CHAPTER 6

λ-LINKED BLOCK DESIGNS THROUGH NON-ADAPTIVE HYPERGEOMETRIC GROUP TESTING DESIGNS FOR IDENTIFYING AT MOST TWO DEFECTIVES AND THEIR OPTIMALITIES

6.0 Introduction

Group testing, known also as factor screening and search for significant factors, is a vast area with many works developing both theoretical and applied aspects. A general statement of the group testing problem that covers many particular statements is as follows. Assume that \( n \) factors (items, elements, varieties, etc.) \( x_1, x_2, \ldots, x_n \) are given and some of them are defective (significant, important, etc.). The problem is to determine which factors are defective by testing several factor groups, that is subsets of the set \( X = \{x_1, x_2, \ldots, x_n\} \) in a series of \( v^* \) or \( t \) tests. Each group test can provide only one of two outcomes; a positive outcome indicates the presence of one or more defective factors among the factors tested while a negative outcome indicates no defective factors among the tested factors. The problems differ in

(i) Prior information concerning the number of defective factors

(ii) Constraints on the test groups, and

(iii) Information we are getting by inspecting the groups

The problem in group testing is to devise a sequential sampling scheme which minimizes the expected number of tests required to classify all the \( n \) factors as defective or non-defective.
The technique of group testing was originally proposed by Dorfman (1943) in the context of blood testing. The common assumption in the group testing problem is that there is no test error. The group testing problem has been studied by several authors, viz., Sobel and Groll (1959, 1966), Hwang (1978), Hwang and Sos (1981) Weideman (1984), and Weideman and Raghavarao (1987,a,b). Dorfman’s (1943) original procedure has been modified and extended to other screening situations.

The first application of group testing (Dorfman, 1943)) in the literature was to the problem of pooling blood samples in order to classify each one of a large group of people (e.g. soldiers in an Army Unit) as to whether or not they have a particular disease (e.g. Syphilis).

An interesting feature about the applications of group testing is the variety of different fields in which they appear. The binary model describes the blood testing and many other practical problems, where by inspecting a group $Z$ one receives 1 if there is at least one defective in $Z$ and 0 otherwise. This model is perhaps the most important and well studied. Since more than 50 years of the history of the group testing, many other mathematical models have been also developed and advanced results have been achieved. Among many classical works in the field of group testing, only several dealing with statistical and probabilistic aspects (Sobel and Groll, 1959; Sobel, 1968; Srivastava, 1975; Renyi, 1965) are mentioning here. Sobel and Groll (1959) have mentioned many industrial applications of group testing, like checking leaking devices or the existence of a current in a sequential circuit etc.
Sobel (1968) suggested classifying group testing problems according to the joint distribution of the \( n \) items tested. If the joint distribution is binomial with parameters \((q, n)\), where \( 0 < q < 1 \) is the constant probability of each item being good, then we call it a binomial group testing (BGT) problem. If the joint distribution is hypergeometric with parameters \((d, n)\), where \( d \) is the number of defectives among \( n \) items and each item is equally likely to be a defective one, then we call it a hypergeometric group testing (HGT) problem. These definitions can be extended so that for the binomial case \( q \) is allowed to vary from one item to another, and for the hypergeometric case \( d \) is allowed to be an upper-bounded on the number of defectives. We call them the generalized BGT problem and the generalized HGT problem respectively.

When the number of defectives \( d \) is not known prior to testing, the situation is binomial problem and has been discussed extensively (for example, Sobel and Groll (1959, 1966)). If the number of defectives \( d \) is known, a hypergeometric problem arises, as discussed by Hwang (1978) and Hwang and Sos (1981).

The testing procedure can be sequential, where the information obtained from the initial test or tests is used to determine subsequent test patterns. The test can also be specified simultaneously so that no feedback information is used. This situation is called non-adaptive by Hwang and Sos (1981).

Thus the non-adaptive hypergeometric problem was first introduced by Hwang and Sos (1981). This problem arises when the
number of defectives $d$ is known prior to testing. It can be approached in two ways. The traditional approach in the literature has focused on minimizing the number of tests required to identify the defectives for a given number of items. The second approach introduced by Weideman and Raghavarao (1987,a) is to maximize the number of items that can be tested in a given number of tests. Weideman and Raghavarao (1987,b) considered the problem of constructing group testing designs with $n$ items that can be identify at most two defectives by performing $v^*$ tests through the dual formation of the designs. Ghosh and Thannippara (1991) developed some more such designs from Hypercubic designs and BIB designs.

6.1 $\lambda$-Linked Block Designs

Bose (1963, 1975, 1976) studied the problem of designs and multigraphs. His investigation was based on BIBD, Lattice Designs, Mutually orthogonal Latin square designs etc. In this investigation, Bose had given the definitions of $\lambda$-linked block designs, block multigraph designs, treatment multigraph designs and geometry designs and their constructions. From the lines of Bose (1975) we state the following definitions.

