangle \in P_V(x_0)$ such that $d(v_n, y_n) \to 0$. As $P_V(x_0)$ is compact, there exists a subsequence $y_{n_i} \to y_0 \in P_V(x_0)$. Consider

$$d(v_{n_i}, y_0) \leq d(v_{n_i}, y_{n_i}) + d(y_{n_i}, y_0) \to 0.$$ 

This implies $v_{n_i} \to y_0$. Since $y_0 \in P_V(x_0) \subset W$, $y_0 \in W$ and $W$ is open, $v_n \in W$ for large $n$, a contradiction.

Suppose $V$ is compact. To prove (v) $\Rightarrow$ (i). We show that (v) $\Rightarrow$ (iv) $\Rightarrow$ (i).

Let $v_0 \in P_V(x_0)$ and $\langle x_n \rangle$ be a sequence in $\{W(v_0, x_0, \lambda) : \lambda \in [0, 1]\}$ with $x_n \to x_0$ and $\langle v_n \rangle$ be a sequence with $v_n \in P_V(x_n)$. To prove $d(v_n, P_V(x_0)) \to 0$. Since $P_V(x_n)$ is
a closed subset of the compact set $V$, it is compact and so $(v_n)$ has a subsequence $v_{n_i} \to v$. By hypothesis, $v \in P_V(x_0)$. Consider

$$d(v_{n_i}, P_V(x_0)) \leq d(v_{n_i}, v) + d(v, P_V(x_0)) \to 0$$

i.e. $(iv)$ is true. Since $V$ is compact, $P_V(x_0)$ is compact and so $(iv) \Rightarrow (i)$. Hence $(v) \Rightarrow (i)$. □

For Chebyshev sets, we have

**Theorem 5.4.1.** If $V$ is Chebyshev set in a convex metric space $(X, d)$ then $P_V$ is IRU-continuous.

**Proof.** Let $x_0 \in X$ be arbitrary and $v_0 \in P_V(x_0)$. Let $G$ be an open set with $G \supset P_V(x_0) = \{v_0\}$. Let $x \in \{W(x_0, v_0, \lambda) : \lambda \in [0, 1]\}$ then $v_0 \in P_V(x)$ (Narang(1998)). As $V$ is Chebyshev, $P_V(x) = \{v_0\}$. Let $U$ be any neighborhood of $x_0$ then for all $x \in U \cap \{W(x_0, v_0, \lambda) : \lambda \in [0, 1]\}$, $P_V(x) = \{v_0\} \subset G$. Hence $P_V$ is IRU-continuous. □

## 5.5 IRL and ORU Continuity of Metric Projections

In this section, we discuss the structure of the set having both IRL and ORU-continuous metric projection.

**Definition 5.5.1.** A subset $V$ of a metric space $(X, d)$ is said to be boundedly connected if $V \cap B(x, r)$ is connected for every $x \in X$ and $r > 0$.

Brosowski and Deutsch(1974b) proved that for Banach spaces, a set having both IRL and ORU-continuous metric projection is boundedly connected and have a connected valued metric projection. In this section, we extend this result to M-spaces. We first set up a notation that we use in this section.

Recall that in an M-space, there is a unique metric segment denoted by $G[x, y]$ joining two points $x$ and $y$. Let $L(x, y, \lambda)$ denotes a point on $G[x, y]$ such that $d(x, L(x, y, \lambda)) = (1 - \lambda)d(x, y)$ and $d(y, L(x, y, \lambda)) = \lambda d(x, y)$.
Theorem 5.5.1. Let $V$ be a proximinal subset of an $M$-space $(X, d)$ such that $P_V$ is both IRL and ORU-continuous then $V$ is boundedly connected and $P_V(x)$ is connected for each $x$ in $X$.

Proof. Suppose $V$ is not boundedly connected. Then there exists some $x_0 \in X$ and a positive $r > d(x_0, V)$ such that $V \cap B(x_0, r)$ is not connected ($d(x_0, V) \geq r \Rightarrow V \cap B(x_0, r) = \emptyset$ and so there is nothing to prove). Let $V \cap B(x_0, r) = E \cup F$, where $E$ and $F$ are nonempty disjoint sets which are open in $V$. We claim that $P_V(x_0) \subseteq E \cup F$.

Let $v_0 \in P_V(x_0)$ i.e. $v_0 \in V$ and $d(x_0, v_0) = d(x_0, V) < r$. Therefore $v_0 \in V \cap B(x_0, r) = E \cup F$ and so $P_V(x_0) \subseteq E \cup F$. We may assume $P_V(x_0) \cap E \neq \emptyset$ as $P_V(x_0) \subseteq E \cup F$, where $E$ and $F$ are non-empty disjoint sets open in $V$ implies $P_V(x_0)$ will have some points common with $E$ or $F$. Let $y \in F$ then there is a $\lambda_0 \in (0, 1)$ such that for every $\lambda \in [\lambda_0, 1]$, $P_V(L(y, x_0, \lambda)) \subseteq F$. Let

$$\beta = \inf \{ \lambda \in [0, 1] : P_V(L(y, x_0, \lambda)) \subseteq F \}.$$ 

We first note that $P_V(L(y, x_0, \beta)) \subseteq F$. If not, then $P_V(L(y, x_0, \beta)) \cap E \neq \emptyset$ (we have $P_V(L(y, x_0, \lambda)) \subseteq E \cup F$ for all $\lambda \in [0, 1]$ as $v_0 \in P_V(L(y, x_0, \lambda))$. This implies $d(L(y, x_0, \lambda), v_0) \leq d(L(y, x_0, \lambda), v)$ for all $v \in V$ and therefore $d(L(y, x_0, \lambda), v_0) \leq d(L(y, x_0, \lambda), y) = (1 - \lambda)d(y, x_0) < (1 - \lambda)r$ as $y \in F \subseteq E \cup F = V \cap B(x_0, r)$.

