CHAPTER - 2
Finite Separations of Generalized Topological Spaces

In 1914, Hausdorff introduced the concept of separated sets for two subsets in topological spaces. He defined a set to be connected if it cannot be expressed as the union of two separated sets. In 2003, Császár extended this notion to GTS in the name of $\gamma-$ separated sets, again, for two sets only [Definition 1.29]. If a topological space can be expressed as a disjoint union of a finite number of proper nonempty closed sets, then it can be expressed as a disjoint union of two proper nonempty closed sets. Hence, we called a topological space connected if it cannot be expressed as a disjoint union of two proper nonempty closed sets. This will not hold for a generalized topological space (GTS) [Definition 1.11] introduced and studied by Császár, since finite union of $\gamma-$ closed sets in a GTS $(X, \mu)$ need not be $\gamma-$ closed. A GTS can be expressed as a disjoint union of a finite number of proper nonempty $\gamma-$ closed sets, even when it cannot be expressed as a disjoint union of two proper nonempty $\gamma-$ closed sets. Thus, it is possible to redefine connectedness in the case of GTS. In this chapter, we extend the concept of separated sets to any finite number of subsets and define $n-$ separation. We introduce the concepts of $\gamma-$ finitely disconnected GTS, $\gamma-$ finitely connected GTS and maximum separation. We characterize these concepts and give examples wherever necessary. Most of the results of this chapter are from [2].
The following Example 2.1 gives a GTS \((X, \gamma)\) in which \(\gamma \in \Gamma_1\) and \(X\) cannot be written as the union of either two disjoint nonempty \(\gamma\)-open sets or two disjoint nonempty \(\gamma\)-closed sets and so \(X\) is \(\gamma\)-connected [Definition 1.32]. Note that, \(X\) cannot be written as a disjoint union of \(n\) nonempty \(\gamma\)-open sets but \(X\) can be written as a disjoint union of \(n\) nonempty \(\gamma\)-closed sets. This motivates us to define \(n\)-separation of a GTS for any positive integer \(n \geq 2\) and \(\gamma\)-finitely disconnected GTS.

**Example 2.1.** Let \(X = I_n = \{1, 2, 3, \ldots, n\}\). Define \(\eta : \wp(I_n) \to \wp(I_n)\) by
\[
\eta(A) = A \text{ if } I_n - \{i\} \subseteq A \text{ for some } i \in I_n \text{ and } \eta(A) = \emptyset, \text{ if otherwise.}
\]
Then \(\eta \in \Gamma_1\) and \(\mu = \{\emptyset, X\} \cup \{A \subset I_n \mid A = I_n - \{i\}, i = 1, 2, 3, \ldots n\}\). We will call this generalized topology as the *co-singleton generalized topology defined on a finite set*. The only \(\eta\)-closed sets are \(\emptyset, X\) and singleton subsets of \(I_n\). Similarly, we can define *co-singleton generalized topology on any countable or uncountable set*. Note that, when \(n = 2\), \(\mu\) turns out to be the discrete topology.

**Definition 2.2.** Let \((X, \gamma)\) be a GTS where \(\gamma \in \Gamma\). A family \(\{A_1, A_2, A_3, \ldots, A_n\}\) of nonempty subsets of \(X\) where \(n \geq 2\) is called an \(n\)-separation of \(X\) if
\[
X = \bigcup \{A_i \mid i = 1, \ldots, n\} \text{ and } c_\gamma(A_i) \cap A_j = \emptyset \text{ for all } i \neq j.
\]

When \(n = 2\), \(\{A_1, A_2\}\) becomes the classical 2-separation. The following Theorem 2.3 characterizes \(n\)-separation of \(X\). The proof of Corollary 2.4 below follows from Theorem 2.3.
Theorem 2.3. Let \((X, \gamma)\) be a GTS where \(\gamma \in \Gamma\). Then the following are equivalent.

(a) \(\{A_1, A_2, A_3, \ldots, A_n\}\) is an \(n\)-separation of \(X\).

(b) \(\{A_1, A_2, A_3, \ldots, A_n\}\) is a collection of nonempty, mutually disjoint \(\gamma\)-closed subsets of \(X\) whose union is \(X\).