**Definition 6.1.1**

Given a design $d$, we can obtain another design $d^*$, the dual of $d$ by interchanging the blocks and the treatments. Thus the treatments $t_1, t_2, \ldots t_v$ of $d$ become blocks of $d^*$ and the blocks $B_1, B_2, \ldots B_b$ of
d become the treatments of \( d^* \). Actually this process is known as dualization. If the treatment \( \theta_i \) of \( d \) occurs in the block \( B_i \) of \( d \) then the treatment \( B_i \) of \( d^* \) occurs in the block \( \theta_i \) of \( d^* \). Thus the dual of a BIBD with parameters \((v, b, r, k, \lambda)\) is configuration \((b, v, k, r)\) in which any two blocks intersect in \( \lambda \) treatments. This is called \( \lambda \)-linked block design. In particular if \( \lambda = 1 \), the design is called a singly linked block (SLB) design.

**Definition 6.1.2**

A design \( d \) will be called a geometry if any two treatments of \( d \) cannot occur together in more than one block. In this case the treatments may be called the points, and the blocks may be called the lines of the geometry.

**Definition 6.1.3**

Given a design \( d \), we can obtain from it two multi-graphs, the treatment multi-graph \( G(d) \) and the block multi-graph \( G^*(d) \). The treatment multi-graph of the design can be obtained in the following manner. Let, the two treatments \( t_i \) and \( t_\alpha \) of \( d \) occur together in exactly \( \lambda_{i\alpha} \) blocks then \( m(t_i, t_\alpha) = \lambda_{i\alpha} \) where \( m \) represents the multiplicity function. In particular \( d \) is a geometry, then \( \lambda_{i\alpha} = 0 \) or 1. Hence \( G(d) \) is a graph. A graph \( G \) is a multi-graph in which the multiplicity function can take only two values 0 and 1.

In the same way we can define the block multi-graph also. If the blocks \( B_j \) and \( B_\beta \) of \( d \) intersect in exactly \( \mu_{j\beta} \) treatments then \( m(B_j, B_\beta) = \mu_{j\beta} \).
In this investigation we have shown that non-adaptive hypergeometric group testing designs for identifying at most two defectives for \( v^* = 0 \pmod{6} \) and \( v^* = 2 \pmod{6} \) are \( \lambda \)-linked block designs. Here we have also shown that such designs are Type-I optimal designs and geometry designs.

6.2 Type I Optimality of Group Testing Designs for \( v^* = 0 \pmod{6} \) and \( v^* = 2 \pmod{6} \)

Blocking is an experimental technique commonly used in agricultural, industrial and biological experiments to eliminate heterogeneity in one direction. In any experimental situation requiring usage of a block design, it is desirable to maximize the amount of information gained on the treatments being studied by using an 'Optimal Block Design'. Several results are known concerning the Type I Optimality of block designs in classes \( d(v, b, k) \). For example, it is well known that a BIBD is optimal in \( d(v,b,k) \) under all Type I optimality criteria. Certain types of Regular Graph Designs (RGD'S) which are not BIBD'S have also been shown to be optimal under various Type I criteria in a number of classes and subclasses of \( d(v, b, k) \), for example, see Conniffee and Stone (1975); Shah, Raghavarao and Khatri (1976); Williams, Patterson and John (1977) and Cheng (1978, 1979). However those designs which are Type I optimal in a vast majority of classes \( d(v, b, k) \) remain unknown.

Jacroux (1985) studied the Type I optimality of Semi-Regular Graph Designs (SRGD), we call \( d \) a SRGD if \( d \) is binary and if \( N_dN_d^1 \)
has all of its diagonal elements and off-diagonal elements differing by at most one (Jacroux, (1985)). The notion of an SRGD is a generalization of the definition given by Mitchel and John (1977) for a Regular Graph Design (RGD) and reduces to their definition when \( bk/v \) is an integer. If \( d \) is an RGD its concurrence matrix has the additional property that all of its off-diagonal elements are equal, then \( d \) is called a Balanced Block Design (BBD) (Jacroux, (1980)).

In this investigation we consider the determination of Type I optimal block designs from \( v^* \equiv 0 \mod 6 \) and \( v^* \equiv 2 \mod 6 \) group testing designs having \( v \) treatments arranged in \( b \) blocks of size \( k \). For simplicity, it is assumed throughout this section that \( v > k \).

Let \( d \) be a block design such as described above. Then \( d \) has associated with it \( v \times b \) incidence matrix \( N_d \) whose entries \( n_{dij} \) give the number of times the \( i \)th treatment occurs in the \( j \)th block. When \( n_{dij} = 1 \) or \( 0 \) for all \( i \) and \( j \), the design is said to be binary. The \( i \)th row sum of \( N_d \) is denoted by \( r_{di} \) and represents the number of times the treatment \( i \) is replicated in the design. The matrix \( N_d N_d^t \) where \( N_d^t \) is the transpose of \( N_d \) is referred to as the concurrence matrix of \( d \), and its entries are denoted by \( h_{dij} \). Hence we consider the design under the two-way additive model. A design \( d \) is connected if and only if its C-matrix has rank \( v - 1 \).

