CHAPTER - 4
Connected subsets of GTS
and separations of product of GTS

In this chapter, we define $\gamma -$ generalized connected or shortly $GC$ subsets and prove that the closure of a $GC$ subset is again a $GC$ subset. Next, we define generalized components containing a point $x$ and prove that the family of all generalized components is the maximum separation. We study the properties of images and pre images of generalized connected sets. Also, we prove the intermediate value theorem for generalized topological spaces. We prove that arbitrary product of strong GTS is generalized connected if and only if each coordinate space is generalized connected. Also, we study the separations of generalized disconnected product spaces. All these results of this chapter are from [4]. At the end of this chapter, we study the lattice structure of the collection $\mathcal{G}(X)$ of all the GT on a nonempty set $X$. All these results are from [6].

**Definition 4.1.** Let $(X, \gamma)$ be a GTS and $A \subset X$. A collection $\mathcal{A} = \{A_\alpha \mid \alpha \in \Delta\}$ of nonempty disjoint $\gamma -$ closed subsets of $X$ is a separation of $A$ if $A \subset \bigcup \{A_\alpha \mid \alpha \in \Delta\}$ and $A \cap A_\alpha \neq \emptyset$ for every $\alpha \in \Delta$ where $|\Delta| \geq 2$.

$A \subset X$ is said to be a generalized disconnected or GD subset if $A$ has a separation. $A \subset X$ is said to be a generalized connected or GC subset if $A$ has no separation. $X$ is said to be generalized connected if $X$ is generalized connected as a subset of $X$. 
One may ask a question, why not we define the separation \( \mathcal{A} \) of a subset as follows:

"A collection \( \mathcal{A} = \{ A_\alpha \mid \alpha \in \Delta \} \) of nonempty disjoint \( \gamma \)-closed subsets of \( X \) is a separation of \( A \) if \( A = \bigcup \{ A_\alpha \mid \alpha \in \Delta \} \) and \( A \cap A_\alpha \neq \emptyset \) for every \( \alpha \in \Delta \) where \( |\Delta| \geq 2 \)."

Will it make any difference? We answer this question in the following Example 4.2.

**Example 4.2.** [Example 2.7] Let \( X = \{a, b, c, d, e\} \) and the family of all \( \gamma \)-open sets be \( \mu = \{\emptyset, \{a, b\}, \{a, b, c\}, \{c, d, e\}, \{a, b, d, e\}, \{a, c, d, e\}, \{b, c, d, e\}, X\} \). Then the family of all \( \gamma \)-closed sets is given by \( \emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{d, e\}, \{c, d, e\}, X \). If \( A = \{a, b, d\} \), then \( A \subset \{a, b\} \cup \{d, e\} \) but \( A \neq \{a, b\} \cup \{d, e\} \). Thus \( A \) is a GD subset, if we take inclusion in the definition of separation and \( A \) is a GC subset, if we take equality in the definition of separation

"A subset \( Y \) of a topological space \( (X, \tau) \) is connected if it is connected as a subspace." Our definition is equivalent to the above definition, since \( A = \bigcup \{(A_\alpha \cap A) \mid \alpha \in \Delta \} \) if and only if \( A \subset \bigcup \{A_\alpha \mid \alpha \in \Delta \} \).

**Theorem 4.3.** Let \( (X, \gamma) \) be a GTS and \( S \neq \emptyset \) be any GC subset of \( X \). If \( \mathcal{A} = \{A_\alpha \mid \alpha \in \Delta \} \) is any collection of proper, nonempty disjoint \( \gamma \)-closed subsets of \( X \) such that \( S \subset \bigcup \{A_\alpha \mid \alpha \in \Delta \} \), then \( S \subset A_\beta \) for some \( \beta \in \Delta \).

**Proof.** The proof follows from the definition.
The following Example 4.4 illustrates this.

**Example 4.4.** Let $X = \{a,b,c,d\}$ and $\mu = \{\emptyset, \{d\}, \{a,b,c\}, \{a,c,d\}, \{b,c,d\}, X\}$. Then $\mu$– closed sets are $\{ \emptyset, \{a\}, \{b\}, \{d\}, \{a,b,c\}, X\}$. $S = \{a,b,c\}$ is a GC subset, since $S$ cannot be expressed as a disjoint union of proper nonempty $\mu$– closed sets. If $S_1 = \{a,b\}$, then, $\{a,b\} \subset \{a\} \cup \{b\}$ but $S_1$ is not a subset of either of the two $\mu$– closed sets. $S_1$ is a GD subset.

**Theorem 4.5.** Let $(X, \gamma)$ be a GD space. If $S \subset X$, $S$ is generalized connected and $S \subset T \subset c_\gamma(S)$, then $T$ is generalized connected.

**Proof.** Suppose $T$ is not GC. Then $T$ has a separation $A = \{A_\alpha : \alpha \in \Delta\}$. $T \subset \cup\{A_\alpha : \alpha \in \Delta\}$ implies that $S \subset \cup\{A_\alpha : \alpha \in \Delta\}$. $S$ is generalized connected implies that $S \subset A_\beta$ for some $\beta \in \Delta$. Hence $c_\gamma(S) \subset A_\beta$ and so $T \subset A_\beta$, a contradiction. $T$ is generalized connected.

**Corollary 4.6.** Let $(X, \gamma)$ be a GD space and $S \subset X$. If $S$ is generalized connected, then $c_\gamma(S)$ is generalized connected.

**Corollary 4.7.** Let $(X, \gamma)$ be a GD space and $S \subset X$. If $S = \cup\{S_\lambda : \lambda \in \Lambda\}$ where $S_\lambda$ is generalized connected for every $\lambda \in \Lambda$ and $S_\lambda \cap S_{\lambda'} \neq \emptyset$ for $\lambda, \lambda' \in \Lambda$, then $S$ is generalized connected.

**Proof.** Assume that $S$ is GD. Then $S$ has a separation $A = \{A_\alpha : \alpha \in \Delta\}$. $S_\lambda$ is generalized connected implies that $S_\lambda \subset A_\beta$ for some $\beta \in \Delta$. $S_\lambda \cap S_{\lambda'} \neq \emptyset$ for
λ, λ’ ∈ Λ implies that S_λ ⊂ A_β for every λ ∈ Λ and so S ⊂ A_β, a contradiction.

