Chapter 5

Extrapolation to Gravity

Duality is a fascinating symmetry which keeps appearing in many contexts. It was originally developed for electromagnetism, where duality invariance of the Maxwell equations leads to the introduction of magnetic sources and the quantization of electric charge. It also has been at the origin of many remarkable developments in Yang-Mills theory. More recently duality has revolutionised the understanding of string theory by providing non-perturbative insight. These latter developments indicate that duality should play a central role in gravitational theory as well. Three dimensional higher derivative theories of gravity have received considerable attention over the years. The first example of such a higher derivative theory is topologically massive gravity (TMG) model. The TMG lagrangian consists of the usual Einstein-Hilbert term, which by itself does not describe any degrees of freedom in three dimensions, a Lorentz Chern-Simons (LCS) term which is parity odd and third order in the derivatives. The two terms together describe a single massive state of helicity +2 or −2, depending on the relative sign between the EH and LCS terms. Last couple of years there has been intense activity on this subject of higher spin theories in different dimensions and their dual formations.

In this part of our work we will consider first order self dual model suggested in [26] which is the helicity +2 analogue of the helicity +1 self dual model in D=2+1. Drawing analogy with the spin-1 vector models, here we shall implement the notions
developed in the previous chapters to illuminate different features of such rank two tensor models which arise in linearised approximation of gravity models. As in Chap. 3 where we illustrated the combination of a doublet of self-dual models with distinct masses and spins ±1 to yield an effective Maxwell-Chern-Simons-Proca model, here we will consider the combination of such SD doublet of helicity ±2 tensor gravity models. Due to the involved algebra we have opted only the method based on the equation of motions technique which is a completely lagrangian formalism.

In Sec.5.1 we will investigate the self dual massive spin-2 tensor model including the source term with identical mass parameters but with opposite sign (signifies opposite spin). We will discuss the relevant factorisation properties. Sec.(5.2) revisits this similar calculation but with the distinct mass parameters. For sake of simplicity we will drop the source terms in this respective analysis. Section ends with the conclusion.

5.1 Spin-2 self dual tensor model with same mass

Following similar procedure, let us start with the action of first order self dual massive spin 2 model as suggested in [26]

\[ S = \int d^3x \left( \frac{m}{2} e^{\mu \lambda} f_{\mu}^{\alpha} \partial_{\nu} f_{\lambda \alpha} + \frac{m^2}{2} (f^{2} - f_{\mu \nu} f^{\mu \nu}) \right) \]  

(5.1)

where \( f = \eta^{\mu \nu} f_{\mu \nu} \). The metric is flat \( :\eta^{\mu \nu} = \text{diag} (-,+,+) \). In this model we use second rank tensor fields, like \( f_{\alpha \beta} \) with no symmetry with respect to their indices. Replacing \( m \) by \( -m \) in (5.1) implies helicity change from +2 to −2. The first term in (5.1) is reminiscent of a spin one topological Chern-Simons term which will be called a Chern-Simons term of first order. The second term in (5.1) is the Fierz-Pauli(FP) mass term [60] which is the spin two analogue of a spin one Proca mass term. Note that FP term breaks the local invariance of the Chern-Simons term. The above first order self dual massive spin-2 field action can be easily found after writing topologically massive gravity in an intrinsically geometric form language and then linearizing it [61].
Let us then consider the following doublet of first order lagrangian densities in presence of source terms,

\[ \mathcal{L}(f) = \frac{m}{2} \epsilon^{\mu\nu\lambda} f_\mu \partial_\nu f_\lambda + \frac{m^2}{2} (f^2 - f_{\mu\nu} f^{\mu\nu}) - \frac{m}{2} f_{\mu\nu} J^{\mu\nu} \]  (5.2)

\[ \mathcal{L}(g) = -\frac{m}{2} \epsilon^{\mu\nu\lambda} g_\mu \partial_\nu g_\lambda + \frac{m^2}{2} (g^2 - g_{\mu\nu} g^{\mu\nu}) + \frac{m}{2} g_{\mu\nu} J^{\mu\nu} \]  (5.3)

where \( f_{\mu\nu} \) and \( g_{\mu\nu} \) are distinct fields. Note that although the helicites are ±2, the mass term is identical for both lagrangian (5.2) and (5.3). The case of different masses will be dealt in the next section. Now the equations of motion are given by,

\[ \epsilon_\mu \nu \lambda \partial_\nu f_\lambda + m (f_{\eta\mu\alpha} - f_{\alpha\mu}) = \frac{1}{2} J_{\alpha\mu} \]  (5.4)

\[ \epsilon_\mu \nu \lambda \partial_\nu g_\lambda - m (g_{\eta\mu\alpha} - g_{\alpha\mu}) = \frac{1}{2} J_{\alpha\mu} \]  (5.5)