For \( d(v, b, k) \), let \( 0 = Z_{do} < Z_{d1} \leq Z_{d2} \ldots \leq Z_{d,v-1} \) denote the eigenvalues of the associated C-matrix \( C_d \). A design \( d \) is said to be \( \phi_r \)-optimal in \( d(v,b,k) \) provided
$$\phi_{f}(C_d) = \sum_{i=1}^{r_{d}} f(Z_{di}) \quad (6.2.1)$$

is minimal over all designs $d(v,b,k)$ where $f$ is non-increasing and convex real valued function. The well-known A-and D-optimality criteria correspond to taking $f(x) = \frac{1}{x}$ and $-\log x$ in equation (6.2.1) respectively.

6.2.1 Type I optimality

$\phi_{f}$ of equation (6.2.1) is called a Type I Optimality criterion if $f$ satisfies the following conditions.

(i) is continuously differentiable on $(0, \max \{C_d\})$ and $f' < 0, f'' > 0, f''' < 0$ on $(0, \max \{C_d\})$

(ii) $f$ is continuous at 0 or $\lim_{x \to 0^+} f(x) = f(0) = \frac{1}{x}$

Note that the A-and D-optimality criteria mentioned above are Type I optimality criteria.

This definition is due to Jacroux (1985).

6.3 Preliminary Results

Lemma 6.3.1

If a group testing design $d$ for identifying at most two defectives exists, then it satisfies the following conditions.

(i) $B_i^* \cup B_j^* = B_k^* \cup B^*$, $i, j, k, l = 1, 2, \ldots n$, $(i, j) \neq (k, l)$

where $B_i^*$ denotes the numbers of the tests in $d$ in which the $i^{th}$ item is tested.
(ii) In \( d^* \) any pair of treatments can appear at most once.

(iii) \( n \leq \lceil v^* (v^* + 1)/6 \rceil \) where \( \lceil x \rceil \) denotes the greatest integer contained in \( x \).

The conditions discussed in Lemma 6.3.1 is given by Weideman and Raghavarao (1987,a).

**Definition 6.3.1**

A group testing design \( d \) is said to be a \( \lambda \)-linked block design if the following conditions are satisfied.

(i) \( v^* \equiv 0 \pmod{6} \) and \( v^* \equiv 2 \pmod{6} \) where \( v^* \) denotes the number of treatments in the dual design \( d^* \).

(ii) All blocks of \( d \) have the same size \( k \).

(iii) \( r_{d1}^* = r_{d2}^* = \ldots = r_{dv}^* = r_d^* \), that is, the replication of the test treatments in the dual design \( d^* \) remains the same.

(iv) \( B_i \cap B_j = B_k \cap B_l = \lambda, \ i, j, k, l = 1, 2, \ldots, b, (i, j) \neq (k, l) \), that is, any two blocks intersect in \( \lambda \) treatments, where \( B_i \) denotes the \( i^{th} \) block of group testing design \( d \).

**6.3.1 Some Results related to Type I Optimality**

In this section we are discussing some results related to Type I optimality which are proven in Jacroux (1985).
\[ A = \text{tr } C_d = \sum_{i=1}^{r-1} Z_{d_i} \]

\[ \text{Tr } C_d^2 = \sum_{i=1}^{r-1} Z_{d_i}^2 \]

\[ B = \text{tr } C_d^2 + \min \left( \frac{1}{2}, \frac{4}{k^2} \right) \]

\( m_1 \) and \( m_1^* \) are non negative constants (to be defined) such that

\[ (A - m_1)^2 \geq B - m_1 \geq (A - m_1^2) / (v - 2) \]

\[ (A - m_1^* )^2 \geq B - m_1^* \geq (A - m_1^* )^2 / (v - 2) \]

\[ P_1 = [(B - m_1) - (A - m_1)^2 / (v - 2)]^{\frac{1}{2}} \]

\[ m_2 = \{(A - m_1) - [v - 2] / (v - 3)\}^{\frac{1}{2}} P_1 / (v - 2) \]

\[ m_3 = \{(A - m_1) + [v - 2] (v - 3)\}^{\frac{1}{2}} P_1 / (v - 2) \]

\[ m_4 = (A - (2/k) - m_1^*) / (v - 2) \] \hspace{1cm} (6.2.2)

**Lemma 6.3.2**

Let \( d(v, b, k) \) such that \( bk / v \) is not an integer and let \( d(v,b,k) \) be an SRGD with \( C \)-matrix \( C_d \) having non zero eigenvalues

\[ Z_{d1} \leq Z_{d2} \leq \ldots \leq Z_{d, v-1} \]

