CHAPTER - 5
Separation axioms and convergence in GTS

Cluster points and limits of sequences are well developed concepts in topological spaces. We intend to extend these concepts to generalized topological spaces (GTS) in this chapter. Distance concept in metric spaces is replaced by neighborhoods in topological spaces. In topological spaces, every point has a neighborhood. The strength as well as weakness of a GTS is that not all points have neighborhoods containing them. Also, the intersection of any two neighborhoods of a point \( x \) need not be a neighborhood of \( x \). The task before us is checking whether the existing definition can be interpreted to suit the new situation. Most of the results in this chapter are from [3]. A sequence \((x_n)\) converges to \( x \) if \((x_n)\) is eventually in every neighborhood of \( x \). In this case, \( x \) is called a limit of the sequence \((x_n)\). \( x \) is called a limit point of \((x_n)\) if it is frequently in every neighborhood of \( x \). Throughout this chapter, the GTS \((X, \gamma)\) for \( \gamma \in \Gamma \) may also be denoted by \((X, \mu)\) where \( \mu \) is the family of all \( \gamma \)-open sets.

Consider \( X = \mathbb{R} \), the set of all real numbers with the \( \mathbb{Z} \) forbidden GTS where \( \mathbb{Z} \) is the set of all integers. \( \mu = \varnothing (\mathbb{R} - \mathbb{Z}) \) is the family of all \( \mu \)-open sets. Note that \( X \not\in \mu \), \( c_\mu(A) = A \cup \mathbb{Z} \) for every subset \( A \) of \( X \) and \( \mu(z) = \emptyset \) for every \( z \in \mathbb{Z} \). This justifies the following definition of limit point of a set \( A \) in a GTS \((X, \mu)\).
Definition 5.1. Let \((X, \mu)\) be a GTS and \(A \subset X\). A point \(x \in X\) is a limit point or cluster point of \(A\) if \(A \cap (U - \{x\}) \neq \emptyset\) for every neighborhood \(U\) of \(x\). \(x\) is not a limit point or cluster point of \(A\) if \(A \cap (U - \{x\}) = \emptyset\) for some neighborhood \(U\) of \(x\). The collection of all limit points of a subset \(A\) of \(X\) is denoted by \(A'\) and is called the derived set of \(A\). It is but natural to expect the following results in any GTS.

Theorem 5.2. Let \((X, \mu)\) be a GTS and \(A, B \subset X\). Then the following hold.
(a) If \(A \subset B\), then \(A' \subset B'\).
(b) \(A' \cup B' = (A \cup B)'\).
(c) \((A \cap B)' \subset A' \cap B'\).

Proof. (a) Let \(x \in X\). If \(x \notin B'\), then there exists a neighborhood \(U\) of \(x\) such that \(B \cap (U - \{x\}) = \emptyset\). Therefore \(U\) is a neighborhood of \(x\) such that \(A \cap (U - \{x\}) = \emptyset\). Hence \(x \notin A'\).
(b) Since \(A \subset A \cup B\), by (a), \(A' \subset (A \cup B)'\). Similarly, \(B' \subset (A \cup B)'\). Hence \(A' \cup B' \subset (A \cup B)'\). Conversely, suppose \(x \in (A \cup B)'\). Then for every \(\mu\)– open set \(V\) containing \(x\), \((A \cup B) \cap (V - \{x\}) \neq \emptyset\). This implies that either \(A \cap (V - \{x\}) \neq \emptyset\) or \(B \cap (V - \{x\}) \neq \emptyset\) and so either \(x \in A'\) or \(x \in B'\). Hence \(x \in A' \cup B'\) and so \((A \cup B)' \subset A' \cup B'\).
(c) Since \(A \cap B \subset A, B\), by (a), \((A \cap B)' \subset A', B'\) and so \((A \cap B)' \subset A' \cap B'\).