Following our previous approach, let us introduce new fields \( F \) and \( G \) as,

\[ F_{\mu\alpha} = f_{\mu\alpha} + g_{\mu\alpha}; \quad G_{\mu\alpha} = f_{\mu\alpha} - g_{\mu\alpha}; \]

\[ F = \eta^{\mu\alpha} F_{\mu\alpha} = f + g; \quad G = \eta^{\mu\alpha} G_{\mu\alpha} = f - g \]  (5.6)

Now adding (5.4) and (5.5) and substituting old fields by new ones defined in (5.6) leads to

\[ \epsilon_\mu \nu \lambda \partial_\nu F_\lambda - m (G_{\mu\alpha} - \eta_{\alpha\mu} G) = J_{\alpha\mu} \]  (5.7)

Our motivation now is to express the above equation solely in terms of the G-field. To achieve this we abstract certain results from (5.4).

- Contraction by \( \eta^\alpha_\mu \) of (5.4) yields

\[ \epsilon^{\alpha\nu\lambda} \partial_\nu f_\lambda + 2mf = \frac{1}{2} J; \quad J = \eta^{\alpha\alpha} J_{\mu\alpha} \]  (5.8)

- Contraction by \( \epsilon^{\mu\alpha\rho} \) of (5.4) leads to

\[ -\partial_\alpha f^{\rho\alpha} + \partial^{\rho} f - m \epsilon^{\mu\alpha\rho} f_{\alpha\mu} = \frac{1}{2} \epsilon^{\mu\alpha\rho} J_{\mu\alpha} \]  (5.9)

- Operating (5.4) by \( \partial^\mu \) on both sides gives

\[ -m (\partial^\mu f_{\alpha\mu} - \partial_\alpha f) = \frac{1}{2} \partial^\mu J_{\alpha\mu} \]  (5.10)
Taking the difference of (5.4) and (5.5) and exploiting (5.6) leads to
\[ \varepsilon_\mu^{\nu\lambda} \partial_\nu G_{\lambda\alpha} - m (F_{\alpha\mu} - \eta_{\alpha\mu} F) = 0 \] (5.11)

Taking the trace yields the identity,
\[ F = -\frac{1}{2m} \varepsilon^{\mu\nu\lambda} \partial_\mu G_{\lambda\nu} \] (5.12)

Using (5.12) in (5.11) we obtain
\[ F_{\alpha\mu} = \frac{1}{m} \varepsilon_\mu^{\nu\lambda} \partial_\nu G_{\lambda\alpha} - \frac{1}{2m} \eta_{\alpha\mu} \varepsilon^{\rho\sigma\omega} \partial_\rho G_{\omega\sigma} \] (5.13)

Now substituting \( F_{\alpha\mu} \) in (5.7) gives,
\[ \frac{1}{m} \varepsilon_\mu^{\nu\lambda} \partial_\nu [\varepsilon_\sigma^{\rho\sigma} \partial_\rho G_{\sigma\lambda} - \frac{1}{2} \eta_{\lambda\sigma} \varepsilon^{\rho\omega\eta} \partial_\sigma G_{\omega\rho}] - m [G_{\alpha\mu} - \eta_{\alpha\mu} G] = J_{\alpha\mu} \] (5.14)

Combining (5.8) and (5.9) we obtain,
\[ \varepsilon^{\mu\alpha} \rho J_{\alpha\mu} = \frac{1}{2m^2} \partial_\mu J_{\rho\mu} - \frac{1}{2m} \varepsilon^{\mu\alpha} \rho J_{\alpha\mu} \] (5.15)

The corresponding equation for \( g \) follows by replacing \( m \) by \(-m\),
\[ \varepsilon^{\mu\alpha} \rho g_{\alpha\mu} = \frac{1}{2m^2} \partial_\mu J_{\rho\mu} + \frac{1}{2m} \varepsilon^{\mu\alpha} \rho J_{\alpha\mu} \] (5.16)