Now let \( m_1 = m_1^* = r(k - 1) v / (v - 1) k \)

If \( m_1 \leq m_2, \quad m_1^* \leq m_4 \) and

\[ \sum_{i=1}^{r-1} f(Z_{d_i}) < \min \{ f(m_1) + (v - 3) f(m_2) + f(m_3), f(m_1^* ) + (v - 2) f(m_4) \} \]

then a \( \phi_j \) -optimal design in \( d(v,b,k) \) must be an SRGD.
6.3.2 Examples for λ-linked Block Designs

Hence we have shown that non-adaptive hypergeometric group testing designs for identifying at most two defectives constructed by Weideman and Raghavarao (1987,a) from $v^* = 0 \pmod{6}$ and $v^* = 2 \pmod{6}$ are λ-linked block designs. The dual design $d^*$ with block sizes of 2 and 3 is constructed such that it satisfies the conditions of Lemma 6.3.1.

Example 6.3.1 (Using definition 6.3.1)

Consider the dual design $d^*$ with $v^* = 6 \equiv 0 \pmod{6})$ and blocks

\begin{align*}
B_1^* &= (1, 2), & B_2^* &= (3, 4), & B_3^* &= (5, 6), & B_4^* &= (1, 3, 5), \\
B_5^* &= (1, 4, 6), & B_6^* &= (2, 3, 6), & B_7^* &= (2, 4, 5).
\end{align*}

The corresponding Group Testing Design (GTD) $d$ is the dual design of $d^*$. Since there are seven blocks and six treatments in $d^*$, the Group Testing design $d$ having six blocks and seven treatments. In $d^*$, the treatment '1' occurs in blocks 1, 4 and 5, so the first block of $d$ contain the treatments 1, 4, and 5. In a similar way all the blocks of $d$ are obtained. Thus the blocks of the Group Testing design $d$ are

\begin{align*}
B_1 &= (1, 4, 5), & B_2 &= (1, 6, 7), & B_3 &= (2, 4, 6) \\
B_4 &= (2, 5, 7), & B_5 &= (3, 4, 7), & B_6 &= (3, 5, 6)
\end{align*}

This design is given in Weideman and Raghavarao (1987,a) (Example 3.1).
In this example we found that all blocks of \(d\) have the same size \(3(=k)\), the replication of the test treatments in the dual design \(d'\) remains the same and

\[ B_i \cap B_j = B_k \cap B_l = 1 (=\lambda) \text{ for } i, j, k, l = 1, 2, \ldots, 6 \text{ and } (i, j) \neq (k, l), \]

that is, any two blocks of \(d\) intersect in \(\lambda (=1)\) blocks.

This GTD satisfies all the conditions of definition 6.3.1. Since \(\lambda = 1\), it is a singly linked block design. Clearly, it is also a block multi-graph design, since \(m(B_j, B_\beta) = 1\), for all \(j, \beta = 1, 2, \ldots, 6, j \neq \beta\). That is, any two blocks of \(d\) intersect in exactly one treatment.

Note that any two treatments \(t_i\) and \(t_j\) of \(d\) occur together in \(\lambda_{ij}\) blocks, where \(\lambda_{ij} = 0\) or 1, so \(d\) is also a geometry design.

**Example 6.3.2 (Using definition 6.3.1)**

Consider the dual design \(d'\) with \(v^* = 8\) \((= 2 \text{ (mod } 6))\) and blocks 

\[
\begin{align*}
B_1^* &= (1, 2) & B_2^* &= (3, 4) & B_3^* &= (5, 6) & B_4^* &= (7, 8), \\
B_5^* &= (1, 3, 5) & B_6^* &= (2, 4, 6) & B_7^* &= (2, 3, 7) & B_8^* &= (1, 4, 8) \\
B_9^* &= (1, 6, 7) & B_{10}^* &= (2, 5, 8) & B_{11}^* &= (3, 6, 8) & B_{12}^* &= (4, 5, 7)
\end{align*}
\]

The corresponding Group testing design \(d\) is obtained by considering the dual design of \(d'\). The block of the GTD \((d)\) are

\[
\begin{align*}
B_1 &= (1, 5, 8, 9) & B_2 &= (1, 6, 7, 10) & B_3 &= (2, 5, 7, 11) \\
B_4 &= (2, 6, 8, 12) & B_5 &= (3, 5, 10, 12) & B_6 &= (3, 6, 9, 11) \\
B_7 &= (4, 7, 9, 12) & B_8 &= (4, 8, 10, 11).
\end{align*}
\]
This design is given in Weideman and Raghavarao (1987,a, eg. 3.2).

