CHAPTER VII

I. General Theory of Analysis of Co-variance

For the analysis of covariance in the case of one or more concomitant variates what is generally done is that the sum of squares due to treatment contrasts plus error and that due to error only are found out separately, each being properly corrected for the concomitant variates; then the latter sum of squares is subtracted from the former to get the correct sum of squares due to treatment-contrasts. But to test for a number of individual treatment contrasts, it is better to suggest a general form of their variances, which, of course, follows from an extension of the Markoff's theorem (Rao, 1946). This paper is rather an applied one to the case of two-way heterogeneity.

Here the method of linear estimation as suggested by R. C. Bose in his class-lectures and in Bose (1946) will be followed, a short account of which is given below.

Let \( \gamma_1, \gamma_2, \ldots, \gamma_n \) be independent stochastic variates with common variance \( \sigma^2 \) such that

\[
E(\gamma_i) = \alpha_{i1} \beta_1 + \alpha_{i2} \beta_2 + \cdots + \alpha_{i\lambda} \beta_\lambda;
\]

where \( \alpha_{ij} \) are known and \( \beta_{ij} \) are unknown parameters, or

\[
E(\gamma) = \beta_1 \alpha_{11} + \beta_2 \alpha_{21} + \cdots + \beta_\lambda \alpha_{\lambda 1} \]

where \( \beta_1 = (\beta_{11}, \beta_{12}, \ldots, \beta_{1\lambda}) \) and \( \alpha = (\alpha_{11}, \alpha_{12}, \ldots, \alpha_{1\lambda}) \) the stochastic vector. Then any linear function \( Y = \mathbf{H}^T \mathbf{\alpha} \) where \( \mathbf{H} = (h_{ij}) \) and \( \mathbf{\alpha} = (\alpha_{11}, \alpha_{12}, \ldots, \alpha_{1\lambda}) \) means the scalar product of the two vectors, i.e.

\[
(Y, \gamma) = \gamma_1 \alpha_{11} + \gamma_2 \alpha_{12} + \cdots + \gamma_\lambda \alpha_{1\lambda},
\]

is called an unbiased linear estimate of any parametric function \( \tau = \sum_i \beta_i \alpha_{ij} \) where \( \beta_i, \alpha_{ij} \) being known, if

\[
E(Y) = \tau,
\]

independently of the parameters \( \beta_{ij} \). Thus the necessary and sufficient condition for \( \tau \) being estimable, i.e., having an unbiased estimate, is that the equation \( (\beta_j, \gamma) = 0, \gamma = (\gamma_1, \gamma_2, \ldots, \gamma_\lambda) \) is solvable. Thus in general there may be an infinity of unbiased linear estimates \( \gamma' \) of \( \tau \), of which the one having the minimum variance may be said to be the best estimate.

A linear function \( Y' = (Y, \gamma) \) is said to belong to error, if

\[
E(Y') = 0,
\]

independently of the parameters. Thus the necessary and sufficient condition for \( Y' \) belonging to error is that \( (Y', \gamma_j) = 0, j = 1, 2, \ldots, \lambda \); i.e., \( Y' \) is orthogonal to the vector space generated by \( \gamma_1, \gamma_2, \ldots, \gamma_\lambda \), called the estimation space. The space completely orthogonal to it, is called
the error-space. Thus for any \( \gamma' \) belonging to error-space, the linear function represented by it, viz. \( (\gamma', \gamma) \), belongs to error. Thus it follows that \( E(\gamma', \gamma) = E(\gamma', \gamma_0) \), where \( \gamma_0 \) is the projection of \( \gamma' \) along the estimation-space, and it can also be proved that there is a unique vector \( \gamma_0 \) in the estimation-space such, for a given \( \gamma' \), \( E(\gamma', \gamma) \equiv \pi \).

So the best linear estimate of \( \gamma' \) is \( \gamma_0 = \sum \beta_j \gamma_j \) such that \( E(\gamma') = \pi \), \( \sum \beta_j \gamma_j = \gamma_0, \gamma_1, \ldots, \gamma_j \), are the equations for the solution of \( \gamma_j \)'s. It can be shown that if \( \hat{\gamma} = \sum \beta_j \hat{\gamma}_j \) where \( \hat{\gamma}_j \) are solutions of the normal equations \( \sum \hat{\gamma}_j \gamma_j = \beta_j \gamma_j, j = 1, 2, \ldots, n \), then \( \hat{\gamma} = \gamma \) identically. Then \( \gamma(\gamma') = \sigma^2 \sum \hat{\gamma}_j \gamma_j \) be the solution of the normal equations. If there is an one to one correspondence between the parametric functions and their best estimates, the degrees of freedom carried by a set of the former are the same as those carried by their best estimates.

The sum of squares due to the degree of freedom carried by a linear function \( \gamma = (\gamma', \gamma) \) is \( \frac{\gamma^T \gamma}{\hat{\gamma}^T \hat{\gamma}} \) which is the square of the projection of \( \gamma \) on \( \gamma' \). Thus the sum of squares due to the degrees of freedom carried by a set of linear functions of \( \gamma_j \) is the square of the projection of \( \gamma \) on the vector-space generated by the co-efficient vectors of the linear functions. Then the expectation of the per degree sum of squares corresponding to the linear functions belonging to error is \( \sigma^2 \). The sum of squares corresponding to the estimates of all possible estimable parametric functions is \( \sum \hat{\gamma}_j \gamma_j \) called the sum of squares due to estimates, \( \hat{\gamma}_j \gamma_j \) being the solutions of the normal equations. Hence that due to error is \( \sum \hat{\gamma}_j \gamma_j \). The hypothesis of the simultaneous disappearance of a number of parametric functions is tested by \( F = \frac{\hat{\gamma} \gamma^2 / \hat{\gamma} \gamma^2}{\hat{\gamma} \gamma^2 / \hat{\gamma} \gamma^2} \), where \( \hat{\gamma} \gamma^2 \) is the sum of squares due to their best estimates and \( \hat{\gamma} \gamma^2 \) that due to error, and \( \gamma, \gamma \) are the respective degrees of freedom. This \( F \) follows \( F \)-distribution.
A treatment and a block are said to be associated, if the treatment is contained in the block. Two treatments, two blocks, or a treatment and a block are said to be connected, if it is possible to pass from one to the other by means of a chain of alternate blocks and treatments such that any two consecutive members are associated. If the treatments and blocks of a design are all connected, it is said to be a connected design. For a connected design any treatment contrast is estimable.

