Chapter 1

Introduction

Self duality is a powerful notion in classical mechanics, classical field theory, quantum mechanics as well in quantum field theory. In these theories, the interactions have particular forms and special strength so that the second order equations of motion reduce to first order equations which are simpler to analyse. The minimization of energy or action leads to the "self dual point" at which the interactions and coupling strengths take their special self dual effects. This signifies the self dual theories physically. For example the self dual Yang-Mills equations have minimum action solutions known as instantons, the Bogomol'nyi equations of self dual Yang-Mills Higgs theory have minimum solutions known as 't Hooft-Polyakov monopoles, the Planar Abelian Higgs model has minimum energy self dual solutions known as Nielsen-Olesen vortices. Thus instantons, monopoles and vortices have become paradigms of topological structures in field theory and quantum mechanics, with important applications in particle physics, astrophysics, condensed matter physics and mathematics.

We have discussed here the self-dual Chern-Simons theory specially in (2+1) dimensions (i.e two spatial dimensions). The physical context in which these self-dual Chern-Simons models arise is that of anyonic quantum field theory [1, 2, 3], with direct applications to such planar models as the quantum Hall effect [4, 5, 6], anyonic superconductivity [7] and Aharonov-Bohm scattering. Self dual models in (2+1)dimensions have certain distinct features which are essentially connected with
The possibility of describing gauge theories with a Chern-Simons (CS) term is a special feature of odd dimensional space time. More on that, the (2+1) dimensional case is distinguished in the sense that the derivative part of CS lagrangian is quadratic in gauge fields.

To review the significant properties of the CS lagrangian density let:

\[ \mathcal{L}_{CS} = \epsilon^{\mu\nu\rho} \text{tr} (\partial_\mu A_\nu A_\rho + \frac{2}{3} A_\mu A_\nu A_\rho) \]

where the gauge field is \( A_\mu \) and \( \epsilon \)-symbol stands for anti-symmetric tensor and normalised with \( \epsilon^{012} = +1 \) In an Abelian theory gauge fields commute and so the trilinear term in the above expression vanishes. Also the action, \( S = \int d^3x \mathcal{L}_{CS} \) is gauge invariant and so we expect that a sensible gauge theory may be formulated. Another important feature of CS theories is that the CS term describes a topological gauge field theory in the sense that there is no explicit dependence on the space time metric. Thus the action is independent of the space time metric and the Chern-Simons lagrangian density \( \mathcal{L}_{CS} \) does not contribute to the energy momentum tensor. Moreover the special feature of the first order in space time derivative of \( \mathcal{L}_{CS} \) modifies the structure of the theory leading to many interesting features in CS theories.

The CS lagrangian density when coupled to an external matter current \( J^\mu \) as

\[ \mathcal{L} = \frac{k}{2} \mathcal{L}_{CS} - \text{tr}(A_\mu J^\mu) \]

(where \( k \) is CS coupling coefficient) yields crucial features when applied to condensed matter systems such as the quantum Hall effect.

Most prominent aspect of CS theory evolves when above lagrangian is coupled with Maxwell term to form a gauge model which describe a massive dynamical gauge mode, with mass determined by the CS coupling parameter \( k \) and with spin \( \pm 1 \) given by the sign of \( k \). This system has been termed "topologically massive gauge theory" [8]. The equations of motion when expressed in terms of the dual to the field tensor manifest a self duality. An equivalent version of this model also exists, where the self duality is revealed in the equations of motion for the basic field [10, 11, 12]. More
recently, another possibility has been considered where instead of the first derivative CS term a parity violating third derivative term is added to the Maxwell term [13].

An intriguing fact first observed in [8] and briefly discussed in [14, 15, 16] is that topologically massive CS-doublets, with identical mass parameters having opposite sign, are equivalent to a parity preserving vector theory with an explicit mass term. This is the Proca model. The invariance of the CS doublet under the combined parity and field interchanges is thereby easily understood from the equivalent theory. These equivalent descriptions of the same physical theory become useful and play a significant role in expanding our understanding. Aspects of a theory that are hidden in one formulations become transparent in some other formulation. Most common example is bosonization technique in (1+1)dimension in this context [17].

In recent times the role of duality as a qualitative tool in the investigation of physical systems is being gradually realised in different contexts [18]. Several technical aspects of duality symmetric actions have been explored [19]. Specially the technique able to work with distinct manifestations of duality symmetry proposed by Stone [20] is relevant to our thesis work, where a soldering technique has been developed and applied to different models. This technique for fusing together opposite aspects of duality symmetries provides a new formalism that includes the quantum interference effects between the independent components. This leads to a unique way of obtaining physical results.

