CHAPTER 5
APPLICATIONS OF HYPERCUBIC DESIGNS

5.0 Introduction

Here we have shown that some Semi-Regular Group Divisible designs (SR-1 and SR 23) are possible to construct from certain types of hypercubic designs. However, the class of SRGD designs is known, which can be considered as an application of hypercubic design. Another application of hypercubic design in our investigation is the construction of complete, partial and balanced confounded experiments for symmetrical factorial experiments and then to identify the confounded main efforts and interaction effects.

5.1 Construction of Semi-Regular Group Divisible Designs (SRGD designs) from hypercubic designs

A group Divisible Design (GD Design) is an arrangement of \( v = mn \) treatments in \( b \) blocks, each block contains \( k (< v) \) distinct treatments, each treatment is replicated \( r \) times, and the treatments can be divided into \( m \) groups of \( n (\geq 2) \) treatments each, such that any two treatments occurring together in \( \lambda_1 \) blocks if they belong to the same group and occurring together in \( \lambda_2 \) blocks if they belong to different groups, that is, two treatments of the same group are first associates and two treatments from different groups are second associates. For the usual incidence matrix \( N_d \) of the GD design, \( N_d N_d^t \) has eigenvalues
r - \lambda _1 (= \theta _1, \text{say}) and rk - \lambda _2 v (= \theta _2, \text{say}) other than rk, with the respective multiplicities m (n - 1) and m - i. If \theta _1 > 0 and \theta _2 = 0 then the GD design is called Semi-Regular Group Divisible Design (SRGD design).

Semi-Regular group Divisible designs can be constructed from Hypercubic designs with (i) \( v = 2^2 \), \( k = 2 \) and (ii) \( v = 3^2 \), \( k = 3 \), which are illustrated below.

**Example 5.1.1**

Consider the hypercubic design with parameters \( v = 2^2 \), \( k = 2 \), \( r = 2 \), \( b = 4 \), \( \lambda _1 = 1 \), and \( \lambda _2 = 0 \), obtained by Theorem 3.2.1 of Chapter 3 (Example 3.2.2.).

The blocks of the hypercubic design obtained are

\[
[(11, 12), \quad (21, 22)] \\
[(11, 21), \quad (12, 22)]
\]

Here we decode the treatment combinations as follows

\[
11 \rightarrow 1, \quad 12 \rightarrow 2, \quad 21 \rightarrow 3 \quad \text{and} \quad 22 \rightarrow 4
\]

The following table shows the blocks of the hypercubic design after decoding, here columns treated as blocks.

**Table 5.1.1**

\[
\begin{array}{cccc}
1 & 3 & 1 & 2 \\
2 & 4 & 3 & 4
\end{array}
\]

This is a Semi-Regular Group Divisible design with parameters \( v = 4, \ b = 4, \ r = 2, \ k = 2, \ m = 2, \ n = 2, \ \lambda _1 = 0 \) and \( \lambda _2 = 1 \).
This SRGD design is affine resolvable and the complement of this design is given in the following table.

<table>
<thead>
<tr>
<th>Table 5.1.2</th>
</tr>
</thead>
<tbody>
<tr>
<td>3 1 2 1</td>
</tr>
<tr>
<td>4 2 4 3</td>
</tr>
</tbody>
</table>

However, this SRGD design is already reported in Clatworthy (1973) as SRI and it is truly self complementary.

Example 5.1.2

Consider the hypercubic design with parameters $v = 3^2$, $k = 3$, $r = 2$, $b = 6$, $\lambda_1 = 0$ and $\lambda_2 = 1$, obtained by Theorem 3.2.2 of Chapter 3 (Example 3.3.5).

The blocks of the hypercubic design obtained are

\[
[(11 \ 22 \ 33), \quad (11 \ 23 \ 32),
(12 \ 23 \ 31), \quad (12 \ 21 \ 33),
(13 \ 21 \ 32)], \quad (13 \ 22 \ 31)].
\]

Here we decode the treatment combinations as follows.