Section 1.

Now let us consider a connected design with \( T \) treatments in \( U \) blocks applied in such a manner that \( n_{ij} \) is the number of plots in the \( j \)th block to which the \( j \)th treatment has been applied. Let \( N_j \) be the total number of plots in the \( j \)th block and \( n_{ij} \) the replication of the \( j \)th treatment. Then \[ \sum_{j=1}^{U} n_{ij} = N_j, \quad \sum_{i=1}^{T} n_{ij} = n_{ij} \] and let \( N \) be the total number of plots in the design. Now consider that there are \( K \) concomitant variates acting in each plot of the design. Let \( y_{ij}^{(t)} \) be the observed variate and \( x_{ij}^{(t)} \) the corresponding \( p \) concomitant variate in some one of the plots of the \( j \)th block to which the \( j \)th treatment has been applied, where \( t = 1, 2, \ldots, K \) and \( j = 1, 2, \ldots, T \). Assume that \( y_{ij}^{(t)} \) are mutually independent and have got the same variance \( \sigma^2 \) and that

\[ E(y_{ij}^{(t)}) = d_{ij} + t_{ij} + \sum_{k=1}^{K} \beta_{ik} x_{ij}^{(k)} \]

where \( d_{ij} \) is the effect of the \( i \)th block, \( t_{ij} \) that due to \( j \)th treatment and \( \beta_{ik} \) the regression coefficient of the \( k \)th concomitant variate. It can also be written vectorially as,

\[ E(\mathbf{y}) = \mathbf{1}_{ij} \mathbf{d}_{ij} + \sum_{k=1}^{K} \mathbf{1}_{ik} \beta_{ik} \mathbf{x}_{ij}^{(k)} \]

where \( \mathbf{y}_{ij} \) is the vector \((y_{ij}^{(1)}, y_{ij}^{(2)}, \ldots, y_{ij}^{(K)})\), \( \mathbf{1}_{ij} \) the observation-vector \( y_{ij}^{(t)}, t = 1, 2, \ldots, T \), \( \mathbf{x}_{ij} \) is a vector of which an element is unity if the corresponding element of \( \mathbf{1} \) contains the \( j \)th treatment, otherwise it is zero; and \( \beta_{ik} \) is a vector of which an element is unity if the corresponding element of \( \mathbf{1} \) is of the \( i \)th block, otherwise it is zero.
Then the following results can easily be proved:

\[ \sum_{i=1}^{u} \beta_i = \sum_{j=1}^{v} \gamma_j, \quad (\beta_i, \beta_j) = \gamma_i \delta_{ij}, \quad (\gamma_j, \gamma_j') = \gamma_j' \delta_{jj'}, \quad \beta_i, \gamma_j \text{ and } \gamma_j' \text{ are the components of } E_i, \gamma_j \text{ and } \gamma_j', \quad \cdots \cdots \quad (7.1) \]

where \( \delta_{kk} = 1 \) and \( \delta_{kk'} = 0 \) for \( k \neq k' \).

Let \( E_i = (\beta_i, \gamma_i), \quad E_i' = (\beta_i, \gamma_i'), \quad \gamma_i = (\gamma_j, \gamma_i), \quad \gamma_i' = (\gamma_j', \gamma_i), \quad E_i = (\gamma_j, \gamma_i) \), \( \omega_{ii} = (\gamma_i, \gamma_i) \) and \( \omega_{ij} = (\gamma_i, \gamma_j) \), \( \cdots \cdots \quad (7.2) \).

The \( E_i \) is the total yield of the observed variates of all the plots in the \( i \)th block, \( \gamma_i \) that of the plots containing \( j \)th treatment and \( E_i, \gamma_i \) are the corresponding figures of the \( j \)th concomitant variate; \( \omega_{ii} \) and \( \omega_{ij} \) are the sums of products.

From (7.0) it is evident that the estimation-space, as defined in the Introduction, is generated by \( \gamma_i, \gamma_i', \gamma_j, \beta_1, \beta_2, \beta_3, \gamma_j, \gamma_j', \gamma_j'' \) \( \gamma_i \).

So, if \( \gamma = (\gamma_i) \) be the best unbiased linear estimate of any linear function of \( \beta_1, \beta_2, \beta_3 \) only, then \( \gamma \) should lie in this space and \( E(\gamma_i) \geq \gamma_i \) \( \pi \) being the linear function; \( \sum_{i=1}^{u} \chi_i \).

But \( E(\gamma_i) = \sum_{i=1}^{u} t_i (\gamma_i, \gamma_i) + \sum_{i=1}^{u} d_i (\beta_i, \gamma_i) + \sum_{i=1}^{u} e_i (\gamma_i, \gamma_i) \),

and for it is to be identically equal to the linear function of \( \beta_1, \beta_2, \beta_3 \) only, \( (\beta_i, \beta_i') = 0, \quad (\beta_i, \gamma_i') = 0 \), \( \gamma \) should be orthogonal to the vector-space generated by \( \beta_1, \beta_2, \beta_3, \gamma_j, \gamma_j', \gamma_j'' \), which is of rank \( u + v \), if \( \beta_i \)'s are mutually independent.