The following Example 5.3 shows that equality does not hold in Theorem
5.2(c).

**Example 5.3.** Consider \((\mathbb{R}, \eta)\), the co-singleton GTS on \(\mathbb{R}\). \(\{1, 2\}' = \mathbb{R} - \{1, 2\}\), for every neighborhood of \(x \in \mathbb{R} - \{1, 2\}\) contains either 1 or 2, 1 has a neighborhood not containing 2 and 2 has a neighborhood not containing 1. Similarly \(\{3, 4\}' = \mathbb{R} - \{3, 4\}\), \(\{1, 2\}' \cap \{3, 4\}' = \mathbb{R} - \{1, 2, 3, 4\}\), \((\{1, 2\} \cap \{3, 4\})' = \emptyset\).

**Theorem 5.4.** Let \((X, \mu)\) be a GTS and \(A \subset X\). Then \(c_\mu(A) = A \cup A'\).

**Proof.** Clearly, \(A \subset c_\mu(A)\). Next, we prove that \(A' \subset c_\mu(A)\). \(x \notin c_\mu(A)\) implies that there exists a \(\mu\)– open set \(U\) containing \(x\) not intersecting \(A\) by Lemma 1.8.(e) and so \(A \cap (U - \{x\}) = \emptyset\). Therefore, \(x \notin A'\) and hence \(A \cup A' \subset c_\mu(A)\). Conversely, let \(x \in c_\mu(A)\). If \(x \in A\), then there is nothing to prove. Consider \(x \notin A\). If \(x \notin A'\), then there exists a \(\mu\)– open set \(U\) containing \(x\) not intersecting \(A\) at any point other than \(x\) and so \(U \cap A = \emptyset\) which implies that \(x \notin c_\mu(A)\). Hence \(c_\mu(A) = A \cup A'\).

**Corollary 5.5.** Let \((X, \mu)\) be a GTS and \(A \subset X\). Then \(A\) is \(\mu\)– closed if and only if \(A\) contains all its limit points.

The separation axioms \(T_0, T_1, T_2, T_3\) and \(T_4\) are well developed concepts in topological spaces. \(T_0, T_1, T_2\) and normal spaces are smoothly extended to GTS by Á. Császár, [Definition 1.40, 1.42]. As rightly pointed out by Á. Császár [Remark 1.41], most of the results true in topological spaces hold in GTS and they require no separate proofs also. We define regular, \(T_3\) and \(T_4\) spaces.
**Definition 5.6.** A GTS \((X, \mu)\) is called a \textit{regular space} if for every closed subset \(A\) of \(X\) and every \(x \in X\) not in \(A\) there exist open sets \(U, V\) such that \(A \subset U\), \(x \in V\) and \(U \cap V = \emptyset\).

A GTS \((X, \mu)\) is called a \(T_3\)–space if it is a \(T_1\)–space and a regular space.

A GTS \((X, \mu)\) is called a \(T_4\)–space if it is a \(T_1\)–space and a normal space.

From the definitions, it is clear that \(T_4 \Rightarrow T_3 \Rightarrow T_2 \Rightarrow T_1 \Rightarrow T_0\). However \(T_0 \not\Rightarrow T_1 \not\Rightarrow T_2 \not\Rightarrow T_3 \not\Rightarrow T_4\). We give examples of GTS which are not topological spaces to prove this.

**Example 5.7.** (a) The GTS \((I_{2n}, \mu)\), where \(\mu = \{\phi\} \cup \{A \subset I_{2n} \mid |A| \geq n\}\) is a \(T_4\)–GTS and hence \(T_3, T_2, T_1, T_0\) space which is not a topological space for \(n \geq 2\).

\((I_2, \mu)\) is a \(T_4\) topological space.

\((I_{2n}, \mu)\) is a \(T_1\)–GTS: If \(|A| = 2n - 1\), then \(A\) is \(\mu\)–open implies that \(I_{2n} - A\) is \(\mu\)–closed and so every singleton set is \(\mu\)–closed. Hence \((I_{2n}, \mu)\) is a \(T_1\)–GTS.