Here we found that all blocks of $d$ have the same size 4 ($=k$), the replication of the test treatments in the dual design $d^*$ remains the same and

$$B_i \cap B_j = B_k \cap B_l = 1 (=\lambda) \text{ for } i, j, k, l = 1, 2, \ldots 8 \text{ and } (i, j) \neq (k, l),$$

that is, any two blocks of $d$ intersect in $\lambda (=1)$ blocks. Thus, this GT design is a singly linked block design, since $\lambda = 1$. Clearly it is also a block multi-graph design as any two blocks of $d$ intersect in exactly one treatment, that is, $m(B_j, B_\beta) = 1$ for all $j, \beta = 1, 2, \ldots 8$, and $j \neq \beta$. Also, note that any two treatments $t_i$ and $t_\alpha$ of $d$ occur together in $\lambda_{i\alpha}$ blocks where $\lambda_{i\alpha} = 0$ or 1, so $d$ is a geometry design.

### 6.4 Main Results

Now, we state the main results in our investigation in the form of Theorems 6.4.1, 6.4.2 and 6.4.3.

**Theorem 6.4.1**

A group testing design $d$ for $v^* = 0 \pmod{6}$ and $v^* = 2 \pmod{6}$ implies the existence of a semi-regular graph design.

**Proof**

In a group testing design $d$, obtained from a dual design $d^*$ with $v^* = 0 \pmod{6}$ and block sizes of two or three, there are $v$ treatments. The $v$ treatments of $d$ are further divided into two groups. Group I
contains $k$ treatments and Group II containing $v - k$ treatments. Without loss of generality we assume that Group I contains the first $k$ treatments $1, 2, .. k$ and Group II contains the last $v - k$ treatments $k + 1, k + 2, ..., v$. All the treatments in Group I are replicated $r_1$ times and all the treatments in Group II are replicated $r_2$ times where $r_2 = r_1 + 1$. Any pair of treatments from Group I occurs together in $\lambda_1$ blocks, any pair of treatments from Group II occurs together in $\lambda_2$ blocks and any treatments from Group I occurs together with any treatment of Group II in $\lambda_2$ blocks where $\lambda_2 = \lambda_1 + 1$. The concurrence matrix of this group testing design is of the following form.

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3 .........k</th>
<th>k+1 ....... v</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$r_1$</td>
<td>$\lambda_1$</td>
<td>$\lambda_1$ ......... $\lambda_1$</td>
<td>$\lambda_2$ ......... $\lambda_2$</td>
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<tr>
<td>2</td>
<td>$\lambda_1$</td>
<td>$r_1$</td>
<td>$\lambda_1$ ......... $\lambda_1$</td>
<td>$\lambda_2$ ......... $\lambda_2$</td>
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<td>3</td>
<td>$\lambda_1$</td>
<td>$\lambda_1$</td>
<td>$r_1$ ......... $\lambda_1$</td>
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</tr>
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<td>$N_dN_d^t = k$</td>
<td>$\lambda_1$</td>
<td>$\lambda_1$</td>
<td>$\lambda_1$ ......... $r_1$</td>
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<td>$k+1$</td>
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<td>$v$</td>
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</tr>
</tbody>
</table>

In the case of group testing design $d$, obtained from dual design $d^*$ with $v^* = 2 \pmod{6}$ and block sizes of two or three, there are
v treatments. The v treatments of d are further divided into two groups. Group I contains k treatments and Group II contains \( v - k \) treatments. Without loss of generality we assume that Group I contains the first k treatments 1, 2, ..., k and Group II contains the last \( v - k \) treatments \( k + 1, k + 2, \ldots, v \). Here any pair of treatments from Group I occurs together in \( \lambda_1 \) blocks, any pair of treatments from Group II occurs together in \( \lambda_1 \) or \( \lambda_2 \) blocks and any treatment from Group I occurs with any treatment of Group II in \( \lambda_1 \) or \( \lambda_2 \) blocks, where \( \lambda_2 = \lambda_1 + 1 \). Also note that all the treatments of Group I are replicated \( r_1 \) times and all the treatments of Group II are replicated \( r_2 \) times where \( r_2 = r_1 + 1 \). The concurrence matrix of this Group Testing Design having diagonal elements \( r_1 \) or \( r_2 \) and off-diagonal elements \( \lambda_1 \) or \( \lambda_2 \).

Thus, for group testing designs, \( d \) obtained from \( v^* = 0 \mod 6 \) and \( v^* = 2 \mod 6 \), it is obvious that the diagonal elements of the concurrence matrix are \( r_1 \) for \( k \) rows and \( k \) columns and \( r_2 \) for the remaining rows and columns and \( r_1 \) and \( r_2 \) are differ by one. Again the off-diagonal elements of the concurrence matrix are either \( \lambda_1 \) or \( \lambda_2 \) and these \( \lambda_1 \) and \( \lambda_2 \) differ again by one.

Now, it is clear that, in the concurrence matrices of the group testing designs the diagonal elements are differ by at most one and also the off-diagonal elements are differ by at most one. So from the definition of Semi-regular Graph Designs, the Group Testing Designs for \( v^* = 0 \mod 6 \) and \( v^* = 2 \mod 6 \) are Semi-regular Graph Designs.

