Appendix B

Variance-Vectors

Here we shall give a concept of variance-vectors. To each stochastic variate \( X \), we shall associate an element \( \mathbf{x} \) such that \( \mathbf{x} = \mathbf{v}(X) \) and for two stochastic variates \( X_i \) and \( X_j \), \( (\mathbf{x}_i \cdot \mathbf{x}_j) = C_{ij}(X_i, X_j) \). For the present we shall call these elements to be variance-vectors and later on shall show that they follow the properties of vectors under certain assumptions. We shall also see what are meant by \( \mathbf{x}_c + \mathbf{x}_j \) where the latter is the product of a constant \( c \) with \( \mathbf{x}_c \).

We shall assume that \( \mathbf{x}_c + \mathbf{x}_j = \mathbf{x}_c + \mathbf{x}_j \) and \( c(\mathbf{x}_c + \mathbf{x}_j) = c\mathbf{x}_c + c\mathbf{x}_j \), \( \mathbf{x}_c, \mathbf{x}_j, c = (\mathbf{x}_c + \mathbf{x}_j) \cdot c \), and so on; i.e., the elements follow commutative and associated laws of addition and in case of product with constants follow the 1st and 2nd distributive laws like the vectors. We shall also assume that \( \mathbf{x}_c = \mathbf{x}_j \).

Consistent with these assumptions the following holds:

1. \( c \mathbf{x}_c = (c \mathbf{x}_c) \).
2. \( \mathbf{x}_c + \mathbf{x}_j = \mathbf{x}_c + \mathbf{x}_j \).

Proof: \( \mathbf{v}(X) = c\mathbf{v}(X) = c\mathbf{x}_c \) and \( C_{ij}(X_i, X_j) = C_{ij}(X_i, X_j) = C_{ij}(\mathbf{x}_c, \mathbf{x}_j) = (C_{ij}(\mathbf{x}_c, \mathbf{x}_j)) \).

Thus \( (C_{ij}(\mathbf{x}_c, \mathbf{x}_j)) \) is the variance vector of \( cX_i \); hence the relation (1). Also, \( \mathbf{v}(X_c + X_j) = \mathbf{v}(X_c) + \mathbf{v}(X_j) \) and \( C_{ij}(X_c + X_j) = C_{ij}(X_c, X_j) \).

And \( C_{ij}(X_c + X_j) = C_{ij}(X_c, X_j) + C_{ij}(X_j, X_j) \).

Thus the variance-vector of \( X_c + X_j \) is \( \mathbf{x}_c + \mathbf{x}_j \), hence the relation (2).

From these relations it follows that consistent with the above assumptions, the variance-vector of the linear function of a stochastic variate \( X \) is defined by \( \mathbf{x}_c \).
number of stochastic variates with non-stochastic (or constant) co-efficients is the linear function of their variance-vectors with same co-efficients.

The expression \((\mathbf{x}_i \cdot \mathbf{x}_j)\) will be called the scalar product of the two variance vectors. A vector of which the scalar product with any another vector is zero will be called a null vector. Thus the variance vector of a constant is null-vector. Also, the variance-vector of \(\mathbf{x}_c + \mathbf{c}\) is the variance vector of \(\mathbf{x}_c\). Thus the stochastic variates associated to a variance-vector are differing by a constant. In order to make it unique we shall always have the convention that the constant part should be zero. Two vectors \(\mathbf{x}_i\) and \(\mathbf{x}_j\) are said orthogonal if \((\mathbf{x}_i \cdot \mathbf{x}_j) = 0\), i.e. \(\mathbf{c}_i \cdot (\mathbf{x}_i \cdot \mathbf{x}_j) = 0\). We shall assume that no vector unless it is a null vector is self-orthogonal.

A vector \(\mathbf{x}\) will be said to be dependent on \(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\) if \(\mathbf{x}\) can be expressed as a linear function of \(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\), i.e., \(\mathbf{x} = c_1 \mathbf{x}_1 + c_2 \mathbf{x}_2 + \ldots + c_n \mathbf{x}_n\) where \(c_1, c_2, \ldots, c_n\) are constants. The set of all vectors dependent upon \(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\) will be called a vector-space generated by \(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\) and these vectors will be called the generating vectors of the vector-space. A set of vectors will be called a dependent set if at least one vector of the set is dependent upon others; otherwise, it is an independent set. A null vector is a dependent vector and a set of vectors containing a null vector is a dependent set.