$11 \rightarrow 1$, $12 \rightarrow 2$, $13 \rightarrow 3$, $21 \rightarrow 4$, $22 \rightarrow 5$, $23 \rightarrow 6$, $31 \rightarrow 7$, $32 \rightarrow 8$, $33 \rightarrow 9$

The following table shows the blocks of the hypercubic design after this decoding, here columns treated as blocks.
Now, if we add the blocks (1,2,3), (4,5,6), and (7,8,9), we get SRGD design with parameters $v = 9$, $b = 9$, $r = 3$, $k = 3$, $m = 3$, $n = 3$, $\lambda_1 = 0$ and $\lambda_2 = 1$. The following table shows the blocks of the SRGD design.

**Table 5.1.4**

<table>
<thead>
<tr>
<th>1</th>
<th>2</th>
<th>3</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>1</th>
<th>4</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>6</td>
<td>4</td>
<td>6</td>
<td>4</td>
<td>5</td>
<td>2</td>
<td>5</td>
<td>8</td>
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<tr>
<td>9</td>
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<td>8</td>
<td>9</td>
<td>7</td>
<td>3</td>
<td>6</td>
<td>9</td>
</tr>
</tbody>
</table>

This SRGD design is affine resolvable and the complement of this design is given in the following table.

**Table 5.1.5**

<table>
<thead>
<tr>
<th>2</th>
<th>1</th>
<th>1</th>
<th>2</th>
<th>1</th>
<th>1</th>
<th>4</th>
<th>1</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>3</td>
<td>2</td>
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<td>3</td>
<td>2</td>
<td>5</td>
<td>2</td>
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<td>6</td>
<td></td>
</tr>
</tbody>
</table>
However, this SRGD design is already reported in Clatworthy (1973) as SR 23 and its complement as SR 65.

The designs constructed in Example 5.2.1 and 5.2.2 satisfies all the properties of SRGD designs; viz.,

(i) $b \geq v - m + 1$

(ii) $k$ is divisible by $m$.

(iii) $r - \lambda_1 > 0$ and $rk - \lambda_2 v = 0$ that is, $\theta_1 > 0$ and $\theta_2 = 0$

(iv) $P_1 = P_{jk}^1 = \begin{pmatrix} n-2 & 0 \\ 0 & n(m-1) \end{pmatrix}$

$P_2 = P_{jk}^2 = \begin{pmatrix} 0 & n-1 \\ n-1 & n(m-2) \end{pmatrix}$

(v) $n_1 = n - 1$ and $n_2 = n(m - 1)$.

5.2 Identification of Confounded main effects and Interaction effects in Symmetrical Factorial Experiments

In factorial experiments, combinations of two or more levels of more than one factor are taken as the treatments. If the number of levels of each factor in an experiment is the same, the experiment is called symmetrical factorial, otherwise, it is called asymmetrical factorial. These experiments provide an opportunity to study not only the individual effects of each factor but also their interactions. We shall denote the factors in a general way by the letters A, B, C etc. Several types of notations of the levels of the factors are in use. These notations
are always in codes. One type is \( a_0, a_1, \ldots \) for the factor A and similarly for the other factors. One more type of level codes used is 0, 1, 2, \ldots for each factor.

The simplest of the factorials consists of two factors each at two levels. This factorial is called \( 2^2 \) as there are four treatment combinations in the experiment. Denoting the factors by A and B, the treatment combinations can be written either as \( a_0b_0, a_0b_1, a_1b_0 \) and \( a_1b_1 \) or as 00, 01, 10 and 11. Sometimes the treatment combinations are also written as 1, a, b and ab. The symbol 1 denotes that both factors are at the lower levels in this combination. It is also called the control treatment.

The factorial experiments are conducted by using completely randomized, randomized block or Latin square design. If the number of factors or the number of levels of each factor increase then the number of treatment combinations also increase and it is not advisable to adopt a randomized block design or a Latin square design for it, because blocks having large number of plots are too big to ensure homogeneity within them. A new device is, therefore, necessary for designing experiments with a large number of treatments. One such device is to take blocks of size less than the number of treatments and have more than one block per replication. The treatments are then divided into as many groups as the number of blocks per replication. The different groups of treatments are allotted to the blocks.