It should also lie in the estimation-space of rank \( (u + v + w - 1) \) and hence lies in a space of rank \( (v - 1) \). We may consider \( \beta_i \) as the component of \( \gamma_i \) orthogonal to the vector-space \( (\beta_1, \beta_2, \beta_3, \gamma_j, \gamma_j', \gamma_j'') \). Then \( \gamma \) will lie in the vector-space \( (\beta_1, \beta_2, \beta_3) \) which is of rank \( (v - 1) \); because \( \sum_{i=1}^{u} \beta_i = \sum_{i=1}^{u} \beta_i \).

Let \( \gamma_i^{(v)}, \gamma_i^{(v')}, \gamma_j^{(v')} \) and \( \gamma_j^{(v'')} \) be the components of \( \gamma_i, \gamma_j, \gamma_j' \), and \( \gamma \) respectively orthogonal to the vector-space \( (\beta_1, \beta_2, \beta_3) \).

Then \( \gamma_i^{(v)} = \gamma_i - \sum_{i=1}^{u} \beta_i, \beta_i, \gamma_i^{(v')} = \gamma_i - \sum_{i=1}^{u} \gamma_j, \beta_i \) and \( \gamma_j^{(v')} = \gamma_j - \sum_{i=1}^{u} \beta_i, \beta_i \).

where \( \beta_i \) and \( \beta_i \) are as given in (7.2)
Let
\[
T_{ij}^{(v)} = (y_i^{(v)} y_j^{(v)}) \quad \text{and} \quad T_{ij}^{(u)} = (y_i^{(u)} y_j^{(u)}),
\]
\[
\omega_{ij}^{(v)} = (y_i^{(v)} z_j^{(v)}) \quad \text{and} \quad \omega_{ij}^{(u)} = (y_i^{(u)} z_j^{(u)}).
\]

Then
\[
T_{ij}^{(v)} = T_{ij} - \sum_{k=1}^{m} \frac{Y_{ki} r_{kj}}{K_{ki}} B_{ij}, \quad T_{ij}^{(u)} = T_{ij}^{(v)} - \sum_{k=1}^{m} \frac{Y_{ki} r_{kj}}{K_{ki}} B_{ij},
\]
\[
\omega_{ij}^{(v)} = \omega_{ij} + \sum_{k=1}^{m} \frac{Y_{ki} r_{kj}}{K_{ki}} B_{ij}, \quad \omega_{ij}^{(u)} = \omega_{ij}^{(v)} - \sum_{k=1}^{m} \frac{Y_{ki} r_{kj}}{K_{ki}} B_{ij},
\]
\[
\epsilon_{ij}^{(v)} = \gamma_{ij}^{(v)} - \sum_{k=1}^{m} \frac{\gamma_{ki}^{(v)} \gamma_{kj}^{(v)}}{K_{ki}} \quad \text{and} \quad (\gamma_{ij}^{(u)}) = \sum_{k=1}^{m} \frac{\gamma_{ki}^{(u)} \gamma_{kj}^{(u)}}{K_{ki}} - \frac{1}{m} \sum_{k=1}^{m} \frac{B_{ij} K_{ki}}{K_{ki}} - (7.4d).
\]

Thus \(T_{ij}^{(v)}\) and \(T_{ij}^{(u)}\) are the respective block-adjusted yields of the \(j^{th}\) treatment and the \(p^{th}\) concomitant variate corresponding to it.

\(v\) is the within block sum of product of the \(p^{th}\) and \(q^{th}\) concomitant variates, and \(\omega_{ij}^{(v)}\) that between the \(q^{th}\) concomitant and the observed variates.

Evidently \(\gamma_{ij}\) is the projection of \(\gamma_{ij}^{(v)}\) orthogonal to the vector-space

\[
(\gamma_{ij}^{(v)}, \ldots, \gamma_{ij}^{(v)}), \quad i \neq q, \quad \gamma_{ij}^{(v)} = \gamma_{ij} + \sum_{p=1}^{m} \lambda_{ij} \gamma_{ij}^{(v)}
\]

whence multiplying both sides by \(\gamma_{ij}^{(v)}, q = 1, 2, \ldots, K\); we get \(\gamma_{ij}^{(v)}\) as the solution of the equations:

\[
\frac{K}{\lambda_{ij}} \lambda_{ij} \gamma_{ij}^{(v)} = T_{ij}^{(v)}, \quad \sum_{k=1}^{K} \lambda_{ij} \gamma_{ij}^{(v)} = \left(\omega_{ij}^{(v)}\right),
\]

\[
\lambda_{ij} = \left(\gamma_{ij}^{(v)} - \sum_{p=1}^{K} \lambda_{ij} \gamma_{ij}^{(v)}\right).
\]

Thus

\[
\gamma_{ij} = \lim_{m \to \infty} \gamma_{ij}^{(v)} = \frac{\lambda_{ij} \gamma_{ij}^{(v)}}{\lambda_{ij}}, \quad \lambda_{ij} = \lambda_{ij}^{(v)}
\]

It is to be noted that \(\sum_{j=1}^{n} \lambda_{ij} = 0\).