Then we can prove the following results. The necessary and sufficient condition that the variance-vectors \(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\) are independent is that we cannot find constants \(c_1, c_2, \ldots, c_n\) (not simultaneously zero) such that \(c_1 \mathbf{x}_1 + c_2 \mathbf{x}_2 + \ldots + c_n \mathbf{x}_n = \mathbf{0}\), where \(\mathbf{0}\) means the null vector. The vector-space generated by the vectors \(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\) is unchanged (i) if we add a null vector to the set or omit a null vector, if any, from the set, (ii) if we replace, orthogonal to the vector-space, then is called the projection of along , and this projection also is unique. The absolute value of will be called
of the set by where $d = f = 0$ and (iii) if we replace $Z_i$ by $Z_i + c Z_j$, where $Z_i$ and $Z_j$ belong to the set.

These three operations will be called omission or adjunction of a null vector, row multiplication and row addition respectively. We shall call a set of generating vectors of a vector-space to be a basis of the vector-space, if the set is an independent set. Then we can prove that every vector-space $V$ containing non-null vectors has a basis.

The set of vectors in the basis of a vector-space remains independent under the three operations: omission or adjunction, row multiplication and row addition. If $\alpha_1, \alpha_2, \ldots, \alpha_n$ form a basis of $V$ and $\beta_1, \beta_2, \ldots, \beta_m$ form a set of independent vectors belonging to $V$, then they form a basis. The number of independent vectors in any basis of $V$ is constant and is called the rank of $V$. If every vector of a vector-space $V'$ belongs to a vector-space $V$, then $V'$ is said to be a subspace of $V$ and rank $V \geq$ rank $V'$. If $S$ is a set of vectors of which the maximum number of independent vectors is finite and if the vectors of $S$ has got the property that the sum of any two arbitrary vectors of $S$ belongs to $S$, the product of a vector of $S$ with a constant $C(\neq 0)$ belongs to $S$ and $S$ contains a null vector then $S$ is a vector-space.

We have also defined the scalar product and orthogonality of two variance vectors. Now, if a variance-vector is orthogonal to every vector of a vector-space, it is said to be orthogonal to the vector-space.

Thus if a variance-vector is orthogonal to each of the generating vectors of a vector-space, it is orthogonal to the vector-space. If a vector $Z_i$ can be expressed as $Z_i = c Z_1 + Z_2$, where $Z_1$ is orthogonal to $Z_2$, then $c Z_1$ is said to be the projection of $Z_i$ along $Z_2$; and this projection is unique. Similarly, if $Z_i = Z_1 + Z_2$, where $Z_1$ lies in a vector-space $V$ and $Z_2$ is orthogonal to the vector-space $V$, then $Z_1$ is called the projection of $Z_i$ along $V$; and this projection also is unique. The absolute value of $Z_i^2$ will be called the length of $Z_i$ (i.e. the standard deviation of $Z_i$).
Now, we shall show that every variance-vector belonging to a ordered set of elements. Because, in that vector-space of finite rank can be expressed as an ordered set of mutually orthogonal vectors, each of unit length and can express any vector of that vector-space as a linear function of these vectors and the vector can be expressed as an ordered set of co-efficients of these vectors occurring in their linear function giving .

Thus the variance-vectors belong to a vector-space of finite rank follow all the laws of ordinary vectors.

We shall now consider some problems of correlation. If be a stochastic variate and is a set of stochastic variates and if be a linear function of the variates of such that where the correlation between and any variate of is zero, then the correlation between and will be called the multiple correlation of and the variates of . If and are two stochastic variates and is a set of stochastic variates and if and are linear functions of the variates of and are non-correlated with any variate of , then the correlation between and is said to be the partial correlation between and after eliminating the effect of the variates of . If and are two sets of variates and and two linear functions of the variates of and respectively such that where and is non-correlated with any variate of and is non-correlated with any variate of , then the correlation between and will be called the canonical correlation between and . In this way, we can define multiple and partial canonical correlations.

All these correlations can then easily be expressed in terms of variance-vectors.