S is generalized connected.

**Corollary 4.8.** Let (X, γ) be a GD space and S ⊂ X. If S = ∪{S_λ : λ ∈ Λ} where S_λ is generalized connected for every λ ∈ Λ and ∩{S_λ : λ ∈ Λ} ≠ ∅, then S is generalized connected.

We now proceed to define generalized components C_x containing x ∈ X. It is interesting to note that the collection of all generalized components of a generalized disconnected space turns out to be the maximum separation.

**Definition 4.9.** The set C_x = ∪{S ⊂ X : x ∈ S, S is generalized connected} is called the component of X containing x. Note that, by Corollary 4.8, C_x is generalized connected for all x ∈ X.

**Theorem 4.10.** If (X, γ) is a GD GTS, then C = {C_x : x ∈ X} is a separation of X.

**Proof.** By Corollary 4.6, C_γ(C_x) is connected. So C_γ(C_x) is a member of the collection of GC subsets containing x. Therefore C_γ(C_x) ⊂ C_x. C_γ(C_x) is γ−closed for all x ∈ X. x ∈ C_x implies that C_x is nonempty and ∪{C_x : x ∈ X} is X. If C_x ∩ C_y ≠ ∅, then by Corollary 4.7, C_x ∪ C_y is connected. Therefore, C_x ∪ C_y ⊂ C_x and C_x ∪ C_y ⊂ C_y which implies that C_x = C_y. Hence C = {C_x : x ∈ X} is a separation of X.
Theorem 4.11. If a GTS \((X, \gamma)\) is not GC, then \(\mathcal{C} = \{C_x : x \in X\}\) is the maximum separation of \(X\).

Proof. If \(\mathcal{M} = \{M_\alpha : \alpha \in \Delta\}\) is the maximum separation and \(\mathcal{C} = \{C_x : x \in X\}\) is not the maximum separation of \(X\), then \(\mathcal{C} \land \mathcal{M} = \mathcal{M}\). Therefore, there exists \(C_x \in \mathcal{C}\) such that \(C_x\) is the union of two or more members of \(\mathcal{M}\), which is a contradiction.

The following Example 4.12 and 4.13 illustrate Theorem 4.11.

Example 4.12. [Example 4.4] Let \(X = \{a, b, c, d\}\) and \(\mu = \emptyset, \{d\}, \{a, b, c\}, \{a, c, d\}, \{b, c, d\}\). Then \(\mu\)–closed sets are \(\emptyset, \{a\}, \{b\}, \{d\}, \{a, b, c\}, X\).

\(C_a = \{a\} \cup \{a, b, c\} = \{a, b, c\}\), \(C_b = \{b\} \cup \{a, b, c\} = \{a, b, c\}\), \(C_c = \{a, b, c\}\).

\(C_d = \{d\}\). \(\{C_x : x \in X\} = \{\{a, b, c\}, \{d\}\}\) is the maximum separation of \(X\). But, in this example, \((X, \mu)\) has only one separation.

Example 4.13. [Example 2.26] Consider the GTS \((X, \mu)\) where \(X = \{a, b, c, d\}\) and \(\mu = \emptyset, \{a\}, \{b\}, \{c\}, \{d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, X\).

\(\mu\)–closed sets are: \(\emptyset, \{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{a, c\}, \{b, d\}, \{c, d\}\), \(X\).

Then \(\{\{a\}, \{b\}, \{c\}, \{d\}\}\) is the unique 4–separation.

\(\{\{a\}, \{b\}, \{c\}, \{d\}\}\), \(\{\{a\}, \{c\}, \{b, d\}\}\), \(\{\{b\}, \{d\}, \{a, c\}\}\) and \(\{\{c\}, \{d\}, \{a, b\}\}\) are four 3–separations of \(X\). \(\{\{a, b\}, \{c, d\}\}\) and \(\{\{a, c\}, \{b, d\}\}\) are two 2–separations of \(X\). \(C_a = \{a\}\), \(C_b = \{b\}\), \(C_c = \{c\}\), \(C_d = \{d\}\). \(\mathcal{C} = \{C_x : x \in X\} = \{\{a\}, \{b\}, \{c\}, \{d\}\}\) is the maximum separation of \(X\).
**Example 4.14.** For $\mathbb{R}$ with usual topology, $C_x = \{x\}$ for every $x \in \mathbb{R}$. \{$\{x\} : x \in \mathbb{R}\}$ is the maximum separation.

**Theorem 4.15.** Let $(X, \gamma)$ and $(X', \gamma')$ be two GT spaces and $f : X \to X'$ be $(\gamma, \gamma')-$ continuous. If $S \subset X$ is $\gamma-$ generalized connected, then $f(S)$ is $\gamma'$-generalized connected.

**Proof.** Assume that $f(S)$ is not $\gamma'$-generalized connected. Let $B = \{B_\alpha : \alpha \in \Delta\}$ be any separation of $f(S)$. $f$ is continuous implies that $f^{-1}(B_\alpha)$ is closed for every $\alpha \in \Delta$. $x \in f^{-1}(B_\alpha) \cap f^{-1}(B_\beta)$ implies that $f(x) \in B_\alpha \cap B_\beta$, a contradiction. Also $S \subset \cup \{f^{-1}(B_\alpha) : \alpha \in \Delta\}$. We get $\{f^{-1}(B_\alpha) : \alpha \in \Delta\}$ is a separation of $S$, which is a contradiction.

**Corollary 4.16.** Let $(X, \gamma)$ and $(X', \gamma')$ be two GT spaces and $f : X \to X'$ be $(\gamma, \gamma')-$ continuous. If $S \subset X$ is $\gamma-$ connected, then $f(S)$ is $\gamma'$-connected [Lemma 1.34].

**Proof.** If $f(S)$ is not $\gamma'$- connected, then it has a 2-separation and hence $S$ has a 2-separation.

**Corollary 4.17.** Let $(X, \tau)$ and $(X', \tau')$ be two topological spaces and $f : X \to X'$ be $(\tau, \tau')-$ continuous. If $S \subset X$ is connected, then $f(S)$ is connected.

