Chapter 6

Conclusions

The present analysis depicts the important role of symmetry in understanding various models in odd dimensions. The dual nature of symmetry manifested in (left-right) chirality or (anti)self duality was responsible for the properties of the final theory. We have observed that the quantum mechanical example served as the bedrock from where the more involved examples of field theory and gravity were studied. More specifically, the similarity in the structures of the quantum mechanical model and the other models in field theory/gravity naturally suggested this possibility of dual composition. We have also discussed the factorisability of equations of motion of different models. Such a phenomenon illuminates the dual composition of the models. Specially in case of gravity, this factorisation is possible subject to certain conditions following from the equation of motion.

In Chapter 2 we started with the basic oscillator model in two dimensions which was shown to be composed of two chiral oscillators moving in opposite directions. Chirality gets hidden in an ordinary two dimensional oscillator since the opposing effects of chirality in its constituent pieces are cancelled. There were two approaches to visualize this doublet structures of a composite theory—one based on the lagrangian formulation and the other is canonical formalism or the hamiltonian analysis. The first approach was through soldering mechanism which was demonstrated in Sec.(2.2).
In this method the distinct lagrangians were combined through a contact term. In Sec.(2.3) we considered topological quantum mechanical model – such an example was the generalised Landau problem with steady electric and magnetic fields. Initially we considered a particle model describing motion in steady magnetic field but in the absence of electric field. Implementation of the soldering method on the doublet of such model produced a bi-dimensional harmonic oscillator. Expectedly the initial chiral symmetry of the primary models got hidden. Subsequently in Sec.(2.3.1) a hamiltonian analysis was performed. Using canonical transformation the hamiltonian was diagonalized into independent pieces corresponding to the individual hamiltonian of chiral models. Thus these two results actually were complementary with each other.

The soldering formalism eventually becomes technically involved requiring arcane field redefinitions. So before going into the intricacy of the analysis we have introduced a method in Sec.(2.3.2) based on equations of motion necessitating very simple field redefinitions and generic to a wide variety of models. We have considered a variant of the above quantum mechanical model with distinct frequencies and illustrated the features of this process in detail. We have also discussed the factorisability property of the final equation of motion emphasising the chiral nature latent in it. Either of these above techniques led to a new composite model without having symmetry of the basic primary models.

It is now well known that the measurement of space time coordinates at small scale involves unavoidable effects of quantum gravity. This effect can be incorporated in a physical theory by making the space time coordinates non-commutative. We have discussed such effects in Sec.(2.4). The representations of Galilean generators were constructed on a space where both position and momentum coordinates were non-commutative operators. A simple dynamical model invariant under non commutative(NC) phase space transformations was constructed. Analysing the model via Dirac brackets reproduced the original NC algebra. Also the generators in terms of NC phase space variables were abstracted in a consistent manner. Finally the role
of Jacobi identities was emphasised to produce the noncommutating structure that usually occurs when an electron is subjected to a constant magnetic field and Berry curvature.

It is well known that models in quantum mechanics can could be interpreted as a field theory in (0+1)dimension and also the result would serve as precursor to genuine field theory in (2+1)dimension. Chapter.3 provides the complete correspondence between those quantum mechanical models and self dual field theoretical models in odd dimensions. The analysis with respect to quantum models in (0+1)dimension was directly extended to (2+1)dimensional vector field theory. It was observed that all the results and interpretations found in the quantum mechanical examples had the exact analogues in the corresponding field theory. In Sec.(3.1) we studied self(anti-self)dual doublet which were analogues of left(right)chiral oscillators. Here the topological mass parameters were taken to be distinct and with opposite sign ± signifying spin ±1 for vector fields. For generalisation of the model the source term was also included. Following our trodden path we approached the analysis first by the equation of motion technique. Similar field redefinitions and specific algebraic steps led to the final equation of motion corresponding to the effective form of Maxwell-Chern Simons-Proca model. Self dual or anti selfdual symmetry which got hidden in the M-CS-Proca model actually did manifest in the final factored form of the equation of motion. It was reassuring to note that the Proca model was reproduced in the case where the masses were identical. This finding was reported earlier in different article.