Then in \(Y = (\gamma_{ij})\) the best estimate of \(\pi = \sum_{j=1}^{n} \lambda_{ij} \gamma_{ij}\) is of the form:

\[
\lambda_{ij} = \sum_{i=1}^{m} \lambda_{ij} \gamma_{ij}, \quad b_{ij} = \sum_{i=1}^{m} \lambda_{ij} \gamma_{ij} \gamma_{ij}^{(v)}
\]

Thus, \(b_{ij}\) are the solutions of the equation:

\[
\sum_{i=1}^{m} \lambda_{ij} \gamma_{ij}^{(v)} = b_{ij} \gamma_{ij}^{(v)} + \frac{1}{m} \sum_{i=1}^{m} \gamma_{ij}^{(v)} B_{ij} \gamma_{ij}^{(v)}
\]

where \(b_{ij} = \left(\lambda_{ij} \gamma_{ij}^{(v)}\right) = \frac{1}{m} \sum_{i=1}^{m} \gamma_{ij}^{(v)} B_{ij} \gamma_{ij}^{(v)}\), from (7.4b) and (7.4a),

and \(b_{ij} = (\gamma_{ij}^{(v)} \gamma_{ij}^{(v)})\) are as in (7.4b) and \(b_{ij}^{(v)}\) in (7.4a).
If we put \( \alpha_j \gamma_j = a_i, a_i = \sum_{i=1}^{k} r_i \omega_{ij} \) from (7.6b), then (7.4b), (7.4a) and (7.4b); then \( Y = \sum_{j=1}^{n} \theta_j a_j \), where \( \theta_j \) are obtained from the equation (7.6a), this \( Y \) can easily be proved to be identically equal to \( \sum_{j=1}^{n} \theta_j a_j \), where \( \theta_j \) are the solution of the following normal equations.

\[
\sum_{j=1}^{n} c_{ij} \theta_j = a_i, \quad i = 1, 2, \ldots, \sigma.
\]

The case of a particular interest is when \( \eta \) is a contrast, i.e., \( \eta = \sum_{i=1}^{n} c_{ij} \theta_j \). Now, because \( \sum_{i=1}^{n} c_{ij} \theta_j = 0 \) and \( (c_{ij}) \) is of rank (\( \sigma - 1 \)), the general solution of (7.6b) is of the form \( \left\{ t_1, t_2, \ldots, t_{\sigma - 1}, t_0 \right\} \), where \( \left\{ t_1, t_2, \ldots, t_0 \right\} \) is any particular solution; therefore, \( \sum_{j=1}^{n} \theta_j a_j = \sum_{j=1}^{n} \theta_j a_j \), i.e., we can substitute any solution, general or particular, of equation (7.6b) in the expression for \( \eta \) to get its best unbiased estimate. Similar is the case for \( \theta_j \) in \( Y \) from (7.6a), because \( \sum_{j=1}^{n} \theta_j = 0 \). It is to note that no other \( \eta \) excepting a treatment-contrast is estimable, because of the rank of the estimation space being equal to the number of possible independent treatment-contrasts, all of which are estimable, as the design is a connected one.

To find a unique particular solution of (7.6b) we may consider the constraint \( \sum_{j=1}^{n} \alpha_j \theta_j = 0 \), where \( \sum_{j=1}^{n} \alpha_j \theta_j \neq 0 \).

Now, \( N(Y) = \sum_{d, j=1}^{n} c_{ij} \theta_j \), hence \( \theta_j = \frac{\sum_{j=1}^{n} c_{ij} \theta_j}{\sum_{j=1}^{n} c_{ij} \theta_j} \), for (7.7).

\( c_{ij} \theta_j \) are the solution of the normal equation (7.6b); for in that case...
Section 2.

Now we shall find out the sum of squares due to error. For that we are to find out the sum of squares due to estimates that is the square of the projection of \( \tau \) along the estimation space, viz.,

\[ \gamma(\beta_1, \beta_2, \ldots, \beta_p) = \gamma(\beta_1'; \beta_2'; \ldots; \beta_p') = \gamma'(\beta_1', \beta_2', \ldots; \beta_p'). \]

Let \( \tau \) be the projection of \( \tau \) along the estimation-space and \( \gamma_1, \gamma_2, \gamma_3 \) the projections of \( \gamma \) along \( \gamma(\delta \gamma_1, \delta \gamma_2, \ldots, \delta \gamma_p) \), \( \gamma(\delta \gamma_1, \delta \gamma_2, \ldots, \delta \gamma_k) \) and \( \gamma(\delta \gamma_1, \delta \gamma_2, \ldots, \delta \gamma_m) \) respectively that are mutually orthogonal subspaces of the estimation-space.

\[ \gamma_1 = \gamma_1^2 + \gamma_2^2 + \gamma_3^2. \]

Now,

\[ \gamma_1 = \sum_{i=1}^{p} \left( \frac{\gamma_i - \bar{\gamma}}{\beta_i^2} \right)^2 = \sum_{i=1}^{p} \left( \frac{\beta_i^2}{\bar{\gamma}} \right)^2 \]

and

\[ \gamma_2 = \sum_{i=1}^{k} p_i \omega_i \omega_i', \]

where \( p_i, p_i', \ldots, p_k \) are the solutions of \( \omega_1 \omega_1' + \omega_2 \omega_2' + \ldots + \omega_k \omega_k' = \omega_i \omega_i' \ldots \omega_j \omega_j' \).

It is to be noted that the maximum solution of \( \gamma \) is \( \gamma_1 = \gamma_2 + \gamma_3 \), where

\[ \left( \omega_1 \omega_1', \omega_2 \omega_2', \ldots, \omega_k \omega_k' \right) \]

being the inverse of \( \left( \omega_1 \omega_1', \omega_2 \omega_2', \ldots, \omega_k \omega_k' \right) \).