(c) \(\{X - A_1, X - A_2, X - A_3, \ldots, X - A_n\}\) is a collection of proper \(\gamma\)-open subsets of \(X\) such that \(\cap\{X - A_i \mid i = 1, 2, \ldots, n\} = \emptyset\) and the union of any two such sets is \(X\).

Proof. (a) \(\Rightarrow\) (b). If \(\{A_1, A_2, A_3, \ldots, A_n\}\) is an \(n\)-separation of \(X\), then \(X = \cup\{A_i \mid i = 1, \ldots, n\}\) and \(c_\gamma(A_i) \cap A_j = \emptyset\) for all \(j \neq i\). Now, for all \(j \neq i\), \(c_\gamma(A_i) \cap A_j = \emptyset\) implies that \(c_\gamma(A_i) \cap (\cup\{A_j \mid j = 1, 2, \ldots, n, j \neq i\}) = \emptyset\) and so \(c_\gamma(A_i) \cap (X - A_i) = \emptyset\). Hence \(c_\gamma(A_i) \subset A_i\) and so \(A_i\) is \(\gamma\)-closed. By the definition of \(n\)-separation, \(\{A_1, A_2, A_3, \ldots, A_n\}\) is a collection of nonempty, mutually disjoint \(\gamma\)-closed subsets of \(X\) whose union is \(X\).

(b) implies (a), and (b) and (c) are equivalent, are clear.

Corollary 2.4. Let \((X, \gamma)\) be a GTS where \(\gamma \in \Gamma\). Then the following are equivalent.

(a) \(X\) has a \(2\)-separation.

(b) \(X\) is the union of two disjoint nonempty \(\gamma\)-closed sets.

(c) \(X\) is the union of two disjoint proper \(\gamma\)-open sets.

(d) \(X\) has a proper nonempty set which is both \(\gamma\)-open and \(\gamma\)-closed.
Corollary 2.5. If \((X, \gamma)\) is a GTS with an \(n\)-separation, then \(X \in \mu\).

**Proof.** The Proof follows from Theorem 2.3(c).

The converse of Corollary 2.5 is not true, since the real line with the usual topology has no \(n\)-separation for any positive integer \(n \geq 2\). In Example 2.1, \(X\) has an \(n\)-separation but has no \(m\)-separation, \(2 \leq m \leq n - 1\). The following Theorem 2.6 shows that if one of the elements of an \(n\)-separation of \(X\), \(n > 2\), is \(\gamma\)-open, then \(X\) has a \(2\)-separation.

**Theorem 2.6.** Let \((X, \gamma)\) be a GTS with an \(n\)-separation \(\{A_1, A_2, A_3, \ldots, A_n\}\) of \(X\). If one of the elements \(A_i\) of the \(n\)-separation of \(X\) is \(\gamma\)-open for some \(i\), then \(X\) has a \(2\)-separation.

**Proof.** If one of the elements \(A_i\) of the \(n\)-separation \(\{A_1, A_2, A_3, \ldots, A_n\}\) of \(X\) is \(\gamma\)-open for some \(i\), then it is a proper nonempty subset of \(X\) which is both \(\gamma\)-open and \(\gamma\)-closed. The proof follows from Theorem 2.4(d).

Example 2.7 below shows that \(X\) has a \(4\)-separation as well as a \(2\)-separation, but no element of the \(4\)-separation is \(\gamma\)-open and so the converse of Theorem 2.6 is not true.

**Example 2.7.** Let \(X = \{a, b, c, d, e\}\) and the family of all \(\gamma\)-open sets be \(\mu = \{\emptyset, \{a, b\}, \{a, b, c\}, \{c, d, e\}, \{a, b, d, e\}, \{a, c, d, e\}, \{b, c, d, e\}, X\}\). Then the family of all \(\gamma\)-closed sets are given by \(\{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{d, e\}, \{c, d, e\}, X\}\).
\{a\}, \{b\}, \{c\}, \{d, e\} is a 4-separation of \(X\), \{a, b\}, \{c, d, e\} is a 2-separation of \(X\) and no element of the 4-separation of \(X\) is \(\gamma\)-open.

**Theorem 2.8.** Let \((X, \gamma)\) be a GTS with an \(n\)-separation \(\{A_1, A_2, A_3, \ldots, A_n\}\) of \(X\). If there exists \(r\) such that \(1 \leq r \leq n-1\) and the union of some \(r\) elements of the separation is both \(\gamma\)-open and \(\gamma\)-closed, then \(X\) has a 2-separation.