\((I_{2n}, \mu)\) is a normal GTS: If \(A\) and \(B\) are nonempty disjoint \(\mu\)–closed subsets of \(X\), then \(|A| \leq n\) and \(|B| \leq n\). If \(|A| = n = |B|\) then \(A\) and \(B\) themselves are open subsets of \(X\) satisfying the required condition. If \(|A| < n\) and \(|B| \leq n\), then also there are \(\mu\)–open subsets of \(X\) satisfying the required condition.

(b) \(T_0 \not\Rightarrow T_1\): \((X, \mu)\) where \(X = \{1, 2, 3, 4\}\) and \(\mu = \{\phi, \{1\}, \{3\}, \{1, 3\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, X\}\) is a GTS which is a \(T_0\)–space but not a \(T_1\)–space.
(c) $T_1 \not \Rightarrow T_2$: The GTS $(I_n, \eta)$, the co-singleton GTS on $I_n = \{1, 2, 3, \ldots n\}$, is a $T_1$-space which is not a $T_2$-space for $n \geq 3$.

(d) $T_2 \not \Rightarrow T_3$: Consider the GTS $(X, \mu)$ where $X = I_{2n+1}$, $n \geq 2$ and $\mu = \{\phi\} \cup \{A \subseteq X \mid |A| \geq n+1\} \cup \{|i+j-n, i+j-(n-1), i+j-(n-2), \ldots, i+j-1\}$ for each fixed $j$, $1 \leq j \leq n$ and for each fixed $i$, $1 \leq i \leq 2n+1$.

$(I_{2n+1}, \mu)$ is a $T_1$-space: If $|A| = 2n$ then $|A| > n+1$ and hence $A$ is $\mu$-open. Therefore, the complement which is a singleton set is $\mu$-closed. So $(I_{2n+1}, \mu)$ is a $T_1$-space.

$(I_{2n+1}, \mu)$ is a $T_2$-space: Let $r, s$ where $r < s$ be any two distinct points of $X$. Let $s = r+m$. If $m \geq n$ then $\{r, r+1, \ldots, r+(n-1)\}$ and its complement are two disjoint $\mu$-open sets of $r$ and $s$ where the addition is addition modulo $2n+1$. If $m \leq n$ then $\{s, s+1, \ldots, s+(n-1)\}$ and its complement are the disjoint $\mu$-open sets of $s$ and $r$ where the addition is addition modulo $2n+1$. Therefore, $(X, \mu)$ is a $T_2$-space.

$(X, \mu)$ is not a regular space: If $A = \{2, 4, 6, \ldots, 2n\}$, then $A$ is a $\mu$-closed set not containing 3. The only $\mu$-open set containing 3 and disjoint from $A$ is $N = \{1, 3, 5, \ldots, 2n-1, 2n+1\}$. Every $\mu$-open set containing $A$ will intersect $N$. Therefore, $A$ and $3$ have no disjoint $\mu$-open sets containing them. Hence $(X, \mu)$ is not a regular space.

For instance if $n = 2$, $X = \{1, 2, 3, 4, 5\}$.

$\mu = \{\phi\} \cup \{\{1, 2\}, \{1, 5\}, \{2, 3\}, \{3, 4\}, \{4, 5\}\} \cup \{A \subseteq X : |A| \geq 3\}$ is a GTS which is a
$T_2$– space but not a $T_3$– space.

$(X, \mu)$ is a $T_2$– space: $r, s$ be any two distinct points of $X$. Assume that $r < s$ and $s = r + m$. If $m \geq 2$, \( \{r, r+1\} \) and its complement will be disjoint neighborhoods of $r$ and $s$. If $m \leq 2$, \( \{r-1, r\} \) and its complement will be disjoint neighborhoods of $r$ and $s$.