**Theorem 4.18.** Let $(X, \gamma)$ and $(X', \gamma')$ be two GT spaces and $f : X \to X'$ be a $(\gamma, \gamma')-$ closed, injective mapping. If $S \subset X$ and $f(S)$ is $\gamma'$-generalized
connected, then $S$ is $\gamma-$generalized connected.

**Proof.** Assume that $S$ is not $\gamma-$generalized connected. Let $\mathcal{A} = \{A_\alpha : \alpha \in \Delta\}$ be any separation of $S$. Then $f(A_\alpha)$ is $\gamma'-$closed for every $\alpha \in \Delta$. $f(A_\alpha) \cap f(A_\beta) \neq \emptyset$ implies that there exist $x \in A_\alpha$ and $y \in A_\beta$ such that $f(x) = f(y)$ which implies that $x = y$. Therefore $A_\alpha \cap A_\beta \neq \emptyset$, a contradiction. Also $y \in f(S)$ implies that there exists $x \in S \subset \bigcup \{A_\alpha : \alpha \in \Delta\}$ which implies that $y \in \bigcup \{f(A_\alpha) : \alpha \in \Delta\}$. We get $\{f(A_\alpha) : \alpha \in \Delta\}$ is a separation of $f(S)$, which is a contradiction.

**Corollary 4.19.** Let $(X, \gamma)$ and $(X', \gamma')$ be two GT spaces and $f : X \to X'$ be a $(\gamma, \gamma')-$closed (open) and injective mapping. If $S \subset X$ and $f(S)$ is $\gamma'-$connected, then $S$ is $\gamma-$connected [Lemma 1.35].

**Corollary 4.20.** Let $(X, \tau)$ and $(X', \tau')$ be two topological spaces and $f : X \to X'$ be $(\tau, \tau')-$continuous and injective. If $S \subset X$ and $f(S)$ is connected, then $S$ is connected.

The following Example 4.21 shows that the condition *injective* on $f$ in Theorem 4.18 cannot be dropped.

**Example 4.21.** Consider the GTS $(X, \eta)$ where $X = I_n$ and $\eta$ is the co-singleton generalized topology. Let $S \subset I_n$ where $S$ contains more than one element. Define $f : I_n \to I_n$ by $f(x) = 1$ for every $x$. $f$ is $(\eta, \eta)-$closed but not injective. $S = \bigcup \{\{x\} : x \in S\}$ implies that $S$ is not generalized connected but $f(S) = \{1\}$ is generalized connected.
Theorem 4.22. (Intermediate value theorem for \(\gamma\)-FC GTS) Let the GTS \((X, \gamma)\) be FC and \((Y, \tau)\) be an order topological space. If \(f : X \rightarrow Y\) is \((\gamma, \tau)\)-continuous, \(a, b\) are two distinct points of \(X\) and \(r\) is a point of \(Y\) lying between \(f(a)\) and \(f(b)\), then there exists a point \(c\) in \(X\) such that \(f(c) = r\).

**Proof.** If \(S = \{y \in Y \mid y \leq r\}\) and \(T = \{y \in Y \mid y \geq r\}\), then \(S\) and \(T\) are \(\tau\)-closed subsets of \(Y\). If \(f(c) \neq r\) for any \(c \in X\), then \(\{f^{-1}(S), f^{-1}(T)\}\) is a 2-separation of \(X\), a contradiction.

Corollary 4.23. (Intermediate value theorem for \(\gamma\)-connected GTS) Let the GTS \((X, \gamma)\) be \(\gamma\)-connected and \((Y, \tau)\) be an ordered space. If \(f : X \rightarrow Y\) is \((\gamma, \tau)\)-continuous, \(a, b\) are two distinct points of \(X\) and \(r\) is a point of \(Y\) lying between \(f(a)\) and \(f(b)\), then there exists a point \(c\) in \(X\) such that \(f(c) = r\).

Corollary 4.24. (Intermediate value theorem for topological space) Let \((X, \tau)\) be a connected topological space and \((Y, \tau')\) be an ordered space. If \(f : X \rightarrow Y\) is \((\tau, \tau')\)-continuous, \(a, b\) are two distinct points of \(X\) and \(r\) is a point of \(Y\) lying between \(f(a)\) and \(f(b)\), then there exists a point \(c\) in \(X\) such that \(f(c) = r\).

Recall that, if \((X, \gamma)\) is a GTS and \(\mu\) is the collection of all \(\gamma\)-open sets, then \((X, \gamma)\) and \((X, \mu)\) mean the same GTS, \(i_\gamma = i_\mu\) and \(c_\gamma = c_\mu\) [Lemma 1.13].

Theorem 4.25. Let \((X, \mu)\) be the product GT where \(X = \prod_{\alpha \in \Delta} X_\alpha\) and \((X_\alpha, \mu_\alpha)\) is a strong GTS for each \(\alpha \in \Delta\). If \((X, \mu)\) is a GC space, then \((X_\alpha, \mu_\alpha)\)
is a $GC$ space for every $\alpha \in \Delta$.

**Proof.** By lemma 1.39(f), the projection $p_\alpha : X \to X_\alpha$ is $(\mu, \mu_\alpha)$-continuous, for every $\alpha \in \Delta$. Since generalized continuous image of every $\gamma - GC$ space is $\gamma - GC$, by Theorem 4.15, $(X_\alpha, \mu_\alpha)$ is $GC$ for each $\alpha \in \Delta$.

**Theorem 4.26.** Finite product of $GC$ strong GTS is a $GC$ GTS.

**Proof.** It is enough, if we prove that the product of two $GC$ strong GTS is $GC$.

Let $(X_1, \mu_1)$ and $(X_2, \mu_2)$ be two strong GTS. Fix a point $(a, b) \in X_1 \times X_2$. $X_1 \times b = \{ (x, b) \mid x \in X_1 \}$ is homeomorphic to $X_1$ implies that $X_1 \times b$ is $GC$.

Similarly, $x \times X_2$ is $GC$ for every $x \in X_1$. By corollary 4.7, $T_x = (X_1 \times b) \cup (x \times X_2)$ is $GC$, since $(x, b)$ is in common. $\bigcup_{x \in X_1} T_x$ is $GC$, since $(a, b)$ is in common. $X_1 \times X_2$ is $GC$. By induction, finite product of $GC$ strong GTS is $GC$.