The parameters are obvious.
Theorem 6.4.2

A BIBD obtained from a Group Testing design \( d \) for \( v^* \equiv 0 (\text{mod } 6) \) by adding a single block.

Proof

In a group testing design \( d \) for \( v^* \equiv 0 (\text{mod } 6) \), the \( v \) treatments are divided into two groups. Group I contains the first \( k \) treatments, all of which are replicated \( r_1 \) times and Group II contains the last \( v - k \) treatments all of which are replicated \( r_2 \) times where \( r_2 = r_1 + 1 \). Any pair of treatments from Group I occurs together in \( \lambda_1 \) blocks, any pair of treatments from Group II occurs together in \( \lambda_2 \) blocks and any treatment from Group I occurs together with any treatment of Group II in \( \lambda_2 \) blocks, where \( \lambda_2 = \lambda_1 + 1 \).

Now, adding a simple block, which contains all the \( k \) treatments of the Group I. In the new design, all the treatments of Group I replicated \( r_2 \) times and any pair of treatments from Group I occurs together in \( \lambda_2 \) blocks. So in the new design, all the \( v \) treatments are replicated an equal number \( (r_2) \) of times and any pair of treatments occurs together an equal number \( (\lambda_2) \) of times. Hence the concurrence matrix of this new design has all diagonal elements are equal and all of its off-diagonal elements are also equal. So the new design is a BIBD.

Lemma 6.4.1

A Semi Regular Graph Design obtained from a BIBD by adding a single block.

This Lemma is due to Jacoux (1985).
Theorem 6.4.3

A Semi Regular Graph Design obtained from a Group Testing design for \( v^* = 0 \pmod{6} \) by adding two identical blocks.

Proof

The proof of this theorem obtained from Theorems 6.4.1 and 6.4.2 and Lemma 6.4.1.

6.4.1 Numerical Examples

Theorem 6.4.1 can be illustrated by the following examples.

Example 6.4.1

Consider the group testing design \( d \) for \( v^* = 0 \pmod{6} \) discussed in example 6.3.1. The blocks of \( d \) are

\[
\begin{align*}
B_1 &= (1, 4, 5) & B_2 &= (1, 6, 7) & B_3 &= (2, 4, 6) \\
B_4 &= (2, 5, 7) & B_5 &= (3, 4, 7) & B_6 &= (3, 5, 6).
\end{align*}
\]

The concurrence matrix of \( d \) is

\[
N_d N_d^{-1} = \begin{pmatrix}
2 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 2 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 2 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 3 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 3 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 3 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 3
\end{pmatrix}
\]
From the concurrence matrix of $d$ it is clear that all of the diagonal elements and all the off-diagonal elements are differing by at most one. So by the definition of semi-regular graph design the Group testing design for $v^* = 0 \pmod{6}$ implies the existence of a semi-regular graph design.

**Example 6.4.2**

Consider the group testing design $d$ for $v^* = 2 \pmod{6}$ discussed in example 6.3.2. The blocks of $d$ are

- $B_1 = (1, 5, 8, 9)$
- $B_2 = (1, 6, 7, 10)$
- $B_3 = (2, 5, 7, 11)$
- $B_4 = (2, 6, 8, 12)$
- $B_5 = (3, 5, 10, 12)$
- $B_6 = (3, 6, 9, 11)$
- $B_7 = (4, 7, 9, 12)$
- $B_8 = (4, 8, 10, 11)$.

The concurrence matrix of the above group testing design $d$ is

$$
N_d N_d^\top = \begin{pmatrix}
2 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\
0 & 2 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 2 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 2 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 0 & 3 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 0 & 0 & 3 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 0 & 1 & 1 & 1 & 3 & 0 & 1 & 1 & 1 & 1 \\
1 & 1 & 0 & 1 & 1 & 1 & 0 & 3 & 1 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 & 1 & 1 & 1 & 3 & 0 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 3 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 3 & 0 & 1 \\
0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 3
\end{pmatrix}
$$
From the concurrence matrix of $d$, it is obvious that all of the diagonal elements and off-diagonal elements differing by at most one. So by the definition of semi-regular graph design, the group testing design $d$ for $v^* = 2 \mod 6$) implies the existence of a semi-regular graph design.

The following example illustrates Theorem 6.4.2.

*Example 6.4.3*

Consider the Group testing design $d$ for $v^* = 0 \mod 6$, (see example 6.3.1). The following table shows the blocks of $d$.

<table>
<thead>
<tr>
<th>Table 6.4.1</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 1 2 2 3 3</td>
</tr>
<tr>
<td>4 6 4 5 4 5</td>
</tr>
<tr>
<td>5 7 6 7 7 6</td>
</tr>
</tbody>
</table>

Here we adding the single block (1, 2, 3) to the design $d$. The new design obtained is shown in Table 6.4.2.