$(X, \mu)$ is not a $T_3$– space: $A = \{2, 4\}$ is a closed set not containing 3. Only neighborhood of 3 disjoint from $A$ is $N = \{1, 3, 5\}$. Every $\mu$– open set containing $A$ will intersect $N$. $A$ and 3 have no disjoint neighborhoods.

e) $T_3 \nRightarrow T_4$: Take $X = I_{2n+1}$, $n \geq 3$. $\mu = \{\emptyset\} \cup \{A \subset X : |A| \geq n+1\} \cup \{A \subset X : |A| = n, 1 \in A\}$. $\mu$ is closed under union implies it is a GT.

$(X, \mu)$ is a $T_1$ space: $|A| = 2n$ implies that $A$ is $\mu$– open and hence every singleton set is $\mu$– closed.

$(X, \mu)$ is a regular space: Let $x \in X$ be any point and $F \subset X$ be any $\mu$– closed set not containing $x$. $F$ is $\mu$– closed implies that $|F| \leq n + 1$. If $|F| = n + 1$ then $1 \notin F$ and $F$ is $\mu$– open. $1 \in F^c$ implies $F^c$ is a $\mu$– open set containing $x$. If $|F| = n$ then $|F^c| = n + 1$ implies that $F^c$ is a $\mu$– open set containing $x$. If $1 \in F$ then $F$ is also $\mu$– open. If 1 is not in $F$ there exists $y \in F^c$ such that $y \neq x, 1$. Now $F^c - \{y\}$ and $F \cup \{y\}$ are $\mu$– open.

$(X, \mu)$ is not a normal space: $A = \{2, 4, 6,..., 2n\}$ and $B = \{3, 5,..., 2n-1, 2n+1\}$ are disjoint $\mu$– closed sets. The only neighborhood of $B$ disjoint from $A$ is $N = \{1, 3, 5,..., 2n-1, 2n+1\}$. Every neighborhood of $A$ will intersect $N$. $A$ and $B$ have
Theorem 5.8. Let \((X, \mu)\) be a GTS. \((X, \mu)\) is a \(T_1\)-space if and only if each singleton set is \(\mu\)-closed.

Proof. Let \(x \in X\). Then, \((X, \mu)\) is a \(T_1\)-space if and only if every \(y \in X, y \neq x\) has a neighborhood not containing \(x\) if and only if every \(y \in X, y \neq x\) has a neighborhood not intersecting \(\{x\}\) if and only if \(y\) is not a limit point of \(\{x\}\) if and only if \(\{x\}\) is \(\mu\)-closed.

It must be noted that every finite set need not be \(\mu\)-closed in a \(T_1\)-GTS, since finite union of \(\mu\)-closed sets need not be \(\mu\)-closed. In a \(T_1\) topological space \((X, \tau)\), \(x \in X\) is a limit point of \(A \subseteq X\) if and only if every neighborhood of \(x\) contains infinitely many points of \(A\). Such a result does not hold in a \(T_1\)-GTS, as the following Example 5.9 shows.

Example 5.9. The co-singleton GTS \((I_n, \eta)\) on \(I_n = \{1, 2, 3, \ldots, n\}\) is a \(T_1\)-space. \(c\eta(\{1, 2\}) = I_n\). Every neighborhood of every limit point contains \(n-1\) or \(n\) points only.

If a GTS \((X, \mu)\) has no disjoint \(\mu\)-closed subsets, then vacuously, it is normal [Lemma 1.43]. Even though every \(T_4\)-space is a \(T_3\)-space, every normal GTS need not be a regular GTS. This is shown in Example 5.10.