**Theorem 4.27.** Arbitrary product of $GC$ strong GTS is $GC$.

**Proof.** Let $\Delta \neq \emptyset$ be an index set and for each $\alpha \in \Delta$, $(X_\alpha, \mu_\alpha)$ be strong $GC$ GTS. Let $(X, \mu)$ be the product GT. $a = (a_\alpha)$ be any fixed point of $X$ where $a_\alpha \in X_\alpha$ for each $\alpha \in \Delta$. Given any finite subset $K$ of $\Delta$, let $X_K = \{ x \in X \mid x_\alpha = a_\alpha, \alpha \notin K \}$. $X_K$ is homeomorphic to $X = \prod_{k \in K} X_k$. By Theorem 4.26, finite product of $GC$ strong GTS is $GC$. Therefore, $X_K$ is $GC$ for every finite subset $K$ of $\Delta$. $Y = \cup \{ X_K \mid X_K$ is a finite subset of $\Delta \}$ is $GC$, since $a \in X_K$ for every finite subset $K$ of $\Delta$. By Corollary 4.5, $c_\mu(Y)$ is $GC$.

It remains to prove that $c_\mu(Y) = X$. Since $c_\mu(Y) \subset X$, it is enough to prove that
Given \( x \in X \), any basic neighborhood \( B \) of \( x \) will be of the form
\[
B = \prod_{\alpha \in \triangle} U_\alpha
\]
where \( x_\alpha \in U_\alpha \subset \mu_\alpha \) for \( \alpha \in K_0 \) and \( U_\alpha = X_\alpha \) for \( \alpha \in \triangle - K_0 \) where \( K_0 \) is a finite subset of \( \triangle \). \( x \in B \cap X_{K_0} \) implies that \( B \cap X_{K_0} \neq \emptyset \) and hence \( B \cap Y \neq \emptyset \). Therefore, \( x \in c_\mu(Y) \) and so \( X \subset c_\mu(Y) \). Hence \( X \) is GC.

Theorems 4.25 and 4.27 together imply that a product of strong GTS is GC if and only if each coordinate space is GC. If \( X \) is FD with an \( m - \) separation \( \{A_1, A_2, A_3, ..., A_m\} \) and \( Y \) is a GTS, then \( \{A_1 \times Y, A_2 \times Y, A_3 \times Y, ..., A_m \times Y\} \) is an \( m - \) separation for \( X \times Y \). If \( \{M_1, M_2, M_3, ...M_m\} \) is an \( m - \) separation for \( \prod_{i \leq n} X_i \), then \( \{p_j(M_j) \mid j = 1, 2, ..., n\} \) need not be a separation for \( X_j \).

Definition 4.28. A separation of a GD product space \( \prod_{\alpha \in \triangle} X_\alpha \) is called a non-trivial separation if in each member of the separation, each coordinate set is a proper nonempty closed subset of the respective space.

Theorem 4.29. If \( (X_i, \mu_i) \) for \( i \in I_n = \{1, 2, ..., n\} \) where \( r_i \geq 2 \) are FD GTS with \( r_i - \) separations, then \( \prod_{i \in I_n} X_i \) has a nontrivial separation having \( r_1 r_2 ... r_n \) elements.

Proof. \( A_i = \{A_{i_1}, A_{i_2}, A_{i_3}, ..., A_{i_{r_i}}\} \) be an \( r_i \) separation of \( X_i \), \( 1 \leq i \leq n \). Then \( A = \{\prod_{i \in I_n} A_{i_m} \mid 1 \leq m_i \leq r_i\} \) is an \( r_1 r_2 ... r_n \) separation of \( X \).

Theorem 4.30. Let \( (X, \mu) \) be the product GTS where \( X = \prod_{i \in I_n} X_i \) and \( I_n = \{1, 2, ..., n\} \). If \( (X, \mu) \) has a nontrivial \( r - \) separation, then \( (X_i, \mu_i) \) has an \( r_i - \) separation where \( r_i \mid r \).
Proof. If $\mathcal{A} = \{A_1, A_2, ..., A_r\}$ is a nontrivial $r-$ separation of $X$, then $\{p_i(A_j) | 1 \leq j \leq n\}$ is an $r_i-$ separation for $(X_i, \mu_i)$ for some $r_i \geq 2$. By Theorem 4.24, $\mathcal{A} = \{\prod_{i \in I_n} A_i^{m_i} | 1 \leq m_i \leq r_i\}$ is an $r_1r_2...r_n$ separation of $X$. Hence $r_i \mid r$.

Theorem 4.31. Let $(X, \mu)$ be a product GTS where $X = \prod_{i \in I_n} X_i$ and $I_n = \{1, 2, ..., n\}$. If $(X_i, \mu_i)$ is $m_i-$ separated, then $(X, \mu)$ is $m_1m_2...m_n-$ separated.

Proof. If $(X_i, \mu_i)$ is $m_i-$ separated, then $(X, \mu)$ has an $m_1m_2...m_n-$ separation. If it is not the maximum separation, then $X$ has an $m$ separation where $m > m_1m_2...m_n$. Also, $p_j(X) = X_j$ has an $r_j-$ separation where $r_j > m_j$ for some $j$, $1 \leq j \leq n$, which is a contradiction.

Theorem 4.32. The finite product of FD spaces is FD.

Theorem 4.33. If a finite product of GTS is FD, then at least one co-ordinate space is FD.

Theorem 4.34. The countable product of FD spaces is FD and CD.

Proof. Let $(X_i, \mu_i)$ be $m_i-$ separated for each $i \in N$. Let $\mathcal{M}_i = \{M_{i_1}, M_{i_2}, M_{i_3}, ..., M_{i_{m_i}}\}$ be the unique maximum separation of $X_i$. Then $\mathcal{A} = \{\prod A_{ij} | A_{ij} = M_{1j}, 1 \leq j \leq m_1 \text{ and } X_i \text{ for } i \neq 1 \text{ and for all } j\}$ is an $m_1-$ separation of $X$ and $\mathcal{M} = \{\prod M_{ij} | 1 \leq j \leq m_i, i \in N\}$ is an $\aleph_0-$ separation.