To find out \( \gamma_1, \gamma_2, \gamma_3 \) because, \( k+1 \) out of \( \lambda_1, \lambda_2, \ldots, \lambda_k \) are independent, \( \lambda_1, \lambda_2, \ldots, \lambda_k \) may be taken to form the basis of \( \gamma(\lambda_1, \lambda_2) \).

\[ \gamma = \gamma_1 + \gamma_2 + \gamma_3 + \ldots + \gamma_n = \gamma_1 + \gamma_2 + \gamma_3 + \ldots + \gamma_n = \gamma_1 + \gamma_2 + \gamma_3 + \ldots + \gamma_n = \gamma_1 + \gamma_2 + \gamma_3 + \ldots + \gamma_n. \]

Multiplying both sides of \( \gamma \) by \( \lambda_j \), \( j = 1, 2, \ldots, (k+1) \).
because \((\lambda_j, \gamma_j) = c_{ijj}^j\) and \((\lambda_j, \gamma_j) = d_{ij}^j\).

Now, if \((m_1, m_2, \ldots, m_{j-1})\) be the solution of (7.10),

\[(m_1 + \theta, m_2 + \theta, \ldots, m_{j-1} + \theta, \theta)\]

is a solution of normal equations (7.6b),

the general solution of the normal equations (7.6b) is

\[(m_1 + \theta, m_2 + \theta, \ldots, m_{j-1} + \theta, \theta)\]

Now, because \(\sum_{j=1}^{n} \lambda_j = 0\) from (7.6b),

\[\gamma'' = \sum_{j=1}^{n} \lambda_j\]

where \((m_1, m_2, \ldots, m_{j-1})\) are solutions of (7.10)

\[\gamma'' = (m_1 + \theta) \lambda_1 + \ldots + (m_{j-1} + \theta) \lambda_{j-1}\]

because \(\sum_{j=1}^{n} \lambda_j = 0\).

\[= t_0 \lambda_1 + t_0 \lambda_2 + \ldots + t_0 \lambda_j\]

where \((t_0, t_1, t_2, \ldots, t_k)\) is any particular solution of (7.6b) is

the general solution of (7.6b),

\[\gamma'' = t_0 \lambda_1 + t_1 \lambda_2 + \ldots + t_k \lambda_j\]

Hence \(\gamma''' = (t_0'' \gamma) = t_0 (\lambda_1, \gamma) + t_1 (\lambda_2, \gamma) + \ldots + t_k (\lambda_j, \gamma)\)

\[= t_0 \delta_1 + t_1 d_{j} + \ldots + t_k d_{j}\]

which is also the sum of squares due to the estimates of all possible treatment-contrasts.

Thus the sum of squares due to estimated is:

\[\sum_{i=1}^{K} \frac{\sum_{j=1}^{m} \omega_j}{\hat{V}_{ij}} + \sum_{j=1}^{K} \frac{\sum_{i=1}^{m} \omega_i}{\hat{V}_{ij}} + \sum_{j=1}^{K} \sum_{i=1}^{m} \omega_j d_{ij}\]

sum of squares due to Error,

\[\sum_{j=1}^{K} \sum_{i=1}^{m} \omega_j d_{ij} - \sum_{i=1}^{K} \frac{\sum_{j=1}^{m} \omega_j}{\hat{V}_{ij}} - \sum_{j=1}^{K} \sum_{i=1}^{m} \omega_j d_{ij}

Then we may consider the following hypotheses:
(a) To test \( H^r \) \( \mu = \mu_0 \) \( \sigma = \sigma_0 \), the proper statistic is
\[
F = \frac{\sum_{j=1}^{r} \frac{t_j^2}{\sigma_j^2}}{\sum_{j=1}^{r} \frac{1}{\sigma_j^2}} \text{ distributed as F-distribution }
\]
with \((r-1)\) and \((n-u-d-K+1)\) d.f., where \( t_j \) are the solutions of (7.6b).

(b) To test \( H^t \) \( \mu = \mu_0 \) \( \sigma = \sigma_0 \), the proper statistic is
\[
t = \frac{\sum_{j=1}^{r} \frac{t_j^2}{\sigma_j^2} - \bar{X}}{\sum_{j=1}^{r} \frac{1}{\sigma_j^2}} \gamma \frac{1}{2}
\]
distributed as t-distribution with \((n-\mu-\sigma-d-K+1)\) d.f., \( t \) are the solutions of (7.6b) and \( C_{j,j} \) are as given in (7.7).

The sum of squares due to Estimate can also be obtained in the following alternative way:

Let \( \bar{y}_i^{(v)} \) be the projection of \( \bar{y}_i \) minimizing \( \sum (\bar{y}_i - \bar{y}) \bar{y}_i \).

Then \( \sum (\beta_1, ..., \beta_v, \gamma_1, ..., \gamma_v, \bar{y}_i, ..., \bar{y}_v) = \sum (\beta_1, ..., \beta_v, \gamma_1, ..., \gamma_v, \bar{y}_i, ..., \bar{y}_v) \).

Now \( \bar{y}_i^{(v)} = \bar{y}_i + \gamma_1 \bar{y}_1 + ... + \gamma_v \bar{y}_v \).

Multiplying both sides by \( \gamma_1^{(v)}, \gamma_2^{(v)}, ..., \gamma_v^{(v)} \),
\[
\sum_{j=1}^{v} \sum_{j=1}^{v} t_j \bar{y}_j + \sum_{j=1}^{v} \sum_{j=1}^{v} t_{j,v} \bar{y}_j = \sum_{j=1}^{v} \sum_{j=1}^{v} t_{j,v} \bar{y}_j \quad \text{ (7.12)},
\]

where \( t_{j,v}, \gamma_{j,v} \) are given in (7.4a).