The factorisation of equation of motion of this final composite M-CS-Proca model revealed two distinct mass modes with two degrees of freedom as known earlier. On the other hand computations of the correlation functions with respect to vector fields in the final model manifested the contrast between the true selfdual nature of fields of the basic models and of the M-CS-Proca model. We concluded the lagrangian formulations by implementing the soldering mechanism in Sec.(3.2).
So far we discussed about the self dual (anti selfdual) models. But the equivalence between the SD and MCS model is well known. We have reviewed this feature in Sec.(3.3) first by implementing the soldering method. We have illustrated briefly how a MCS doublet with distinct mass parameters culminated in the required M-CS-Proca theory. Since the result was similar to that obtained via SD doublet so from this perspective an equivalence is established. Alternately in Sec.(3.4) this equivalence was interpreted from path integral approach. A generating functional was constructed for the doublet of SD and ASD models yielding M-CS-Proca theory. Similar analysis was then carried out for the doublet of MCS model. 

In Sec.(3.5) we have investigated in details the hamiltonian form of this M-CS-Proca vector model. Illustrations of the constraints and the Dirac brackets of the phase space coordinates had been explored. On the other hand a suitable canonical mapping enabled us to decouple the composite hamiltonian into its constituent pieces $H_{\pm}$. In Sec.(3.6) an elaborate study of Energy-Momentum Tensor ensured the the spin components of the vector fields in the final model to be $\pm 1$ depending on the sign of the mass term.

So far we discussed the first order abelian topological Chern-Simons (CS)term. But the coupling of higher derivative order (here third order derivative)of CS term with either usual Maxwell term or ordinary CS term or with both of these terms suggests interesting features. We have investigated in Chapter.4 such models in $(2+1)$dimensions. The polarisation vectors in these models unveiled an identical structure with the corresponding expressions for usual models which contain at most quadratic structures. We also studied hamiltonian structure of these models and revealed how Wigner’s Little group acted as a gauge generator.

In various chapters we exploited dual descriptions where a particular theory is interpreted as a combination or a doublet of theories. A typical illustration is the Proca model in $(2+1)$ dimensions. The two massive modes of this model were known to be obtained from a doublet of self-dual models with helicity $\pm 1$. In Chapter.5 we exploited similar notions and concepts to study a new version of topologically
massive gravity in (2+1)dimensions. The propositions made for spin-1 vector models in (2+1)dimensions were extended to this part. We considered first order self dual massive spin-2 tensor model in terms of tensor fields like $f_{\alpha\beta}$ with no symmetry with respect to their indices. We investigated specially the combination of a doublet of spin $\pm 2$ models that arise in linearised gravity. To avoid repetition we have chosen only the method based on equations of motion to analyse these doublet. In the first part, we analysed identical mass parameters $m$ where replacing $m$ by $-m$ implied helicity change from $+2$ to $-2$. Strikingly here also the generic field redefinitions worked properly.

Systematic steps combined the doublets into an effective model containing the quadratic Einstein-Hilbert term with the Pauli-Fierz mass term applicable for spin-2 particle which could be easily recognised as an analogue of Proca model for spin-1 case in vector theory. Similar calculations were repeated by considering doublets with distinct mass parameters $m_1$ and $m_2$. The final equation of motion yielded the composite model having - an Einstein-Hilbert term, a Fierz-Pauli mass term and interestingly a generalised first order Chern-Simons term. This CS-term contained, apart from the standard CS term, two other similar terms with a different orientation of indices. Thus the action for the effective model with distinct mass parameters could be interpreted as an analogue of Maxwell-Chern-Simons-Proca theory for the spin-1 particle.

Let us visualise in general terms the obtention of a new theory from a combination of chiral ones. Chiral theories occur in doublets corresponding to the left and right degrees of freedom. We may take the equation of motion approach as symbolic and try to explain this point. The equations of motion following from a doublet are form invariant, differing only by a sign in the chiral piece. Adding and subtracting these equations naturally leads to a combination which is either a sum or a difference of the original variables. Renaming this 'sum' and 'difference' as new fields yields a pair of coupled differential equations. It is then possible to eliminate one of these new fields in favour of the other using these differential equations. The final outcome is
an equation of motion involving only the new fields. Furthermore, the symmetrical treatment implies that we obtain identical equations of motion for both the new fields. Consequently we are led to a unique new theory obtained by a composition of the chiral degrees of freedom.

The other approaches are more sophisticated leading to fresh insights, nevertheless this basic idea runs as a common string.