**Proof.** For \(r = 1\), the theorem is proved in Theorem 2.6. If there exists \(r\) such that \(2 \leq r \leq n-1\) and the union of some \(r\) elements of the separation \(\{A_1, A_2, A_3, \ldots, A_n\}\) of \(X\) is both \(\gamma\)-open and \(\gamma\)-closed, then \(X\) has a proper nonempty subset which is both \(\gamma\)-open and \(\gamma\)-closed. The proof follows from Theorem 2.4(d).

In fact Theorem 2.8 can be stated as follows.

"Let \((X, \gamma)\) be a GTS with an \(n\)-separation \(\{A_1, A_2, A_3, \ldots, A_n\}\). If there exists \(r\) such that \(1 \leq r \leq \left\lfloor \frac{n}{2} \right\rfloor\), where \(\left\lfloor \frac{n}{2} \right\rfloor\) is the greatest integer less than or equal to \(\frac{n}{2}\) and the union of some \(r\) elements of the separation is both \(\gamma\)-open and \(\gamma\)-closed, then \(X\) has a 2-separation."

**Definition 2.9** Let \((X, \gamma)\) be a GTS and \(A\) be a subset of \(X\). A subset \(A\) of \(X\) is said to be an \(F_s\)-set if \(A\) is a finite union of \(\gamma\)-closed sets. \(A\) is said to be an \(F^{*}_s\)-set if \(A\) is a finite union of nonempty mutually disjoint \(\gamma\)-closed sets. \(A\) is said to be a \(G_d\)-set if \(A\) is a finite intersection of \(\gamma\)-open sets. \(A\) is said to be a \(G^{*}_d\)-set if \(A\) is a finite intersection of proper \(\gamma\)-open sets such that the union of any two sets is \(X\). Clearly, every \(\gamma\)-closed set is an \(F_s\)-set, every nonempty
γ-closed set is an $F_s^*$-set, every γ-open set is a $G_d$-set and every proper γ-open set is a $G_d^*$-set.

**Example 2.10.** Consider the GTS given in Example 2.7.

\{b, c\} = \{b\} \cup \{c\} is an $F_s$ (resp. $F_s^*$)-set which is neither γ-open nor γ-closed.

\{a, b, c\} = \{a\} \cup \{b\} \cup \{c\} is an $F_s$ (resp. $F_s^*$)-set which is γ-open but not γ-closed. \{a, b\} = \{a\} \cup \{b\} is an $F_s$ (resp. $F_s^*$)-set which is both γ-open and γ-closed. \{a, d, e\} = \{a\} \cup \{c, d, e\} \cap \{a, b, d, e\} is a $G_d$ (resp. $G_d^*$)-set which is neither γ-open nor γ-closed. \{d, e\} = \{b, c, d, e\} \cap \{a, c, d, e\} \cap \{a, b, d, e\} is a $G_d$ (resp. $G_d^*$)-set which is γ-closed but not γ-open. \{c, d, e\} = \{b, c, d, e\} \cap \{a, c, d, e\} is a $G_d$ (resp. $G_d^*$)-set which is both γ-open and γ-closed.

Note that, in topological spaces every $F_s$ (resp. $F_s^*$)-set is a closed set and every $G_d$ (resp. $G_d^*$)-set is an open set.

**Theorem 2.11.** Let $(X, \gamma)$ be a GTS. Then the following hold.

(a) If $A \subset X$, then $A$ is an $F_s$-set if and only if $X - A$ is a $G_d$-set.

(b) If $A$ is a nonempty subset of $X$, then $A$ is an $F_s^*$-set if and only if $X - A$ is a $G_d^*$-set.

**Proof.** (a) $A$ is an $F_s$-set if and only if $A = \bigcup \{A_i \mid 1 \leq i \leq n\}$ for some positive integer $n$ where each $A_i$ is γ-closed, if and only if $X - A = [X - \bigcup \{A_i \mid 1 \leq i \leq n\}] = \bigcap \{X - A_i \mid 1 \leq i \leq n\}$, if and only if $X - A$ is a $G_d$-set.