Example 5.10. Consider the \(Z\) forbidden GT on \(X = \mathbb{R}\). In this GTS, no two
closed sets are disjoint implies that it is normal. If \( A \) is any \( \mu \)-closed set and \( x \in X - A \), then \( A \supseteq Z \). \( A \) has no \( \mu \)-open set containing it and \( \{x\} \) itself is \( \mu \)-open. Therefore, \( A \) and \( x \) have no disjoint \( \mu \)-open sets containing them. Hence the space is not regular.

We have the following Theorem 5.11 and Corollary 5.12 on normal spaces.

**Theorem 5.11.** If \((X, \mu)\) is an \( A \) repulsion GTS [Definition 3.29] for some nonempty subset \( A \) of \( X \), then \((X, \mu)\) is a normal space.

**Proof.** If \((X, \mu)\) is an \( A \) repulsion GTS for some nonempty subset \( A \) of \( X \), then any two proper nonempty \( \mu \)-closed sets are intersecting.

**Corollary 5.12.** If \((X, \mu)\) is an \( A \) forbidden GTS [Definition 3.33] for some nonempty subset \( A \) of \( X \), then \((X, \mu)\) is a normal space.

The converse of Theorem 5.11 and Corollary 5.12 are not true follows from Example 5.7.(a). The GTS \((I_{2n}, \mu)\), where \( \mu = \{\phi\} \cup \{A \subset I_{2n} \mid |A| \geq n\} \) is a \( T_4 \)-GTS and hence \( T_1 \)-space. The intersection of all the nonempty \( \mu \)-closed sets is \( \emptyset \) implies it is not a repulsion GTS or forbidden GTS.

We now proceed to study convergency of sequences in GTS. Let \((X, \mu)\) be a GTS. A sequence \((x_n)\) in \((X, \mu)\) does not converge to a point \( x \) if there exists a neighborhood \( U \) of \( x \) such that \((x_n)\) is not eventually in \( U \). In this case, we denote \((x_n) \not\rightarrow x\). We say that \((x_n) \to x\), if otherwise. If \((x_n) \to x\), then \( x \) is called a
limit of the sequence. In general, a sequence can have no limit, unique limit or more limits as the following Example 5.13 shows. Example 5.14 below shows these cases for the co-singleton generalized topology.

Example 5.13. Consider $X = \mathbb{R}$, the set of all real numbers and $\mathbb{Z}$, the set of all integers. If $\mu = \wp(R - Z)$, the $\mathbb{Z}$ forbidden GT on $X$, then $\mu(x) = \{A \subset R - Z | x \in A\}$, if $x \in R - Z$ and $\mu(x) = \emptyset$, if otherwise. We observe the following:

(a) Every sequence in $\mathbb{R}$ converges to every $z \in \mathbb{Z}$.

It is interesting to note that the sequence 1, 2, 3, 4, ... converges to every $z \in \mathbb{Z}$.

(b) No sequence in $\mathbb{Z}$ converges to any element of $\mathbb{R} - \mathbb{Z}$.

(c) A sequence in $\mathbb{R} - \mathbb{Z}$ converges to an element $x \in \mathbb{R} - \mathbb{Z}$ if and only if it is eventually a constant sequence.

Example 5.14. Consider $\mathbb{R}$ with the co-singleton generalized topology, $\eta$. Then for each $x \in \mathbb{R}$, $\eta(x) = \{\mathbb{R}\} \cup \{\mathbb{R} - \{y\}|y \in \mathbb{R}, y \neq x\}$. If $(x_n)$ is any sequence, then $\{x_n|n \in \mathbb{N}\}$ is written as $\{x_n\}$. We observe the following:

(a) If $\{x_n\}$ is finite and only one element of $(x_n)$ occurs infinite number of times or in other words, if $(x_n)$ is eventually a constant sequence, then $(x_n)$ converges to that only element of $\mathbb{R}$ which occurs infinite no of times.

(b) If $\{x_n\}$ is finite and more than one element of $\{x_n\}$ occurs infinite number of times, then $(x_n)$ does not converge to any element of $\mathbb{R}$. For instance 1, 2, 1, 2, 1, 2, 1, ... does not converge to any element of $\mathbb{R}$.