The proof of the following Theorems are similar and hence, are omitted.
**Theorem 4.35.** If a countable product of GTS is $FD$, then at least one co-ordinate space is $FD$.

**Theorem 4.36.** The uncountable product of $FD$ spaces is $FD$, $CD$ and $UD$.

**Theorem 4.37.** If an uncountable product of GTS is $FD$, then at least one co-ordinate space is $FD$.

**Theorem 4.38.** The countable product of $CD$ spaces is $CD$.

**Theorem 4.39.** If a countable product of GTS is $CD$, then countable number of co-ordinate spaces are $FD$ or at least one co-ordinate space is $CD$.

**Theorem 4.40.** The uncountable product of $CD$ spaces is $CD$ and $UD$.

**Theorem 4.41.** If an uncountable product of GTS is $CD$, then countable number of co-ordinate spaces are $FD$ or at least one co-ordinate space is $CD$.

**Theorem 4.42.** The uncountable product of $UD$ spaces is $UD$.

**Theorem 4.43.** If an uncountable product of GTS is $UD$, then uncountable number of co-ordinate spaces are $FD$ or $CD$ or at least one co-ordinate space is $UD$.

In the rest of this chapter, we discuss the lattice of all generalized topologies defined on a nonempty set $X$. Let $\mathcal{G}(X)$ be the collection of all generalized topologies on $X$. $\mu \in \mathcal{G}(X)$ is said to be coarser than $\lambda \in \mathcal{G}(X)$ or $\lambda \in \mathcal{G}(X)$ is finer than $\mu \in \mathcal{G}(X)$ if $\mu \subset \lambda$. In the following Theorem 4.44, we prove that the intersection
of two GT is a GT. We shall denote it by $\mu \land \lambda$. Example 4.45 below shows that the union of two GT need not be a GT. In Theorem 4.46 below, we define the join of two GT and prove that the join of any two GT on $X$ is again a GT on $X$.

**Theorem 4.44.** The intersection of two GT on $X$ is the largest GT on $X$ contained in both.

**Proof.** Let $\mu$ and $\lambda$ be any two GT on $X$. Clearly, $\emptyset \in \mu \cap \lambda$. $\{ A_\alpha \mid \alpha \in \Delta \} \subset \mu \cap \lambda$ implies that $A_\alpha \in \mu$ and $A_\alpha \in \lambda$ for every $\alpha \in \Delta$. Hence $\cup \{ A_\alpha \mid \alpha \in \Delta \} \in \mu \cap \lambda$. Let $\delta$ be a GT on $X$ contained in both $\mu$ and $\lambda$. $A \in \delta$ implies that $A \in \mu$ and $A \in \lambda$ which implies that $A \in \mu \cap \lambda$. Hence $\delta \subset \mu \cap \lambda$.

**Example 4.45** Let $X = \mathbb{R}$. Then $\{ \emptyset, \{1\} \}$ and $\{ \emptyset, \{2\} \}$ are GT on $X$ but their union $\{ \emptyset, \{1\}, \{2\} \}$ is not a GT on $X$.

**Theorem 4.46.** Let $\mu$ and $\lambda$ be two GT on $X$. Then their join $\mu \lor \lambda = \{ A \cup B \mid A \in \mu, B \in \lambda \}$ is the smallest GT on $X$ containing both $\mu$ and $\lambda$.

**Proof.** Clearly $\emptyset \in \mu \lor \lambda$. If $\{ C_\alpha \mid \alpha \in \Delta \} \subset \mu \lor \lambda$, then $C_\alpha = A_\alpha \cup B_\alpha$ where $A_\alpha \in \mu$ and $B_\alpha \in \lambda$ for each $\alpha \in \Delta$. $\cup \{ C_\alpha \mid \alpha \in \Delta \} = \cup \{ A_\alpha \cup B_\alpha \mid \alpha \in \Delta \} = [\cup \{ A_\alpha \mid \alpha \in \Delta \}] \cup [\cup \{ B_\alpha \mid \alpha \in \Delta \}] \in \mu \lor \lambda$. Hence $\mu \lor \lambda$ is a GT. If $A \in \mu$, then $A = A \cup \emptyset \in \mu \lor \lambda$. Hence $\mu \subset \mu \lor \lambda$. Similarly, $\lambda \subset \mu \lor \lambda$. Let $\eta$ be any GT containing $\mu$ and $\lambda$. If $C \in \mu \lor \lambda$, then $C = A \cup B$ where $A \in \mu, B \in \lambda$ and so $A, B \in \eta$ which implies that $C = A \cup B \in \eta$. Hence $\mu \lor \lambda \subset \eta$. Therefore, $\mu \lor \lambda$ is the smallest GT on $X$ containing both $\mu$ and $\lambda$. 

The following Theorem 4.47 lists all the properties of ∨ and ∧ in \( G(X) \).

**Theorem 4.47.** Let \( X \) be a nonempty set, and \( \mu, \lambda, \lambda_1, \lambda_2 \) and \( \eta \) be any GT on \( X \). Then the following hold.

(a) \( \mu \lor \mu = \mu \).
(b) \( \mu \land \mu = \mu \).
(c) \( \mu \lor \lambda = \lambda \lor \mu \).
(d) \( \mu \land \lambda = \lambda \land \mu \).
(e) \( (\mu \lor \lambda) \lor \eta = \mu \lor (\lambda \lor \eta) \).
(f) \( (\mu \land \lambda) \land \eta = \mu \land (\lambda \land \eta) \).
(g) \( \mu \lor \{\emptyset\} = \mu \).
(h) \( \mu \land \varnothing(X) = \mu \).
(i) \( \mu \lor \lambda_1 = \mu \lor \lambda_2 \nsim \lambda_1 = \lambda_2 \).
(j) \( \mu \land \lambda_1 = \mu \land \lambda_2 \nsim \lambda_1 = \lambda_2 \).
(k) \( \mu \lor (\mu \land \lambda) = \mu \).
(l) \( \mu \land (\mu \lor \lambda) = \mu \).
(m) \( (\mu \lor \eta) \lor (\lambda \land \eta) \subset (\mu \lor \lambda) \land \eta \).
(n) \( (\mu \lor \lambda) \land \eta \neq (\mu \land \eta) \lor (\lambda \land \eta) \).
(o) \( (\mu \land \lambda) \lor \eta \subset (\mu \lor \eta) \land (\lambda \lor \eta) \).
(p) \( (\mu \land \lambda) \lor \eta \neq (\mu \lor \eta) \lor (\lambda \lor \eta) \).