(b). A nonempty subset $A$ of $X$ is an $F_s^*$-set if and only if $A = \bigcup \{A_i \mid 1 \leq i \leq n\}$
for some positive integer \( n \) where \( A_i, \quad 1 \leq i \leq n \) are nonempty mutually disjoint \( \gamma \)-closed sets if and only if \( X - A = [X - \cup\{A_i \mid 1 \leq i \leq n\}] = \cap\{X - A_i \mid 1 \leq i \leq n\} \) where \( X - A_i, \quad 1 \leq i \leq n \) are proper \( \gamma \)-open sets such that the union of any two sets is \( X \) if and only if \( X - A \) is a \( G^*_d \)-set.

**Definition 2.12** A GTS \( (X, \gamma) \) is said to be a \( \gamma \)-finitely disconnected space or \textit{FD space} if \( X \) has an \( n \)-separation for some \( n \geq 2 \).

The following Theorem 2.13 gives characterizations of \textit{FD spaces}.

**Theorem 2.13.** Let \( (X, \gamma) \) be a GTS. Then the following are equivalent.

(a) \( X \) is an \textit{FD space}.

(b) There exists a proper \( \gamma \)-open \( F^*_s \)-set in \( X \).

(c) There exists a nonempty \( \gamma \)-closed \( G^*_d \)-set in \( X \).

**Proof.** (a) \( \Rightarrow \) (b). Suppose \( X \) is an \textit{FD space}. Let \( \{A_1, A_2, A_3, \ldots, A_n\} \) be an \( n \)-separation of \( X \). By Theorem 2.3(b), \( \{A_1, A_2, A_3, \ldots, A_n\} \) is a collection of nonempty, mutually disjoint \( \gamma \)-closed subsets of \( X \) whose union is \( X \). For \( 1 \leq j \leq n, X - A_j \) is \( \gamma \)-open and \( X - A_j = \cup\{A_i \mid i = 1, 2, \ldots, n, i \neq j\} \). So \( X - A_j \) is a proper \( \gamma \)-open \( F^*_s \)-set in \( X \).

(b) \( \Rightarrow \) (a). If \( A \) is a proper \( \gamma \)-open \( F^*_s \)-set in \( X \), then \( A = \cup\{A_i \mid i = 1, 2, \ldots, n\} \) where each \( A_i \) is a proper nonempty \( \gamma \)-closed set such that \( A_i \cap A_j = \emptyset \) for \( i, j = 1, 2, \ldots, n \) and \( i \neq j \). Then \( \{X - A, A_1, A_2, A_3, \ldots, A_n\} \) is an \( (n+1) \)-separation of \( X \). Therefore, \( X \) is an \textit{FD space}.
(b) and (c) are equivalent by Theorem 2.11(b).

Let \((X, \gamma)\) be a GTS. Then \(X\) is not \(\gamma\)-connected if and only if \(X\) has a 2- separation. Hence we get the following corollary.

**Corollary 2.14.** Let \((X, \gamma)\) be a GTS. Then \(X\) is not \(\gamma\)-connected if and only if there exists a proper nonempty subset of \(X\) which is both \(\gamma\)-open and \(\gamma\)-closed.

Let \((X, \tau)\) be a topological space. If \(\gamma = \text{int}\), the interior operator, then the family of all \(\gamma\)-open sets coincides with \(\tau\) and \(c_\gamma = cl\), the closure operator. Therefore, \((X, \tau)\) is disconnected if and only if \(X\) has a 2- separation. Since in a topological space, each \(F_s^*\)-set is a closed set, the following Corollary 2.15 follows from Theorem 2.8.

**Corollary 2.15.** Let \((X, \tau)\) be a topological space. Then \(X\) is a disconnected space if and only if there exists a proper nonempty subset of \(X\) which is both \(\tau\)-open and \(\tau\)-closed.

**Theorem 2.16.** A GTS \((X, \gamma)\) is an FD space if and only if there exists a \((\gamma, \eta)\)-continuous onto function \(f : (X, \gamma) \rightarrow (I_n, \eta)\) for some \(n \geq 2\).