(c) If $\{x_n\}$ is countable and each element of $\{x_n\}$ occurs only a finite number
of times, then \((x_n)\) converges to every element of \(\mathbb{R}\). For instance 1, 2, 3, 4, ... converges to every element of \(\mathbb{R}\).

(d) If \(\{x_n\}\) is countable and only one element of \(\{x_n\}\) occurs infinite number of times, then \((x_n)\) converges to that only element of \(\mathbb{R}\) which occurs infinite no of times. For instance 1, 1, 2, 1, 3, 1, 4, 1, ... converges to 1.

(e) If \(\{x_n\}\) is countable and more than one element of \(\{x_n\}\) occurs infinite number of times, then \((x_n)\) does not converge to any element of \(\mathbb{R}\). For instance 1, 2, 1, 2, 3, 4, 1, 2, 5, 6, 1, 2, ... does not converge to any element of \(\mathbb{R}\).

**Theorem 5.15.** Let \(X\) be any set, \(A \subseteq X\) such that \(A \neq \emptyset\) and \(\mu = \{G \subseteq X|G \subseteq X - A\}\). Then the following hold.

(a) Every sequence \((x_n)\) in \(X\) converges to every \(a \in A\).

(b) A sequence \((x_n)\) in \(X - A\) converges to \(x \in X - A\) if and only if \((x_n)\) is eventually a constant sequence.

**Proof.** (a) Let \((x_n)\) be any sequence in \(X\). Take any \(a \in A\). Then \(\mu(a) = \emptyset\) implies that there exists no neighborhood \(U\) of \(a\) such that \((x_n)\) is not eventually in \(U\). Therefore, \((x_n) \to a\).

(b) Let \((x_n)\) be any sequence in \(X - A\). Then \((x_n) \to x \in X - A\) if and only if \((x_n)\) is eventually in every neighborhood of \(x\) if and only if \((x_n)\) is eventually in \(\{x\}\) if and only if \((x_n)\) is eventually a constant sequence.

We shall now study the relationship between limits of a sequence in \(A\) and
limit points of $A$ in a GTS.

**Theorem 5.16.** Let $(X, \mu)$ be a GTS and $A \subset X$. If $x \notin A$ and there is a sequence in $A$ converging to $x$, then $x \in A'$.

**Proof.** If $x \notin A$ is not a limit point of $A$, then there exists a neighborhood $U$ of $x$ not intersecting $A$ and so no sequence in $A$ is eventually in $U$. Hence no sequence in $A$ converges to $x$, a contradiction.

**Corollary 5.17.** Let $(X, \mu)$ be a GTS and $A \subset X$. If there is a sequence in $A$ converging to $x$, then $x \in c_\mu(A)$.

**Proof.** If $x \in A$, then $x \in c_\mu(A)$. If $x \notin A$, then $x \in A' \subseteq c_\mu(A)$.

**Corollary 5.18.** Let $(X, \mu)$ be a GTS and $F \subset X$ be $\mu$–closed. If $(x_n)$ is a sequence in $F$ and $(x_n) \to x$, then $x \in F$.

The following Example 5.19 shows that the converse of Corollary 5.17 is not true.

**Example 5.19.** Consider the co-singleton GTS $(\mathbb{R}, \eta)$ of Example 5.14. If $A = \{1, 2\}$, then $c_\eta(A) = \mathbb{R}$. 3 is a limit point of $A$ but there is no sequence in $A$ converging to 3.

Clearly, every constant sequence in a GTS is convergent. In Example 5.13, every constant sequence converges to every integer and in Example 5.14, every
constant sequence converges to a unique limit. The following Theorem 5.20 shows that every constant sequence has a unique limit if and only if the GTS is a $T_1$-space.