**Proof.** The proofs of (a), (b), (c), (d), (e), (f), (g), (h), (k) and (l) are clear.

(i) Let \( X = \{a, b, c, d\} \) and \( \mu = \{\emptyset, \{a\}, \{b, c\}, \{a, b, c\} \). If \( \lambda_1 = \{\emptyset, \{c\} \) and \( \lambda_2 = \{\emptyset, \{b, c\} \).
\{\emptyset, \{c\}, \{b, c\}, \{a, b, c\}\}, \text{ then } \mu \lor \lambda_1 = \{\emptyset, \{a\}, \{c\}, \{a, c\}, \{b, c\}, \{a, b, c\}\} = \mu \lor \lambda_2 \text{ but } \lambda_1 \neq \lambda_2.

(j) Let \(X = \{a, b, c, d\}\) and \(\mu = \{\emptyset, \{a\}\}\). If \(\lambda_1 = \{\emptyset, \{b\}\}\) and \(\lambda_2 = \{\emptyset, \{c\}\}\), then \(\mu \land \lambda_1 = \{\emptyset\} = \mu \land \lambda_2 \) but \(\lambda_1 \neq \lambda_2\).

(m) \(A \in (\mu \land \eta) \lor (\lambda \land \eta)\) implies that \(A = C \cup D\) where \(C \in \mu \land \eta\) and \(D \in \lambda \land \eta\). Therefore, \(C \in \mu, D \in \lambda\) and \(C, D \in \eta\) and hence \(C \cup D \in \mu \lor \lambda\) and \(C \cup D \in \eta\) which implies that \(A \in (\mu \lor \lambda) \land \eta\)

(n) Let \(X = \{a, b, c, d\}\) and \(\mu = \{\emptyset, \{a\}, \{a, c\}, X\}\). If \(\lambda = \{\emptyset, \{b\}, \{b, d\}, X\\) and \(\eta = \{\emptyset, \{a, b\}, \{a, b, c\}\}, \text{ then } \mu \lor \lambda = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c\}, \{b, d\}, \{a, b, c\}\} \text{ and so } (\mu \lor \lambda) \land \eta = \{\emptyset, \{a, b\}, \{a, b, c\}\}. \text{ Since } \mu \land \eta = \{\emptyset\} \text{ and } \lambda \land \eta = \{\emptyset\}, \text{ we have } (\mu \land \eta) \lor (\lambda \land \eta) = \{\emptyset\}. \text{ Hence } (\mu \lor \lambda) \land \eta \neq (\mu \land \eta) \lor (\lambda \land \eta).

(o) \(A \in (\mu \land \lambda) \lor \eta\) implies that \(A = C \cup D\) where \(C \in \mu \land \lambda\) and \(D \in \eta\). Therefore, \(C \in \mu\) and \(C \in \lambda\) and \(D \in \eta\), and hence \(C \cup D \in \mu \lor \lambda\) and \(C \cup D \in \lambda \lor \eta\) which implies that \(A \in (\mu \lor \eta) \land (\lambda \lor \eta)\). Therefore, \((\mu \land \lambda) \lor \eta \subseteq (\mu \lor \eta) \land (\lambda \lor \eta)\).

(p) Let \(X = \{a, b, c, d\}, \mu = \{\emptyset, \{a, b\}\}, \lambda = \{\emptyset, \{a, c\}\}\) and \(\eta = \{\emptyset, \{b\}, \{c\}, \{b, c\}\}\). Then \(\mu \lor \eta = \{\emptyset, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, \{b, c\}\}\) and so \((\mu \lor \eta) \land (\lambda \lor \eta) = \{\emptyset, \{b\}, \{c\}, \{b, c\}, \{a, b, c\}\}\). Since \(\mu \land \lambda = \{\emptyset\}, (\mu \land \lambda) \lor \eta = \{\emptyset, \{b\}, \{c\}, \{b, c\}\}\) = \(\eta\). Hence \((\mu \lor \eta) \land (\lambda \lor \eta) \neq (\mu \lor \lambda) \lor \eta\).

**Definition 4.48.** Let \(\mu \in \mathcal{G}(X)\). \(\lambda \in \mathcal{G}(X)\) is said to be a **complement** of \(\mu\) if
\( \mu \lor \lambda = \wp(X) \) and \( \mu \land \lambda = \{\emptyset\} \). We will denote a complement of \( \mu \) by \( \mu^c \).

**Example 4.49.** (a) Let \( X = \{a, b, c\} \) and \( \mu = \{\emptyset, \{a\}, \{a, b\}\} \). Then \( \lambda = \{\emptyset, \{b\}, \{c\}, \{b, c\}\} \) is a complement of \( \mu \).

(b) The co-singleton GT on \( X \) has no complement.

**Example 4.50.** Let \( X \) be a nonempty set and \( Y \) be any proper nonempty subset of \( X \). If \( \mu = \wp(Y) \), then \( \lambda = \wp(X - Y) \) is a complement of \( \mu \).

The following Theorem 4.51 gives a characterization for the existence of a complement of a GT \( \mu \) on \( X \).

**Theorem 4.51.** Let \( X \) be a nonempty set and \( \mu \in \mathcal{G}(X) \). Then \( \mu^c \) exists if and only if for every nonempty set \( A \in \mu \), there exists \( x_0 \in A \) such that \( \{x_0\} \in \mu \).

**Proof.** Let \( \mu \in \mathcal{G}(X) \) such that \( \mu^c \) exists. Suppose there exists a nonempty subset \( A \) of \( X \) such that \( A \in \mu \) and \( \{a\} \notin \mu \) for every \( a \in A \). Then \( \{a\} \in \mu^c \) for every \( a \in A \) which implies \( A \in \mu^c \), a contradiction to the fact that \( \mu \land \mu^c = \emptyset \).