**Proof.** Suppose \((X, \gamma)\) is a \(\gamma\)-FD space. Then, \(X\) has an \(n\)-separation \(\{A_1, A_2, A_3, \ldots, A_n\}\). Define \(f : X \rightarrow I_n\) by \(f(x) = i\), if \(x \in A_i\). Clearly \(\emptyset, I_n\) and all the singleton subsets are the only \(\eta\)-closed sets. Since \(f^{-1}(\emptyset) = \emptyset, f^{-1}(I_n) = X\) and \(f^{-1}(\{i\}) = A_i\) for every \(i = 1, 2, 3, \ldots, n\), it follows that in-
verse image of every $\eta$– closed set is $\gamma$– closed and hence $f$ is a $(\gamma, \eta)$– continuous surjection. Conversely, if $f$ is a $(\gamma, \eta)$– continuous surjection, then $\{f^{-1}\{i\} \mid i = 1, 2, \ldots, n\}$ is a collection of $n$ nonempty disjoint $\gamma$– closed subsets of $X$ whose union is $X$ and hence is a $n$– separation of $X$. Therefore $(X, \gamma)$ is an $FD$ space.

**Corollary 2.17.** A GTS $(X, \gamma)$ has a $2$– separation if and only if there exists a $(\gamma, \eta)$– continuous onto function $f : X \to I_2$.

**Corollary 2.18.** A topological space $(X, \tau)$ is disconnected if and only if there exists a continuous onto function $f : X \to I_2$.

**Definition 2.19** A GTS $(X, \gamma)$ is said to be a $\gamma$– finitely connected space (simply, FC space) if $X$ has no $n$– separation for any positive integer $n \geq 2$.

The following Example 2.20 gives an example of an FC space. Theorem 2.21 gives characterizations of FC spaces, the proof of which follows from Theorem 2.13. A GTS $(X, \gamma)$ is $\gamma$– connected in the sense of Császár [Definition 1.32] if and only if $(X, \gamma)$ has no $2$– separation and so the proof of Corollary 2.21 which is the essence of Lemma 1.33 follows from Corollary 2.4.

**Example 2.20.** Let $X = \mathbb{R}$, the real line. Define $\gamma : \wp(X) \to \wp(X)$ by $\gamma(A) = A - Z$ for every $A \subset X$, where $Z$ is the set of all integers. Clearly, $G$ is $\gamma$– open if and only if $G \subset \mathbb{R} - Z$ and $F$ is $\gamma$– closed if and only if $F \supset Z$. Hence any two proper $\gamma$– closed sets intersect and so $X$ has no $n$– separation for any $n \geq 2$. Therefore, $X$ is FC.
**Theorem 2.21.** Let \((X, \gamma)\) be a GTS. Then the following are equivalent.

(a) \(X\) is an \(\gamma\)-FC space.

(b) There exists no proper \(\gamma\)-open \(F^*_\gamma\)-set in \(X\).

(c) There exists no nonempty \(\gamma\)-closed \(G^*_\gamma\)-set in \(X\).

**Corollary 2.22.** Let \((X, \gamma)\) be a GTS. Then the following are equivalent.

(a) \(X\) is a \(\gamma\)-connected space.

(b) There exists no proper nonempty subset of \(X\) which is both \(\gamma\)-closed and \(\gamma\)-open.

(c) \(X = A \cup B, \ c_\gamma(A) \cap B = \emptyset = A \cap c_\gamma(B)\) implies that either \(A = X\) or \(A = \emptyset\).

In Example 2.20, \((X, \gamma)\) is an \(FC\) space in which \(X\) is not \(\gamma\)-open but in any topological space \((X, \tau)\), \(X\) is always open. Hence we have the following Corollary 2.23.

**Corollary 2.23.** Let \((X, \tau)\) be a topological space. Then the following are equivalent.

(a) \(X\) is connected.

(b) There exists no proper nonempty subset of \(X\) which is both closed and open.

(c) \(\emptyset\) and \(X\) are the only subsets of \(X\) which are both closed and open.

**Definition 2.24.** A GTS \((X, \gamma)\) is said to be a \(n\)-separated space if \(n\) is the largest integer such that \(X\) has an \(n\)-separation.

The GTS \((\mathcal{I}_n, \eta)\) of Example 2.1 is an \(n\)-separated space and has no
$m$–separation for any $m$ where $2 \leq m \leq n - 1$ and it follows that if $(X, \gamma)$ is an $n$–separated space, then $X$ has at least $n$ elements. Example 2.25 below shows that if a GTS $X$ has an $n$–separation, then $X$ may or may not have $m$–separation where $2 \leq m \leq n - 1$. For topological spaces, the situation is different. An $n$–separated topological space $(X, \tau)$ has $m$–separations for all $m$ where $2 \leq m \leq n - 1$, since finite union of closed sets is closed. The following Example 2.26 shows that $m$–separations, $2 \leq m \leq n - 1$, of an $n$–separated GTS need not be unique when they exist.