The homogeneous equations (7.12) are of rank \((v-1)\), and the solution is \((0, 0, ..., 0)\) if \((t_1, t_2, ..., t_{v-1}) \) be a particular solution of (7.12) the sum of the two will give general solution \( j \).

\[
\bar{y}_i^{(v)} = t_1 \gamma_1^{(v)} + t_2 \gamma_2^{(v)} + ... + t_{v-1} \gamma_{v-1}^{(v)} + \bar{y}_i
\]

\[
= (t_1, t_2, ..., t_{v-1}, 0) \gamma_1^{(v)} + (t_1, t_2, ..., t_{v-1}, 0) \gamma_2^{(v)} + ... + (t_1, t_2, ..., t_{v-1}, 0) \gamma_{v-1}^{(v)} + \bar{y}_i
\]

\[
= t_1 \gamma_1^{(v)} + t_2 \gamma_2^{(v)} + ... + t_{v-1} \gamma_{v-1}^{(v)} + \bar{y}_i
\]
where \((\gamma_1, \gamma_2, \cdots, \gamma_k)\) is a solution, general or particular, of \((7.12)\).

Again, the square of the projection of \(\gamma\) along the estimation-space is
the sum of the squares of the projections of \(\gamma\) along \(\nu (\beta_1, \beta_2, \cdots, \beta_n)\),

\[
(\gamma_1, \gamma_2, \cdots, \gamma_k) \quad \text{and} \quad (\gamma_1, \gamma_2, \cdots, \gamma_k).
\]

The projection of \(\gamma\) along \(\nu (\beta_1, \beta_2, \cdots, \beta_n)\) is \(\frac{\kappa}{\nu} \frac{\beta_j}{\kappa} \),
and the square of the projection of \(\gamma\) along \(\nu (\gamma_1, \gamma_2, \cdots, \gamma_k)\)
is \(\sum_{j=1}^{k} \gamma_j \gamma_j\), where \((\gamma_1, \gamma_2, \cdots, \gamma_k)\) is a solution, general
or particular, of the following equations:

\[
\xi_{\gamma, j} \gamma_j + \xi_{\gamma, j} \gamma_j + \cdots + \xi_{\gamma, j} \gamma_j = \frac{\gamma_j}{\nu} \cdot \gamma_j \quad \cdots \cdot (7.13)
\]

\[
\xi_{\gamma, j} \gamma_j \text{ being given in (7.4b)}
\]

Also, the square of the projection of \(\gamma\) along \(\nu (\xi_{\gamma, 1}, \xi_{\gamma, 2}, \cdots, \xi_{\gamma, k})\)
is \(\gamma_1 (\xi_{\gamma, 1} \gamma_1) + \gamma_2 (\xi_{\gamma, 2} \gamma_2) + \cdots + \gamma_k (\xi_{\gamma, k} \gamma_k)\),

where \(\gamma_1, \gamma_2, \cdots, \gamma_k\) are the solutions of:

\[
\omega^{(1)}_{\gamma, 1} \gamma_1 + \omega^{(1)}_{\gamma, 2} \gamma_2 + \cdots + \omega^{(1)}_{\gamma, k} \gamma_k = \omega^{(1)}_{\gamma, 1} \quad \cdots \cdot (7.14)
\]

where from \((7.12a)\):

\[
\omega^{(1)}_{\gamma, 1} = (\xi_{\gamma, 1} \gamma_1) - \sum_{j=1}^{k} \xi_{\gamma, j} (\gamma_j \gamma_j)
\]

\[
= \omega^{(1)}_{\gamma, 1} - \sum_{j=1}^{k} \xi_{\gamma, j} \gamma_j \gamma_j = \gamma \text{ s.s. } (7.15)
\]

and \(\omega^{(1)}_{\gamma, 2} = (\xi_{\gamma, 2} \gamma_2) - \sum_{j=1}^{k} \xi_{\gamma, j} (\gamma_j \gamma_j) \quad \cdots \cdot (7.15)\)

\[
= \omega^{(1)}_{\gamma, 2} - \sum_{j=1}^{k} \xi_{\gamma, j} \gamma_j \gamma_j = \gamma \text{ s.s. } (7.15)
\]

\(\gamma_{\gamma, 1, 2, \cdots, k}\) being solutions of \((7.12)\).

Thus the alternative form of the sum of squares due to estimate is:

\[
\xi_{\gamma} = \sum_{\gamma = 1}^{k} \xi_{\gamma, 1} \gamma_1 + \sum_{j=1}^{k} \xi_{\gamma, j} \gamma_j + \sum_{\gamma = 1}^{k} \omega^{(1)}_{\gamma, j} \gamma_j
\]

where \((\gamma_1, \gamma_2, \cdots, \gamma_k)\) is a solution of \((7.13)\) and \((\xi_{\gamma, 1}, \xi_{\gamma, 2}, \cdots, \xi_{\gamma, k})\) is the
solution of \((7.14)\)
Hence, the alternative form of sum of squares due to Error is:

\[ S_e^2 = \sum_{i=1}^{k} \sum_{j=1}^{r} \frac{y_i^2}{w_i} - \frac{\mu}{r} \sum_{i=1}^{k} \sum_{j=1}^{r} d_{ij} \bar{T}_j - \frac{K}{r} \sum_{i=1}^{k} \bar{y}_i \bar{y}_j. \]

Again the sum of squares due to treatment-contrasts and Error combined is:

\[ S_{e+t}^2 = S_e^2 + S_{w-1}^2 = \sum_{i=1}^{k} \sum_{j=1}^{r} y_i^2 - \frac{\mu}{r} \sum_{i=1}^{k} \sum_{j=1}^{r} d_{ij} \bar{T}_j - \frac{K}{r} \sum_{i=1}^{k} \bar{y}_i \bar{y}_j. \]

where \((\eta_j, \psi_j, \ldots, \eta_k)\) is the solution of (7.4).