Conversely, suppose that \( \mu \in \mathcal{G}(X) \) satisfies the given condition. Let \( M = \{x \in X \mid \{x\} \in \mu\} \) and \( L = X - M \). Let \( \lambda = \wp(L) \). Then \( \lambda \) is a GT on \( X \). Let \( A \) be a nonempty subset of \( X \) such that \( A \in \mu \land \lambda \). Now \( A \in \mu \) implies that there exists \( x_0 \in A \) such that \( \{x_0\} \in \mu \) which in turn implies that \( x_0 \notin L \) and so \( A \notin \lambda \), a contradiction. Hence \( \mu \land \lambda = \emptyset \). Let \( A \in \wp(X) \). If \( A \in \lambda \), then \( A \in \mu \lor \lambda \). If \( A \notin \lambda \), then \( A = (A \cap M) \cup (A \cap L) \in \mu \lor \lambda \). Therefore, \( \mu \lor \lambda = \wp(X) \). Hence \( \lambda \) is a complement of \( \mu \).
Given $\mu \in \mathcal{G}(X)$, by Example 4.49 (a), (b) $\mu^c$ may or may not exist. Even when $\mu^c$ exists it may or may not be unique. This is illustrated in the following Examples 4.52(a) and 4.52(b).

Example 4.52. (a) Let $X = \{a, b, c, d\}$. $\mu_1 = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$, $\mu_2 = \{\emptyset, \{a\}, \{b\}, \{a, c\}, \{a, b, c\}\}$, $\mu_3 = \{\emptyset, \{a\}, \{b\}, \{b, c\}, \{a, b, c\}\}$, $\mu_4 = \{\emptyset, \{a\}, \{b\}, \{b, d\}, \{a, b, d\}\}$, $\mu_5 = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b, c\}\}$, ... are all complements of $\mu = \wp(L)$ where $L = \{c, d\}$.

(b) Let $X = \{a, b, c, d\}$ and $\mu = \{\emptyset, \{a\}, \{b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}\}$, $\{a, b\}, \{a, c\}, \{b, c\}, \{a, c, d\}, \{a, b, d\}\}$. Then $\mu^c = \wp(L)$ where $L = \{c, d\}$ is the unique complement of $\mu$. In general, for any nonempty subset $L$ of $X$, $\wp(L)$ is the unique complement of $\wp(X) - [\wp(L) - \{\emptyset\}]$.

If $\mu^c$ is not unique, the next natural question is: Is there any smallest complement or largest complement for a given $\mu$? We answer this question in Theorem 4.53 and Example 4.54.

Theorem 4.53. Let $X$ be a nonempty set and $\mu \in \mathcal{G}(X)$ such that $\mu^c$ exists. Then $\wedge\{\mu^c\} = \wp(L)$ is the smallest complement of $\mu$.

Proof. If $\mu = \wp(X)$ then $\{\emptyset\}$ is the only complement. If $\mu \in \mathcal{G}(X)$ and $\{x\} \notin \mu$, then $\{x\} \in \mu^c$. Hence $\{x\} \in \mu^c$ for every $x \in L$. Since $\mu^c$ is a GT, $\wp(L) \subset \mu^c$. Thus $\wp(L) \subset \wedge\{\mu^c\} = \cap\{\mu^c\}$. Since $\wp(L)$ itself is a complement of $\mu$, $\wedge\{\mu^c\} \subset \wp(L)$. Therefore $\wedge\{\mu^c\} = \wp(L)$ is the smallest complement of $\mu$. 

The following Example 4.54 shows that if $\lambda_1$ and $\lambda_2$ are two complements of $\mu \in \mathcal{G}(X)$, then $\lambda_1 \lor \lambda_2$ need not be a complement of $\mu$.

**Example 4.54.** Let $X = \{a, b, c\}$. $\lambda_1 = \{\emptyset, \{c\}, \{b, c\}\}$ and $\lambda_2 = \{\emptyset, \{c\}, \{a, c\}\}$ are two complements of $\mu = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$. $\lambda_1 \lor \lambda_2 = \{\emptyset, \{c\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$ is not a complement of $\mu$. Both $\lambda_1$ and $\lambda_2$ are maximal complements of $\mu$. However, if $\mu = \wp(L)$ for some nonempty subset $L$ of $X$, then $\wp(X) - [\wp(L) - \{\emptyset\}]$ is the maximum complement of $\mu$ and it is the join of all the complements of $\mu$.

**Theorem 4.55.** For a nonempty set $X$, $(\mathcal{G}(X), \lor, \land)$ is a bounded lattice.

**Proof.** By Theorem 4.44, $\mu \land \lambda \in \mathcal{G}(X)$, for all $\mu$ and $\lambda \in \mathcal{G}(X)$. By Theorem 4.46, $\mu \lor \lambda \in \mathcal{G}(X)$, for all $\mu$ and $\lambda \in \mathcal{G}(X)$. By Theorem 4.47 (g),(h) identity elements with respect to $\lor$ and $\land$ exist. Hence $(\mathcal{G}(X), \lor, \land)$ is a bounded lattice.

The lattice $(\mathcal{G}(X), \lor, \land)$ is neither distributive nor complemented. Theorem 4.47 (n), (p) show that it is not a distributive lattice. Example 4.49 (b) shows that it is not complemented.

**Theorem 4.56.** Let $X$ be a nonempty set. If $\mu \in \mathcal{G}(X)$ such that $\mu^c$ exists, then for every subset $A$ of $X$, the following hold.

(a) $i_\mu(A) \cup i_{\mu^c}(A) = A$.

(b) $c_\mu(A) \cap c_{\mu^c}(A) = A$. 
Proof. (a) Let $M = \{x \in X \mid \{x\} \in \mu\}$ and $A \subset X$. Then $\varphi(M) \subset \mu$ and so $A \cap M \in \mu$, which implies $A \cap M \subset i_\mu(A)$. If $L = X - M$, then $\varphi(L) \subset \mu^c$ by Theorem 4.53. Similarly, $A \cap L \subset i_\mu(A)$. Since $M \cup L = X$ and $(A \cap M) \cup (A \cap L) = A$, we have $i_\mu(A) \cup i_\mu(A) = A$.