**Theorem 5.20.** Let $(X, \mu)$ be a GTS. Then every constant sequence $(x_n)$ has a unique limit if and only if $(X, \mu)$ is a $T_1$-space.

**Proof.** Let $(x_n)$ be a sequence such that $x_n = x$ for every $n$. Suppose $(X, \mu)$ is a $T_1$-space. Then $(x_n) \to y$ implies that every neighborhood of $y$ contains $x$ and so every neighborhood of $y$ intersects $\{x\}$. Therefore, $y \in c_\mu(\{x\}) = \{x\}$ which implies that $y = x$. Conversely, suppose $(x_n)$ has a unique limit. Clearly, $(x_n) \to x$. Let $y \in X$ be such that $y \neq x$. Then $(x_n) \not\to y$ and so $y$ has a neighborhood $U$ not intersecting $\{x\}$ which implies that $y \not\in c_\mu(\{x\})$. Hence $c_\mu(\{x\}) = \{x\}$ and so $(X, \mu)$ is a $T_1$-space.

In a GTS $(X, \mu)$, if $\{x\}$ is $\mu$-open for every $x$ in $X$, then $(X, \mu)$ is a discrete topological space and the only convergent sequences in $X$ are the constant sequences. The following Theorem 5.21 gives a necessary condition for a GTS $(X, \mu)$ in which the only convergent sequences are the constant sequences. Example 5.22 below shows that the converse of Theorem 5.21 is not true.

**Theorem 5.21.** Let $(X, \mu)$ be a GTS such that the only convergent sequences in $X$ are the constant sequences. Then $(X, \mu)$ is a $T_1$-space.

**Proof.** Suppose $\{x\}$ is not closed. Then there exists $y \in X$ and $y \neq x$ such that $y \in c_\mu(\{x\})$ and so $\mu(y) = \emptyset$ or every neighborhood of $y$ intersects $\{x\}$ which
implies that $\mu(y) = \emptyset$ or every neighborhood of $y$ contains $x$. Hence the sequence $x, y, x, y, x, \ldots$ converges to $y$, a contradiction.

**Example 5.22.** The GTS $(\mathbb{R}, \eta)$ of Example 5.13 is a $T_1$-space. But the sequence $1, 2, 3, 4, \ldots$ converges to every real number.

We now explore the possibilities of having at most one limit for any sequence in a GTS. We prove that every sequence in a $T_2$-space converges to at most one point.

**Theorem 5.23.** Let $(X, \mu)$ be a $T_2$-space. Then every sequence in $X$ converges to at most one point.

**Proof.** Let $(x_n)$ be a sequence in $X$. If $(x_n)$ is not convergent, then there is nothing to prove. Suppose $(x_n) \to x$. Let $y \in X$ such that $y \neq x$. Then there exist disjoint neighborhoods $U_x$ and $U_y$ containing $x$ and $y$, respectively. Since $(x_n)$ is eventually in $U_x$, $(x_n)$ cannot be eventually in $U_y$ and so $(x_n)$ cannot converge to $y$.

The following Example 5.24 shows that the converse of the above Theorem 5.23 is not true.

**Example 5.24.** Consider $(I_n, \eta)$, the co-singleton GTS on $I_n$. It is a $T_1$-space but not a $T_2$-space. A sequence in $(I_n, \eta)$ is convergent if and only if it is eventually a constant sequence. Also every constant sequence in a $T_1$-space converges
to a unique limit. Hence, every sequence in \((I_n, \eta)\) converges to at most one point.

**Theorem 5.25.** Let \((X, \mu)\) be a GTS. If every sequence in \(X\) converges to at most one point, then \((X, \mu)\) is a \(T_1\)–space.

**Proof.** Every sequence in \(X\) converges to at most one point implies that every constant sequence in \(X\) converges to at most one point and so \((X, \mu)\) is a \(T_1\)–space, by Theorem 5.20.