**Example 2.25.** Consider the GTS $(X, \mu)$ where $X = \{a, b, c, d\}$ and $\mu = \{\emptyset, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, X\}$. Then $\mu$–closed sets are $\emptyset, \{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, X$. Hence $\{\{a\}, \{b\}, \{c\}, \{d\}\}$ is the unique $4$–separation. $\{\{a\}, \{c\}\}$ is the unique $3$–separation. $X$ has no $2$–separation at all.

**Example 2.26.** Consider the GTS $(X, \mu)$ where $X = \{a, b, c, d\}$ and $\mu = \{\emptyset, \{a, b\}, \{a, c\}, \{b, d\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, X\}$. $\mu$–closed sets are: $\emptyset, \{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{a, c\}, \{b, d\}, \{c, d\}, X$. Then $\{\{a\}, \{b\}, \{c\}, \{d\}\}$ is the unique $4$–separation. $\{\{a\}, \{b\}, \{c, d\}\}, \{\{a\}, \{c\}, \{b, d\}\}, \{\{b\}, \{d\}, \{a, c\}\}$ and $\{\{c\}, \{d\}, \{a, b\}\}$ are four $3$–separations of $X$. $\{\{a, b\}, \{c, d\}\}$ and $\{\{a, c\}, \{b, d\}\}$ are two $2$–separations of $X$.

**Example 2.27.** Consider $\mathbb{R}$ with the usual topology. $X = [0, 1] \cup [2, 3] \cup [4, 5]$ is a subspace of $\mathbb{R}$. Note that $X$ is $\gamma$–FD. $\{[0, 1], [2, 3], [4, 5]\}$ is the unique
3-separation. \( \{[0, 1] \cup [2, 3], [4, 5]\}, \{[0, 1], [2, 3] \cup [4, 5]\} \) and \( \{[2, 3], [4, 5] \cup [0, 1]\} \) are three 2-separations of \( X \).

In Theorem 2.28 below, we give a technique to get a finite separation of \((X, \gamma)\) given two finite separations of \((X, \gamma)\).

**Theorem 2.28.** Let \((X, \gamma)\) be a GTS. If \( A = \{A_1, A_2, A_3, \ldots, A_m\} \) and \( B = \{B_1, B_2, B_3, B_4, \ldots, B_n\} \) are two finite separations of \( X \), then \( A \land B = \{A_i \cap B_j \mid A_i \cap B_j \neq \emptyset, i = 1, 2, \ldots, m; j = 1, 2, \ldots n\} \) is a \( k \)-separation of \( X \) where \( k \geq \max\{m, n\} \).

**Proof.** Let \( C = A \land B = \{A_i \cap B_j \mid A_i \cap B_j \neq \emptyset, i = 1, 2, \ldots, m; j = 1, 2, \ldots n\} \). Since intersection of closed sets is closed, \( C \) is a collection of closed sets. By construction, each member of \( C \) is nonempty. \( A_i \) are disjoint and \( B_j \) are disjoint imply that \( A_i \cap B_j \) are disjoint. Also \( \cup\{A_i \cap B_j \mid 1 \leq i \leq m; 1 \leq j \leq n\} = \cup\{A_i \cap B_j \mid 1 \leq j \leq n\} \cup\{A_i \mid 1 \leq i \leq m\} = X \). Clearly, \( C \) is a \( k \)-separation of \( X \) where \( k \geq \max\{m, n\} \).

In Theorem 2.29 below, we prove that the \( n \)-separation of an \( n \)-separated GTS is unique. Hence, it may fittingly be called the maximum separation of \( X \).