(b) In (a), replacing $A$ by its complement $X - A$, we get, $i_\mu(X - A) \cup i_\mu(X - A) = X - A$. Taking complements, we get $[X - i_\mu(X - A)] \cap [X - i_\mu(A - A)] = A$. Therefore, $c_\mu(A) \cap c_\mu(A) = A$, by Lemma 1.10(f).

Example 4.57 clearly shows that $i_\mu(A) \cap i_\mu(A)$ need not be $\emptyset$. Theorem 4.58 below gives a necessary and sufficient condition for $i_\mu(A) \cap i_\mu(A)$ to be $\emptyset$ for any subset $A$ of $X$.

Example 4.57. Let $X = \{a, b, c, d\}$ and $\mu = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{a, b, c\}\}$. Consider $\mu^c = \varphi(Y)$ where $Y = \{c, d\}$. If $A = \{a, b, c\}$, then $i_\mu(A) = \{a, b, c\}$ and $i_\mu(A) = \{c\}$ but $i_\mu(A) \cap i_\mu(A) = \{c\}$. Observe that $A = \{a, b, c\} = A \cup \emptyset = \{a, b\} \cup \{c\}$. $A$ can be expressed as a union of an element of $\mu$ and an element of $\lambda$ in more than one way.

Theorem 4.58. Let $X$ be a nonempty set and $A \subset X$. If $\mu \in \mathcal{G}(X)$, such that $\mu^c$ exists, then $A$ can be expressed as a union of an element of $\mu$ and an element of $\mu^c$ in a unique way if and only if $i_\mu(A) \cap i_\mu(A) = \emptyset$.

Proof. Suppose that $i_\mu(A) \cap i_\mu(A) = \emptyset$. If $i_\mu(A) = \emptyset$, then no nonempty subset of $A$ is $\mu$-open and $i_\mu(A) = A$. Therefore, $A = \emptyset \cup A$ is the unique expression.
Similarly, if $i_{\mu^c}(A) = \emptyset$, then $A = A \cup \emptyset$ is the unique expression. If $i_{\mu}(A) = A_1 \neq \emptyset$, $i_{\mu^c}(A) = A_2 \neq \emptyset$ and $A_1 \cap A_2 = \emptyset$, then clearly $A_1 \cup A_2 = A$. Suppose that there exist a $\mu$–open set $B_1$ and a $\mu^c$–open set $B_2$ such that $B_1 \cup B_2 = A$. $B_1 \subset A$ and $B_1$ is $\mu$–open implies that $B_1 \subset A_1$. Similarly, $B_2 \subset A_2$. Therefore, $A_1 \cap A_2 = \emptyset$ implies that $B_1 \cap B_2 = \emptyset$ and so it follows that $B_1 = A_1$ and $B_2 = A_2$. Conversely suppose that every subset of $X$ can be uniquely expressed as a union of a $\mu$–open set and a $\mu^c$–open set. Suppose $i_{\mu}(A) \cap i_{\mu^c}(A) = C$ where $C \neq \emptyset$ for some nonempty subset $A$ of $X$. If $i_{\mu}(A) = A_1$ and $i_{\mu^c}(A) = A_2$, then $A_1 \cup A_2 = A$. By definition, if $M = \{x \in X \mid \{x\} \in \mu\}$ and $L = X - M$, then $(A \cap M) \cup (A \cap L) = A$ where $(A \cap M) \cap (A \cap L) = \emptyset$, $A \cap M$ is $\mu$–open and $A \cap L$ is $\mu^c$–open. Also, $(A \cap M) \neq A_1$ and $(A \cap L) \neq A_2$ since $A_1 \cap A_2 = C$ and $(A \cap M) \cap (A \cap L) = \emptyset$. Thus, $A$ can be expressed as a union of a $\mu$–open set and a $\mu^c$–open set, in more than one way, which is a contradiction.

**Definition 4.59.** If $\mu$ and $\lambda$ are GT on a nonempty set $X$ such that every $A \in \wp(X)$ can be uniquely expressed as a union of a $\mu$–open set and a $\lambda$–open set, then we say that $\wp(X)$ is the direct sum of $\mu$ and $\lambda$ and we write $\wp(X) = \mu \oplus \lambda$.

**Theorem 4.60.** Let $X$ be a nonempty set. Then $\wp(X) = \mu \oplus \lambda$ where $\mu, \lambda \in \mathcal{G}(X)$ if and only if $\mu = \wp(M)$ and $\lambda = \wp(X - M)$ for some $M \subset X$.

**Proof.** Suppose $\mu = \wp(M)$ and $\lambda = \wp(X - M)$ for some $M \subset X$. Clearly, $\mu, \lambda \in \mathcal{G}(X)$ such that $\mu \lor \lambda = \wp(X)$ and $\mu \land \lambda = \{\emptyset\}$. If $A \subset X$, then
\( i_\mu(A) = A \cap M \), and \( i_\lambda(A) = A \cap (X - M) \) and hence \( i_\mu(A) \cap i_\lambda(A) = \emptyset \) for every subset \( A \) of \( X \). Therefore, by Theorem 4.50, \( A \) can be uniquely expressed as a union of a \( \mu \)-open set and a \( \lambda \)-open set and so \( \wp(X) = \mu \oplus \lambda \).

Conversely, suppose that every \( A \in \wp(X) \) can be uniquely expressed as a union of a \( \mu \)-open set and a \( \lambda \)-open set. Therefore, \( \mu \lor \lambda = \wp(X) \) and \( \mu \land \lambda = \{\emptyset\} \).

Let \( M = \{x \in X \mid \{x\} \in \mu\} \) and \( L = X - M \). Then, \( \wp(M) \subset \mu \) and \( \wp(L) \subset \lambda \).

If there exists \( A \in \mu \) such that \( A \neq A \cap M \), then \( A = (A \cap M) \cup A_1 \) for some \( A_1 \in \wp(L) \). \( A \) can be written as \( A = A \cup \emptyset \) where \( A \in \mu \) and \( \emptyset \in \lambda \) and \( A = (A \cap M) \cup A_1 \) where \( A \cap M \in \mu \) and \( A_1 \in \lambda \), a contradiction to the unique representation. Therefore, \( \mu \subset \wp(M) \). Similarly, we can prove that \( \lambda \subset \wp(L) \).

Hence \( \mu = \wp(M) \) and \( \lambda = \wp(X - M) \).