<table>
<thead>
<tr>
<th>Table 6.4.2</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 1 2 2 3 3 1</td>
</tr>
<tr>
<td>4 6 4 5 4 5 2</td>
</tr>
<tr>
<td>5 7 6 7 6 3</td>
</tr>
</tbody>
</table>
The concurrence matrix of this new design is

\[
N_d N_d^{-1} = \begin{pmatrix}
3 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 3 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 3 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 3 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 3 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 3 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 3
\end{pmatrix}
\]

From the concurrence matrix, it is obvious that all of its diagonal elements are equal and all of its off-diagonal elements are also equal. So the corresponding design is a Balanced Incomplete Block Design. This shows that a BIBD is obtained from a group testing design for \( v^* \equiv 0 \pmod{6} \) by adding a single block. However this BIB design is known.

### 6.5 Determination of Type I Optimal Block Design

In the following example we illustrate Theorem 6.4.3 and determined Type I optimal block design.

**Example 6.5.1**

Consider the group testing design for \( v^* = 0 \pmod{6} \) obtained in example 6.3.1. Let us denote this group testing design by \( d_1 (7, 6, 3) \). The blocks of the group testing design \( d_1 (7, 6, 3) \) is shown in Table 6.5.1.
Now, we are adding the single block \((1, 2, 3)\) to \(d_1\). The blocks of the new design \(d_2 (7, 7, 3)\) is shown in Table 6.5.2.

**Table 6.5.1**

\[
\begin{array}{ccccccc}
1 & 1 & 2 & 2 & 3 & 3 \\
4 & 6 & 4 & 5 & 4 & 5 \\
5 & 7 & 6 & 7 & 7 & 6 \\
\end{array}
\]

**Table 6.5.2**

\[
\begin{array}{ccccccc}
1 & 1 & 2 & 2 & 3 & 3 & 1 \\
4 & 6 & 4 & 5 & 4 & 5 & 2 \\
5 & 7 & 6 & 7 & 7 & 6 & 3 \\
\end{array}
\]

The C-matrix of design \(d_2 (7, 7, 3)\) is

\[
C_{d_2} = \begin{pmatrix}
2 & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\
-\frac{1}{2} & 2 & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\
-\frac{1}{2} & -\frac{1}{2} & 2 & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\
-\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 2 & -\frac{1}{2} & -\frac{1}{2} \\
-\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 2 & -\frac{1}{2} \\
-\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 2 \\
\end{pmatrix}
\]
The eigenvalues of $C_{d_2}$ are 2.3333 with multiplicity 6 and 0 with multiplicity 1.

We again, adding the same block $(1, 2, 3)$ to the design $d_2 (7, 7, 3)$, to obtain design $d_3 (7, 8, 3)$, the block of $d_3 (7, 8, 3)$ are shown in Table 6.5.3.

**Table 6.5.3**

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>1</th>
<th>2</th>
<th>2</th>
<th>3</th>
<th>3</th>
<th>1</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>6</td>
<td>4</td>
<td>5</td>
<td>4</td>
<td>5</td>
<td>2</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>7</td>
<td>6</td>
<td>7</td>
<td>7</td>
<td>6</td>
<td>3</td>
<td>3</td>
<td></td>
</tr>
</tbody>
</table>

The concurrence matrix of $d_3 (7, 8, 3)$ is

$$
N_{d_3}N_{d_3}^{-1} = 
\begin{pmatrix}
4 & 2 & 2 & 1 & 1 & 1 & 1 \\
2 & 4 & 2 & 1 & 1 & 1 & 1 \\
2 & 2 & 4 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 3 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 3 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 3 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 3
\end{pmatrix}
$$

From the concurrence matrix of $d_3 (7, 8, 3)$ it is obvious that all of its diagonal elements and off-diagonal elements differing by at most one. So from the definition, $d_1 (7,8,3)$ is a semi-regular graph design.
That is, from a group testing design for $v^* = 0 \pmod{6}$, a semi-regular graph design is obtained by adding two identical blocks. Thus Theorem 6.4.3 is illustrated.

Consider the $C$-matrix $C_{d_3}$ of the design $d_3 (7, 8, 3)$.