The converse of the above Theorem 5.25 is not true, as the following Example 5.26 shows.

**Example 5.26.** Consider the co-singleton GTS \((R, \eta)\) which is a \(T_1\)–space. The sequence \(1, 1/2, 1/3, \ldots\) converges to every real number.

The following Theorem 5.27 gives the expected relationship between continuous functions between GTS and convergent sequences.

**Theorem 5.27.** Let \((X, \mu)\) and \((Y, \lambda)\) be GT spaces and \(f : (X, \mu) \to (Y, \lambda)\) be \((\mu, \lambda)\)–continuous. Then \((x_n) \to x\) in \(X\) implies that \(f(x_n) \to f(x)\) in \(Y\).

**Proof.** \(f\) is \((\mu, \lambda)\)–continuous if and only if \(f\) is \(GNS\)–continuous. Let \(V\) be a neighborhood of \(f(x)\) in \(Y\). Then \(f^{-1}(V)\) is a neighborhood of \(x\) in \(X\) and so \((x_n)\) is eventually in \(f^{-1}(V)\) which implies that \((f(x_n))\) is eventually in \(V\). Therefore, \((f(x_n)) \to f(x)\) in \(Y\).
The following Example 5.28 shows that the converse of Theorem 5.27 is not true.

**Example 5.28.** Let $X = Y = I_{2n}$, $A = \{2, 4, 6, \ldots, 2n\}$, $\eta$ be the co-singleton generalized topology on $X$ and $\lambda$ be the $A$ forbidden GT on $Y$. If $f : (X, \eta) \to (Y, \lambda)$ is the identity function, then $(x_n) \to x$ in $X$ implies that $(x_n)$ is eventually a constant sequence and so $(f(x_n)) \to f(x)$ in $Y$. If $V = \{1\}$, then $V$ is $\lambda$–open but $f^{-1}(V) = \{1\}$ is not $\eta$–open. Therefore, $f$ is not $(\eta, \lambda)$–continuous.

We now establish that, the convergence in the product generalized topological space (as defined in [Definition 1.37]) is pointwise convergence.

**Theorem 5.29.** Let $(X, \mu)$ be the product GTS where $X = \prod_{\alpha \in \Delta} X_{\alpha}$. $(X_{\alpha}, \mu_{\alpha})$ is a strong GTS for each $\alpha \in \Delta$ and $(x_n)$ be a sequence in $X$. Then $(x_n) \to x$ if and only if $p_{\alpha}(x_n) \to p_{\alpha}(x)$ in $X_{\alpha}$ for every $\alpha \in \Delta$.

**Proof.** Suppose $(x_n) \to x$. By Lemma 1.39(f), $p_{\alpha} : X \to X_{\alpha}$ is $(\mu, \mu_{\alpha})$–continuous for every $\alpha \in \Delta$. By Theorem 5.27, $p_{\alpha}(x_n) \to p_{\alpha}(x)$ in $X_{\alpha}$ for every $\alpha \in \Delta$.

To prove the converse, take any basic neighborhood of $x \in X$. It will be of the form $p_{\alpha_1}^{-1}(U_{\alpha_1}) \cap \ldots \cap p_{\alpha_m}^{-1}(U_{\alpha_m})$ where for each $i$, $1 \leq i \leq m$, $x_{\alpha_i} \in U_{\alpha_i} \in \mu_{\alpha_i}$. Since $p_{\alpha_i}(x_n)$ is eventually in $U_{\alpha_i}$, there exists $n_i$ such that $p_{\alpha_i}(x_n) \in U_{\alpha_i}$ for every $n \geq n_i$. Let $n_0 = max\{n_1, n_2, \ldots, n_m\}$. Clearly $x_n \in p_{\alpha_1}^{-1}(U_{\alpha_1}) \cap \ldots \cap p_{\alpha_m}^{-1}(U_{\alpha_m})$ for every $n \geq n_0$. This completes the proof.