1.1 Outline of the thesis :

We will restrict our study within planar models i.e models in (2+1)dimension. As we know quantum models in (0+1)dimension may be interpreted as toy models useful for studying higher dimensional field theoretic examples. Pursuing this feature we start typically with a relevant topological quantum mechanical model (such as Landau problem consisting of two basic chiral oscillators) and extrapolate the analysis to (2+1)dimensional vector field theory. Also from a contemporary view we will consider
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The aspects of selfdual symmetry in topologically massive gravity model. We will consider here three different approaches to analyse this selfdual doublet structure of a composite theory. The first one is soldering which is solely a lagrangian formulation. The idea is to solder the distinct lagrangians through a contact term [15, 16]. In the hamiltonian approach, on the other hand, there is a canonical transformation which diagonalises the composite hamiltonian into two independent pieces [21, 23, 24]. The third method is based on the exploitation of equations of motion. Simply redefining the old variables by the new ones does the trick. We have organized the chapters as follows.

In Chap. 2 we will focus on the quantum mechanical models. We will first demonstrate how duality symmetric (or chiral) actions are already present in the quantum mechanical examples such as in usual harmonic oscillator. Using the chiral oscillator form, we will briefly develop the key concepts of the soldering mechanism. This is purely a lagrangian formulation where schematic addition of two such independent lagrangians yields a final effective theory. Alternately, a canonical transformation based on hamiltonian analysis demonstrates the splitting of a composite hamiltonian into its constituent basic components. We will briefly discuss the consequence of the factorisability property of the final equation of motion of the soldered lagrangian. Next we introduce another form of the chiral oscillator model where the equation of motion approach is applicable. Necessitating very simple field redefinitions that are generic to a wide variety of models, the results of the lagrangian soldering formalism are reproduced.

We have also discussed the non commutative property of such quantum models. Since we know that non commutativity may be present in both position and momenta co-ordinates, we incorporate these features in non-commutative(NC) quantum mechanics. Some Galilean generators will be constructed out of these NC features of the phase space coordinates. A simple dynamical model will be presented that displays these aspects.

We know the compatibility of the (0+1)dimensional quantum mechanical extracts
with the (2+1) dimensional field theoretic models. So we will establish the correlation between the chiral oscillator (CO) models with that of the self (antiself) dual vector models and the outcome of the lagrangian or the hamiltonian formulations regarding such quantum theories. In Chap. 3 thus we have extended the results obtained in Chap. 2 to the case of spin 1 vector models in (2 + 1) dimensions. We will first exploit the equations of motion technique for the fusion of vector models. The results obtained here will be reproduced using other techniques (soldering formalism) leading to fresh insights. We will also discuss the equivalence of the Maxwell-Chern-Simons (MCS) theory with that of the selfdual model from the aspect of soldering as well as from the path integral interpretations. Subsequently the hamiltonian reduction will be executed. A non-trivial canonical mapping will be introduced to decompose the composite hamiltonian into constituting hamiltonians of the net effective theory. An elaborate description of the spin content of the respective vector theories comes from the analysis of the energy momentum tensor. The calculation of the spin of the excitations follows by taking not just the angular momentum operator, but by including also boosts.

In Chap. 4 we have considered models involving higher order derivative of Abelian CS-term in (2+1) dimensions, specially the leading third order derivative Chern-Simons term). Inclusion of this term with usual Maxwell term or with CS term or to both of these terms reveals many interesting observations. For example polarisation vectors in these models possess an identical structure with corresponding expressions for usual Maxwell-CS, Proca or the MCS-Proca model. We know that higher derivative model poses problem in hamiltonian formulation. So we have considered only the Maxwell-Third order Chern-Simons model for convenience. Due to the presence of third order time derivative term the hamiltonian formulation is very tricky and proper care has been taken to identify appropriate canonical pairs. These issues were bypassed by adopting Ostrogradski’s formalism for higher order lagrangian and successively constructing the momentum as well as hamiltonian. We have also illustrated the constrained features of the model and computed relevant
Dirac brackets.

In Chap. 5 emphasis is on the spin 2 tensor models which appear in discussions [8, 25] of linearised gravity in (2 + 1) dimensions. Taking a doublet of self dual massive spin 2 models considered earlier in [26], we show that the effective theory is a new type of generalised self dual model that has a Fierz-Pauli term, a first order Chern-Simons term and the Einstein-Hilbert term. Subject to a specific condition, it reduces to the model taken in [27].

General conclusions and final remarks are relegated to Chap. 6.