Subtracting (5.16) from (5.15) yields
\[ \varepsilon^{\mu\rho} G_{\alpha\mu} = -\frac{1}{m} \varepsilon^{\mu\rho} J_{\alpha\mu} \] (5.17)

Therefore from (5.17) we can conclude,
\[ \varepsilon^{\mu\rho} \partial_\rho G_{\alpha\mu} = -\frac{1}{m} \varepsilon^{\mu\rho} \partial_\rho J_{\alpha\mu} \] (5.18)
\[ G_{\alpha\mu} - G_{\mu\alpha} = -\frac{1}{m} (J_{\alpha\mu} - J_{\mu\alpha}) \] (5.19)

Substituting (5.18) in (5.14) we get
\[ \varepsilon_\mu^{\nu\lambda} \partial_\nu [\varepsilon_\sigma^{\rho\sigma} \partial_\rho G_{\sigma\lambda}] - m^2 [G_{\alpha\mu} - \eta_{\alpha\mu} G] = -\frac{1}{2m} \varepsilon^{\mu\sigma\rho} \partial_\rho J_{\alpha\mu} + m J_{\alpha\mu} \] (5.20)
The symmetrised version of the above equation reads,
\[ \epsilon_\mu^{\nu\lambda}\partial_\nu[\epsilon_\alpha^{\rho\sigma}\partial_\rho G_{\sigma\lambda}] + \epsilon_\alpha^{\nu\lambda}\partial_\nu[\epsilon_\mu^{\rho\sigma}\partial_\rho G_{\sigma\lambda}] - m^2[G_{\alpha\mu} + G_{\mu\alpha}] \\
+ 2m^2\eta_{\alpha\mu}\bar{G} = m(J_{\alpha\mu} + J_{\mu\alpha}) \] (5.21)

Exploiting (5.19) we obtain the final effective equation of motion,
\[ \frac{1}{2}\epsilon_\mu^{\nu\lambda}\partial_\nu[\epsilon_\alpha^{\rho\sigma}\partial_\rho(G_{\sigma\lambda} + G_{\lambda\sigma})] - m^2[G_{\alpha\mu} - \eta_{\alpha\mu}\bar{G}] = mJ_{\mu\alpha} \] (5.22)

Let us now discuss, in the absence of sources, the factorisability of the equations of motion. Some conditions on the tensor field are necessary to achieve this factorisation.

It is known [58] from a study of the equations of motion of (5.1) that the tensor field \( f_{\mu\nu} \) satisfies a) tracelessness \( f_{\mu\nu} = 0 \), b) transversality \( \partial^\mu f_{\mu\nu} = 0 \) and c) symmetricity \( f_{\mu\nu} = f_{\nu\mu} \). Consequently the composite fields \( F_{\mu\nu}, G_{\mu\nu} \) in (5.6) should also satisfy these properties. Indeed one may also verify this directly from (5.22), in the absence of sources. Under these conditions (5.22) factorises as,
\[ (-\epsilon^{\mu\nu\rho}\partial_\rho + m\eta^{\mu\nu})(\epsilon_\mu^{\nu\lambda}\partial_\nu - m\eta_\mu^{\lambda})G_{\lambda\alpha} = 0 \] (5.23)

We observe that the above equation of motion (5.22) corresponds to an effective theory whose action is given by
\[ S = \int d^2x \frac{1}{4} G.d\Omega(G) + \frac{m^2}{2}(G^2 - G_{\mu\nu}G^{\mu\nu}) - \frac{1}{2}mG_{\mu\nu}J^{\mu\nu} \] (5.24)

where
\[ G.d\Omega(G) = G^{\alpha\mu}\epsilon_\mu^{\nu\lambda}\partial_\nu[\epsilon_\alpha^{\rho\sigma}\partial_\rho(G_{\sigma\lambda} + G_{\lambda\sigma})] \]

Note that the first term in the action (5.24) stands for the quadratic Einstein-Hilbert term while the second one is the Pauli-Fierz mass term applicable for spin 2 particle. In the absence of source this action corresponds to an effective theory which is the analogue of Proca model for spin-1 case in vector theory.