Thus if we find out \(S_e^2\) by second method and \(S_{e+t}^2\) by first method, then \(S_{w-1}^2\) can be obtained as \(S_{e+t}^2 - S_e^2\), this is the method that is used in practice. In this method equations (7.4) and (7.14) are necessary.

The corresponding analysis of variance and co-variance tables for the two cases can easily be drawn.

2. Linear Estimation in a Multivariate Case

Here we shall consider another problem in the analysis of variance. Here also it should be noted that the method suggested by Rao (1946) is quite applicable. But there are certain special cases giving some interesting results. The method of approach also just as in the previous case is different from Rao.

Let it be given that:

\[ E(\gamma^{(i)}_{\alpha}) = \alpha_{\alpha i}^{(i)} \gamma_{\alpha i}^{(i)} + \alpha_{\beta i}^{(i)} \gamma_{\beta i}^{(i)} + \ldots + \alpha_{\kappa i}^{(i)} \gamma_{\kappa i}^{(i)} \]

where \(\alpha_{\alpha i}^{(i)}, \ldots, \alpha_{\kappa i}^{(i)}\) are known and \(\gamma_{\alpha i}^{(i)}, \ldots, \gamma_{\kappa i}^{(i)}\) are unknown parameters,

and \(E(\gamma^{(i)}_{\alpha}, \gamma^{(i)}_{\beta}) = \gamma^{(i)}_{\alpha \beta}, E(\gamma^{(i)}_{\alpha}, \gamma^{(i)}_{\kappa}) = 0\) for \(\alpha \neq \kappa\), \(\beta \neq \kappa\).

Let the vectors:

\[ \gamma_i = (y_{i1}^{(1)}, \ldots, y_{i1}^{(k)}, \ldots, y_{im}^{(1)}, \ldots, y_{im}^{(k)}) \]

\[ \alpha_{ij}^{(i)} = (\alpha_{i1}^{(1)}, \ldots, \alpha_{i1}^{(k)}, \ldots, \alpha_{im}^{(1)}, \ldots, \alpha_{im}^{(k)}) \]

We can also write as \(\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_k)\),

where \(\gamma_i = (y_{i1}^{(1)}, \ldots, y_{i1}^{(k)}, \ldots, y_{im}^{(1)}, \ldots, y_{im}^{(k)})\).
Suppose we are to find out the best unbiased linear estimate of a parametric function \( \pi = \sum_{i=1}^{K} \sum_{j=1}^{N} \theta_i^{(j)} p_j^{(i)} \).

Let \( \gamma \) be vector \( (c_1^{(1)}, c_1^{(2)}, \ldots, c_1^{(N)}, \ldots, c_n^{(m)}) \) or \( \gamma = (\gamma_1, \gamma_2, \ldots, \gamma_m) \),

where \( \gamma_i = (c_i^{(1)}, c_i^{(2)}, \ldots, c_i^{(N)}) \) such that \( (\gamma, \gamma) \) is an unbiased estimate of \( \pi \), then, of course, \( E(\gamma, \gamma) = \pi \).

Again, we know, if \( E(\gamma, \gamma) = 0 \), then \( \gamma \) belongs to Error.

In that case \( \sum_{i=1}^{K} \sum_{j=1}^{N} p_j^{(i)} (\gamma_i \cdot \theta_j^{(i)}) = 0 \), \( \langle \gamma_i, \theta_j^{(i)} \rangle = 0 \).

If \( \gamma = (\gamma_1, \gamma_2, \ldots, \gamma_m) \), then \( \langle \gamma_i, \theta_j^{(i)} \rangle = 0 \), \( j = 1, 2, \ldots, m \),

where \( \gamma_j = \langle \gamma_i, \theta_j^{(i)} \rangle \), and \( \theta_j^{(i)} \) is the vector \( (\gamma_1, \gamma_2, \ldots, \gamma_m) \), and I think there will be no confusion with the \( \theta_j^{(i)} \) having \( m \) elements.

Let \( \gamma \) belong to Error, its components \( \gamma_{j0} \perp \langle \theta_j^{(1)}, \ldots, \theta_j^{(K)} \rangle, j = 1, 2, \ldots, K \).

Now, \( \langle \gamma, \gamma \rangle = \sum_{i=1}^{m} (\gamma_i \cdot \gamma_i) = \sum_{i=1}^{m} \gamma_i^{(1)} \gamma_i^{(1)} = \sum_{i=1}^{K} \langle \gamma_i, \gamma_i \rangle \).

If \( \gamma_{j0} \) be the projection of \( \gamma \) along \( \langle \theta_j^{(1)}, \ldots, \theta_j^{(K)} \rangle \),

and \( \gamma_{j0} \) the projection orthogonal to the vector-space,

then \( \gamma = \gamma_{j0} + \gamma_{j0} \perp \gamma_{j0} \).

In this case \( \langle \gamma, \gamma \rangle = \gamma_{j0} \perp \gamma_{j0} \).

where \( \gamma_{j0} = (\gamma_1, \gamma_2, \ldots, \gamma_m) \) and \( \gamma_{j0} = (\gamma_1, \gamma_2, \ldots, \gamma_m) \).

Thus if \( E(\gamma_{j0}, \gamma_{j0}) = \pi \), then \( E(\gamma_{j0}, \gamma_{j0}) = \pi \).