Consider

$$d(x_0, v_0) \leq d(x_0, L(y, x_0, \lambda)) + d(L(y, x_0, \lambda), v_0)$$ $$< \lambda r + (1 - \lambda)r$$ $$= r$$

and so $v_0 \in V \cap B(x_0, r) = E \cup F)$. Choose $v_0 \in P_V(L(y, x_0, \beta)) \cap E$. Since $P_V$ is IRL-continuous, for any sequence $(x_n) \in G[L(y, x_0, \beta), y]$ satisfying $x_n \to L(y, x_0, \beta)$, there exists $v_n \in P_V(x_n) \subseteq F (x_n \in G[L(y, x_0, \beta), y]$ implies $P_V(x_n) \subseteq F$) such that $v_n \to v_0 \in E$. But this is not possible as $E$ is open in $V$ and $v_n \in F \setminus E$ for every $n$. Thus $P_V(L(y, x_0, \beta)) \subseteq F$.
We now claim that best approximation to \(L(y, x_0, \beta)\) lies on the line joining \(L(y, x_0, \beta)\) and \(y\). Suppose this is not true. Then there exists some \(v \in F\) not on the line segment joining \(L(y, x_0, \beta)\) and \(y\) such that \(v\) is a best approximation to \(L(y, x_0, \beta)\) i.e.

\[
d(L(y, x_0, \beta), v) \leq d(L(y, x_0, \beta), z) \text{ for all } z \in F. \tag{5.5.1}
\]

Since \(\beta = \inf\{\lambda \in [0, 1] : P_V(L(y, x_0, \lambda)) \subseteq F\}\), we can find a sequence \(\beta_n\) such that \(P_V(L(y, x_0, \beta_n)) \subseteq F\) and \(\beta_n \to \beta\). Also \(L(y, x_0, \beta_n)\) cannot be a best approximation to \(L(y, x_0, \beta)\) since it lies on the line segment joining \(L(y, x_0, \beta)\) and \(y\), therefore, using (5.5.1), we get

\[
d(L(y, x_0, \beta), v) < d(L(y, x_0, \beta), L(y, x_0, \beta_n)) \\
= d(L(y, x_0, \beta), y) - d(L(y, x_0, \beta_n), y) \\
= (1 - \beta)d(y, x_0) - (1 - \beta_n)d(y, x_0) \\
= (\beta_n - \beta)d(y, x_0) \\
\to 0.
\]

So, \(d(L(y, x_0, \beta), v) < 0\) as equality is not true in view of definition of \(v\), a contradiction. Thus best approximation to \(L(y, x_0, \beta)\) lies on the line segment joining \(L(y, x_0, \beta)\) and \(y\).

Let \(L(y, x_0, \alpha) \in P_V(L(y, x_0, \beta)) \subseteq F\) with \(\alpha > \beta\). Since \(P_V\) is ORU-continuous, there exists a neighborhood \(B(L(y, x_0, \beta), k)\) of \(L(y, x_0, \beta)\) such that \(P_V(x) \subseteq F\) for all \(x\) in \(G_1(L(y, x_0, \alpha), L(y, x_0, \beta), -) \cap B(L(y, x_0, \beta), k)\). Choose \(\epsilon < \frac{k}{d(x_0, y)}\). Consider

\[
d(L(y, x_0, \beta), L(y, x_0, \beta - \epsilon)) \\
= d(x_0, L(y, x_0, \beta)) - d(x_0, L(y, x_0, \beta - \epsilon)) \\
= \beta d(x_0, y) - (\beta - \epsilon) d(x_0, y) \\
= \epsilon d(x_0, y) \\
< k.
\]
This implies $L(y, x_0, \beta - \epsilon) \in B_k(L(y, x_0, \beta))$. But we also have $L(y, d_0, \beta - \epsilon) \in G_1(L(y, x_0, \alpha), L(y, x_0, \beta), -)$ and so $P_V(L(y, x_0, \beta - \epsilon)) \subseteq F$, which contradicts the definition of $\beta$ and hence $V$ is boundedly connected. The proof that $P_V(x)$ is connected for each $x$ is virtually the same. 

**Corollary 5.5.1.** Let $V$ be a Chebyshev set in an M-space satisfying Property(I) and $P_N$ is ORU-continuous. Then $V$ is boundedly connected and $P_V(x)$ is connected for each $x \in X$.

**Proof.** Since $V$ is Chebyshev, $P_V$ is IRL-continuous (Corollary 5.2.1) and the proof follows. 

**Corollary 5.5.2.** Let $V$ be a Chebyshev sun in an M-space satisfying Property (I) then $V$ is boundedly connected and $P_V(x)$ is connected for each $x \in X$.

**Proof.** Since $V$ is Chebyshev, $P_V$ is IRL-continuous (Corollary 5.2.1). Also, $P_V(x)$ is compact as $V$ is Chebyshev. Since $V$ is a sun and $P_V(x)$ is compact, $P_V$ is ORU-continuous (Theorem 5.3.1). Hence $V$ is boundedly connected.