Proceeding in a likewise manner the equation of motion for the \( F \)-field emerges as,
5.2 Tensor fields with distinct mass

In this section we repeat the analysis for the doublet (5.2) and (5.3) but with distinct mass parameters. To avoid technical complications we drop the source terms. We show that combining this doublet yields an effective theory that has an E-H term, a FP mass term and a generalised first order CS term. This CS term contains, apart from the standard form given in (5.1), two other similar terms with a different orientation of indices. Consider therefore the lagrangian densities,

\[ L_+(f) = \frac{m_1}{2} \epsilon^{\mu \lambda} f_{\mu} \alpha \partial_{\nu} f_{\lambda \alpha} + \frac{m_1^2}{2} (f^2 - f_{\mu} f_{\nu} f_{\mu} f_{\nu}) \]  
\[ L_- (g) = -\frac{m_2}{2} \epsilon^{\mu \lambda} g_{\mu} \alpha \partial_{\nu} g_{\lambda \alpha} + \frac{m_2^2}{2} (g^2 - g_{\mu} g_{\nu} g_{\mu} g_{\nu}) \]  

Now (5.27) and (5.28) yield the equations of motion,

\[ \epsilon_{\mu \nu \lambda} \partial_{\nu} f_{\lambda \alpha} + m_1 (f \eta_{\mu \alpha} - f_{\alpha \mu}) = 0 \]  
\[ \epsilon_{\mu \nu \lambda} \partial_{\nu} g_{\lambda \alpha} - m_2 (g \eta_{\mu \alpha} - g_{\alpha \mu}) = 0 \]
5.2. Tensor fields with distinct mass

Following identical field definitions as (5.6) and analogous steps, it can be shown that the final form of the equation of motion for $G_{\mu\nu}$ is given by,

$$
-\frac{1}{2} \epsilon_{\mu}^{\nu} \partial_{\nu} \left[ \epsilon_{\alpha}^{\rho} \partial_{\rho} \left( G_{\alpha\lambda} + G_{\lambda\alpha} \right) \right] - m_1 m_2 \left( G_{\alpha\mu} - \eta_{\alpha\mu} G \right) + \frac{1}{2} (m_1 - m_2) \left( \epsilon_{\mu}^{\nu} \partial_{\nu} G_{\lambda\alpha} + \epsilon_{\alpha}^{\nu} \partial_{\nu} G_{\lambda\mu} \right) = 0
$$

(5.31)

A similar equation of motion is also obtained for the other variable $F_{\mu\nu}$.

The action from which the above equation of motion (5.31) follows is given by,

$$
S = \int d^3x \left\{ m_1 m_2 \left( G^2 - G_{\mu\nu} G^{\mu\nu} \right) + \frac{1}{4} G.d\Omega(G) + \frac{1}{8} (m_2 - m_1) \left( \epsilon_{\mu}^{\nu} \epsilon_{\lambda}^{\alpha} \delta_{\nu} G_{\lambda\alpha} + \epsilon_{\mu}^{\nu} \epsilon_{\alpha}^{\lambda} \delta_{\nu} G_{\lambda\mu} + 2 \epsilon_{\mu}^{\nu} \epsilon_{\alpha}^{\lambda} \partial_{\nu} G_{\lambda\alpha} \right) \right\}
$$

(5.32)

We have thus successfully combined different mass terms in the spin 2 case to yield the action (5.32) of the effective theory. While the first two terms are the usual FP and E-H terms the last piece, which is a consequence of different masses, is a generalised form of the CS term. As announced earlier it has, apart from the usual structure, two other pieces that may be obtained from a reorientation of indices. In fact it has all possible orientations of indices leading to a first order Chern-Simons term. Furthermore if we impose a condition of symmetricity $G^{\mu\alpha} = G^{\alpha\mu}$, then all pieces become identical and the standard first order C-S term with a coefficient $\frac{1}{2} (m_2 - m_1)$ is obtained. The first term in (5.32) is the Fierz-Pauli(FP) mass term with mass co-efficient $m = \sqrt{m_1 m_2}$. The second term involves the usual kinetic term (defined in the previous section) which is equivalent to linearised Einstein-Hilbert(EH) term upto quadratic order. Thus the action (5.32) for spin-2 particle may be interpreted as an analogue of Maxwell-CS-Proca model for spin-1 particle. Incidentally the C-S term for the vector case has a unique orientation of indices $\epsilon_{\mu\lambda} f^{\mu} \partial^{\nu} f^{\lambda}$ and any changes are absorbed in a trivial normalisation of signs.