**Theorem 2.29.** Let \((X, \gamma)\) be an \( n \)-separated GTS. Then the \( n \)-separation of \( X \) is unique.

**Proof.** Let \((X, \gamma)\) be an \( n \)-separated GTS. Suppose that \( \{A_1, A_2, A_3, A_4, \ldots, A_n\} \)
and \( \{B_1, B_2, B_3, B_4, \ldots, B_n\} \) are two \( n \)-separations of \( X \). If for some \( i \), \( A_i \not\subset B_j \) for any \( j \), then \( U = \{A_i \cap B_j \mid A_i \cap B_j \neq \emptyset, \ j = 1, 2, \ldots n\} \) contains at least two elements, since \( A \subset \cup \{B_j \mid 1 \leq j \leq n\} \). Hence \( \{A_1, A_2, \ldots A_{i-1}, A_{i+1}, \ldots, A_n\} \cup U \) is a separation of \( X \) having more than \( n \) elements which is a contradiction. So \( A_i \subset B_j \) for some \( j \). Similarly, we can prove that \( B_j \subset A_k \) for some \( k \). \( i = k \) for otherwise, \( A_i \subset A_k \), a contradiction. Therefore, \( A_i = B_j \) and the \( n \)-separation of an \( n \)-separated GTS \( X \) is unique.

The converse of the above Theorem 2.29 is not true. A GTS \( (X, \gamma) \) has a unique \( n \)-separation does not imply that it is \( n \)-separated. In Example 2.25, \( X \) has a **unique** \( 3 \)-**separation** but it is \( 4 \)-**separated**. Theorem 2.30 gives the way to get the maximum separation from a given finite separation of a **FD space**. This is illustrated in Example 2.31 and 2.32.

**Theorem 2.30.** Let \( (X, \gamma) \) be a **FD space**. A separation \( B = \{B_1, B_2, \ldots, B_m\} \) of \( X \) is not the maximum separation of \( X \) if and only if \( B_k \) has a finite separation by \( \gamma \)-closed sets for some \( 1 \leq k \leq m \).

**Proof.** If \( A = \{A_1, A_2, A_3, \ldots, A_n\} \) is the maximum separation for \( X \), then \( n > m \) and \( \{A_i \cap B_j : A_i \cap B_j \neq \emptyset, 1 \leq i \leq n, 1 \leq j \leq m\} \) is a finite separation of \( X \) containing at least \( n \) elements. This implies that for some \( k, 1 \leq k \leq m \), \( A_i \cap B_k \neq \emptyset \) for more than one \( i \). Clearly, \( \{A_i \cap B_k : A_i \cap B_k \neq \emptyset, 1 \leq i \leq n\} \) is a finite separation by \( \gamma \)-closed sets for \( B_k \). The proof of the converse is clear.
Example 2.31. Consider the GTS given in in Example 2.26. \(\{\{a\}, \{b\}, \{c,d\}\}, \{\{a\}, \{c\}, \{b,d\}\}, \{\{b\}, \{d\}, \{a,c\}\} \) and \(\{\{c\}, \{d\}, \{a,b\}\}\) are four \(3\) separations of \(X\). In \(\{\{a\}, \{b\}, \{c,d\}\}\), \(\{c, d\}\) can be expressed as a union of two \(\gamma\) closed sets. \(\{\{a, b\}, \{c, d\}\}\) and \(\{\{a, c\}, \{b, d\}\}\) are two \(2\) separations of \(X\). In \(\{\{a, b\}, \{c, d\}\}\), \(\{a, b\} = \{a\} \cup \{b\}\) and \(\{c, d\} = \{c\} \cup \{d\}\). \(\{\{a\}, \{b\}, \{c\}, \{d\}\}\) is the unique \(4\) separation. In \(\{\{a, c\}, \{b, d\}\}\), \(\{a, c\} = \{a\} \cup \{c\}\) and \(\{b, d\} = \{b\} \cup \{d\}\). \(\{\{a\}, \{b\}, \{c\}, \{d\}\}\) is the unique \(4\) separation.

Example 2.32. Consider the topological space given in Example 2.27.

\(\{[0, 1] \cup [2, 3], [4, 5]\}\), \(\{[0, 1], [2, 3] \cup [4, 5]\}\) and \(\{[2, 3], [4, 5] \cup [0, 1]\}\) are three \(2\) separations of \(X = [0, 1] \cup [2, 3] \cup [4, 5]\). Each of \([0, 1] \cup [2, 3]\), \([2, 3] \cup [4, 5]\) and \([4, 5] \cup [0, 1]\) is a union of two disjoint closed subsets. \(\{[0, 1], [2, 3], [4, 5]\}\) is the unique \(3\) separation.