$$
C_{d_3} = \begin{bmatrix}
\frac{3}{2} & -\frac{3}{5} & -\frac{3}{5} & -\frac{3}{5} & -\frac{3}{5} & -\frac{3}{5} & -\frac{3}{5} \\
-\frac{3}{5} & \frac{3}{5} & -\frac{3}{5} & -\frac{3}{5} & -\frac{3}{5} & -\frac{3}{5} & -\frac{3}{5} \\
-\frac{3}{5} & -\frac{3}{5} & \frac{3}{5} & -\frac{3}{5} & -\frac{3}{5} & -\frac{3}{5} & -\frac{3}{5} \\
-\frac{3}{5} & -\frac{3}{5} & -\frac{3}{5} & \frac{3}{5} & -\frac{3}{5} & -\frac{3}{5} & -\frac{3}{5} \\
-\frac{3}{5} & -\frac{3}{5} & -\frac{3}{5} & -\frac{3}{5} & \frac{3}{5} & -\frac{3}{5} & -\frac{3}{5} \\
-\frac{3}{5} & -\frac{3}{5} & -\frac{3}{5} & -\frac{3}{5} & -\frac{3}{5} & \frac{3}{5} & -\frac{3}{5} \\
-\frac{3}{5} & -\frac{3}{5} & -\frac{3}{5} & -\frac{3}{5} & -\frac{3}{5} & -\frac{3}{5} & 2
\end{bmatrix}
$$

The eigenvalues of $C_{d_3}$ are 3.3333 with multiplicity 2, 2.333 with multiplicity 4 and 0 with multiplicity 1.

$$
\text{tr } C_{d_3} = \sum_{i=1}^{6} Z_{d_3i} = 16
$$

$$
\text{tr } C_{d_3}^2 = \sum_{i=1}^{6} Z_{d_3i}^2 = 44
$$

Consider the design $d_3 (7, 8, 3)$. In this design $\frac{bk}{v} = \frac{24}{7}$ is not an integer. Now, since the design $d_3$ is obtained by adding a single block ($b^*$) to $d_2$ where $b^*$ is any positive integer such that $b^*k < v$. Let $Z_{d_2}$ denotes the common of the $v-1$ non-zero eigenvalues of $C_{d_2}$, then $C_{d_3}$ has
two distinct non-zero eigenvalues, the larger of which occurs with
multiplicity \( b^* (k - 1) \) and is equal to \( Z_{d2} + 1 \) while the smaller is equal
to \( Z_{d2} \) and occurs with multiplicity \( v - 1 - b^* (k - 1) \). Using this result
we obtain

\[ Z_{d3,i} = 3.33 \text{ with multiplicity 2 and} \]
\[ Z_{d3,2} = 2.33 \text{ with multiplicity 4.} \]

Now, using (6.2.2) we obtain \( A = 16, B = 44.44, m_1 = m_1^* = 2.33 \)

\( P_1 = 1.28 \quad m_2 = 2.45, \quad m_3 = 3.02, \quad m_4 = 2.60. \)

Put \( f(x) = \frac{1}{x} \) in (6.2.1) then \( \varphi_f(d) = \sum_{i=1}^{r-1} f(Z_{d3,i}) = \sum_{i=1}^{s} \frac{1}{Z_{d3,i}} = 2. \)

\[ f(m_1) + (v - 3) f(m_2) + f(m_3) = 1/m_1 + 4/m_2 + 1/m_3 = 2.39 \]
\[ f(m_1^*) + (v - 2) f(m_4) = 1/m_1^* + 5/m_4 = 2.35 \]

From the above values we can see that \( m_1 < m_2, \quad m_1^* < m_4 \)

\[ \text{Min } \{ f(m_1) + (v - 3) f(m_2) + f(m_3), f(m_1^*) + (v - 2) f(m_4) \} \]

\[ = \min (2.39, 2.35) = 2.35 \]

\[ \sum_{i=1}^{s} f(Z_{d3,i}) < \min \{ f(m_1) + (v - 3) f(m_2) + f(m_3), f(m_1^*) + (v - 2) f(m_4) \} \]

Also, note that \( \frac{bk}{v} \) is not an integer.

This shows that the design \( d_3 \) is A-optional.

Now, put \( f(x) = -\log x \) in equation (6.2.1), then

\[ \phi_f(C_d) = \sum_{i=1}^{r-1} f(Z_{d3,i}) = \sum_{i=1}^{s} (-\log Z_{d3,i}) = -2.51. \]
\[f(m_1) + (v - 3) f(m_2) + f(m_3) = (-\log m_1) + 4(-\log m_2) + (-\log m_3) = -2.40\]

\[f(m_1^*) + (v - 2) f(m_4) = -\log m_1^* + 5 (-\log m_4)\]

\[= -2.44\]

\[\text{Min} \{f(m_1) + (v-3) f(m_2) + f(m_3), f(m_1^*) + (v - 2) f(m_4)\}\]

\[= \text{min} (-2.40, -2.44) = -2.44\]

From the above values we can see that \(m_1 < m_2, m_1^* < m_4\), \(\frac{bk}{v}\) is not an integer and

\[\sum_{i=1}^{6} f(Z_{d3,i}) < \text{min} \{f(m_1) + (v-3) f(m_2) + f(m_3), f(m_1^*) + (v - 2) f(m_4)\}.\]

This shows that design \(d_3\) is D-optimal. It follows from Lemma 6.3.2 that an A- or D-optimal design \(d_3 (7, 8, 3)\) must be a semi-regular graph design. It is Type I optimal design by 6.2.1.

---