And \( \gamma_{j0} \) is the projection of \( \gamma \) along \( \langle \theta_j^{(1)}, \ldots, \theta_j^{(K)} \rangle \),

and \( \gamma = \langle \gamma_{j0}, \gamma_{j0} \rangle \).
If possible let there be another \( \gamma' \) such that its components

\[ (\gamma'_{1}, \ldots, \gamma'_{k}) \]

lies in \( \mathcal{L} \) and \( \mathbb{E}(\gamma'_{j}, \mathcal{L}) = \pi \).

\[ \beta_{j} = (\beta_{1}, \ldots, \beta_{k}) \]

where \( \beta_{j} = (\gamma'_{j} - \gamma_{j}) \).

\( \mathcal{E}(\beta, \mathcal{L}) \in \mathcal{D} \)

\( \mathcal{I} \mathcal{E} \) \( \beta_{i} \perp \mathcal{L}(\gamma_{1}, \ldots, \gamma_{k}) \) but lies in \( \mathcal{L}(\gamma_{1}, \ldots, \gamma_{k}) \).

Hence, it is possible to get such \( \gamma'_{j} \) unless, of course,

\[ \mathcal{L}(\gamma_{1}, \ldots, \gamma_{k}) \in \mathcal{L}(\gamma_{1}, \ldots, \gamma_{k}) \]

or in other words, without any loss of generality, \( \gamma'_{j} \) are independent of \( i \). Thus necessary condition that \( \gamma' \) be unique is that

\[ a_{j} \text{ is independent of } i. \]

Then \( \mathbb{E}(\gamma_{j}) = a_{1} \gamma_{1} + \cdots + a_{n} \gamma_{n} \)

and putting \( \gamma_{j} = (a_{1}, a_{2}, \ldots, a_{k}) \) the best unbiased linear estimate of \( \gamma_{j} \mathcal{L}(\gamma_{1}, \ldots, \gamma_{k}) \) where \( \mathbb{E}(\gamma_{j}, \mathcal{L}) = \gamma_{j} \in (\gamma_{1}, \ldots, \gamma_{k}) \)

\[ \mathcal{L}(\gamma_{1}, \ldots, \gamma_{k}) \sim \mathcal{L}(\gamma_{1}, \ldots, \gamma_{k}) \]

independently of \( i \). Thus necessary condition that \( \gamma' \) be unique is that

\[ a_{j} \text{ is independent of } i. \]

It can easily be proved that if \( \gamma' \) mentioned above be not unique then we cannot find out the best unbiased linear estimate of \( \gamma_{j} \).

Theorem 7.4 (The necessary and sufficient condition for the existence of best unbiased linear estimate of a parametric function \( \gamma_{j} \) independently of \( \gamma_{i} \) is that \( a_{j} \) be independent of \( i \).

Again \( \mathbb{E}(\gamma_{j} - \gamma_{i}) = \sum_{i=1}^{n} \sum_{j=1}^{m} p_{i,j} (\gamma_{i} - \gamma_{j}) \), where \( n = \sum_{i=1}^{n} \sum_{j=1}^{m} p_{i,j} \)

\[ \mathbb{E}(\gamma_{j} - \gamma_{i}) = \sum_{i=1}^{n} \sum_{j=1}^{m} p_{i,j} (\gamma_{i} - \gamma_{j}) \]

given the solution of \( q_{i,j} \).

In the set of equations (7.16), if both sides of the \( i \in \mathcal{L} \) be multiplied by \( p_{i,j}^{(k)} \) where \( p_{i,j}^{(k)} \) are solutions of (7.17) below.
and added we get:
\[ p_j^{(1)} (a_j, a_j') + p_j^{(2)} (a_j', a_j') + \ldots + p_j^{(k)} (a_j', a_j') = \frac{\sum_{\ell=1}^k p_j^{(\ell)} (a_j', a_j')}{\sum_{\ell=1}^k p_j^{(\ell)}}. \]

Also, multiplying both sides of the \( j \)th equation of the \( \ell \)th set of (7.17) by \( q_{i\ell} \), where \( q_{i\ell} \) are solutions of (7.16), and adding we get:
\[ \sum_{\ell=1}^K \sum_{i=1}^r q_{i\ell} \cdot (a_j', q_{i\ell}) = \sum_{\ell=1}^K \sum_{i=1}^r q_{i\ell} \cdot (a_j', q_{i\ell}). \]

Again:
\[ (a_j', q_{i\ell}) = \sum_{\ell=1}^K \sum_{i=1}^r q_{i\ell} \cdot (a_j', q_{i\ell}) = \sum_{\ell=1}^K \sum_{i=1}^r q_{i\ell} \cdot (a_j', q_{i\ell}). \]

But:
\[ \sum_{\ell=1}^K \sum_{i=1}^r q_{i\ell} \cdot (a_j', q_{i\ell}) = \sum_{\ell=1}^K \sum_{i=1}^r q_{i\ell} \cdot (a_j', q_{i\ell}) \quad \text{(identically)}; \]

hence:
\[ (a_j', q_{i\ell}) = \sum_{\ell=1}^K \sum_{i=1}^r q_{i\ell} \cdot (a_j', q_{i\ell}), \]
where \( q_{i\ell} \) are solutions of equations (7.17), called normal equations.

So, the method is the same as the method of \( \psi_{i\ell}^{(\ell)} \) independently from each \( \ell \)th of the \( k \) sets of \( \psi_{i\ell}^{(\ell)} \) and then substituting in the expression for \( \psi_{ij}^{(\ell)} \).

The variance of the best estimate is:
\[ \psi_{ij} = \sum_{\ell=1}^K \sigma^{(\ell)} \cdot (a_j', q_{i\ell}) = \sum_{\ell=1}^K \sigma^{(\ell)} \cdot (a_j', q_{i\ell}). \]

We shall not deal further with this problem, but certain tests of significance in this connection must be considered in chapters ahead.