In general, for the confounded experiment the block size for \( 2^n \) factorials is of the form \( 2^k \). Consider, the concept of confounding by
having only two blocks, for obtaining an interaction contrast, the treatment combinations are divided into two groups. Two such groups representing a suitable interaction, say P, can be taken to form the contents of two blocks, each containing half the total number of treatment combinations. In such cases the contrast of the interaction P and the contrast between the two block totals are mixed up with block effects and cannot be separated. In other words, the interaction P, has been confounded with the blocks. Evidently, P has been lost but the other interactions and main effects can now be estimated with better precision because of reduced block size. This device of reducing the block size by making one or more interaction contrasts identical with block contrasts is known as confounding. The number of blocks can be any power of 2. Preferably, only higher interactions, that is, interactions with three or more factors are confounded, because their loss is immaterial. The designs for such confounded factorials are called incomplete randomized block designs.

When there are two or more replications, sometimes the same interactions are confounded in each replication or different sets of interactions are confounded in different replications. If the same set of interactions is confounded in all the replications, confounding is called complete. If different sets of interactions are confounded in different replications, confounding is called partial. In complete confounding all the confounded interactions are lost. But in partial confounding, the confounded interaction can be recovered from those replications in which they are not confounded. If all effects of a certain order, say all
two-factor interactions, are confounded with incomplete block differences an equal number of times, it is called balanced confounding.

In this investigation we have shown that the hypercubic designs discussed in section 3.2 provide some confounding systems for symmetrical factorial experiments. Identification of confounded main effects and interaction effects is discussed here using counter-examples.

5.2.1 Preliminary Results

Das and Giri (1986) discussed how to identify a confounded interaction. The procedure is as follows. Consider the key block of the replicate of a confounded experiment. Code the treatment combinations in terms of $0, 1, 2, \ldots s - 1$, where $0, 1, 2, \ldots s - 1$ are the elements of the Galois field (GF($s$)) of order $s$. Next, form the identity matrix of order $r \times r$, and then identify the confounded interactions from the extra $(n - r)$ columns.

Kane (1994) has obtained a simple method for identifying confounded interactions in $2^n$ ($n = 2, 3$) factorial experiments, when (i) there are only two blocks per replication and (ii) there are more than two blocks per replication. In this method, the conditions imposed to identify a confounded interaction from two blocks per replication and from more than two blocks per replication are not sufficient. Some more conditions were introduced by Ghosh and Bagui (1998) and they have proposed a new way of identifying the confounded main effects and interaction effects in $2^n$ factorial experiments. The same method is extended to $3^n$ and $s^n$ factorial experiments.
While using Ghosh and Bagui (1998), once you identify the confounded main effects and interaction effects from all the replications, the corresponding confounded experiment can be easily identified.

The method of identifying the confounded designs and confounded interactions given by Ghosh and Bagui (1998) is explained below.

5.2.2 Procedure for Identifying Confounded main effects and Interaction effects in $2^n$ Factorial Experiments

The following definitions and procedures are from Kane (1994).

First, determining the principal block group in the design. The principal block group is a combination (s) of one block (when there are two blocks and only one interaction is to be confounded) or more than one block (when there are more than two blocks and more than one interaction is to be confounded), out of which one block must contain the symbol ‘(i)’ (that is all the factors are at a lower level) and some other block (or blocks) such that the total number of treatment combinations in the entire group must be $2^{n-1}$ (or $s^{n-1}$). The group of all remaining blocks is called the ‘non-principal block group’. To identify the confounded interaction the following steps should be performed:

(i) Decide on the principal block group and non-principal block group;

(ii) Collect the treatment labels which contain a single letter from the non-principal block group;
(iii) Combine these treatment labels multiplicatively to give the confounded interaction;

(iv) Change the labels to corresponding treatment labels to indicate effects, as opposed to treatments;

(v) Each non-principal block group should then be examined to find the corresponding confounded interaction.

Ghosh and Bagui (1998) have shown that the steps for identifying the confounded main effects and interaction effects (Kane, 1994) as mentioned in Section 5.2.2 for the case of two blocks per replication are not sufficient. They have modified the method as follows.

If the multiplication of treatment combinations of the single letter in the non-principal block group is $X_1, X_2, \ldots X_n$, for example, and its geometrical representation $X_1 + X_2 + \ldots X_n = 0 \mod 2$ is satisfied in the principal block group, and $X_1 + X_2 + \ldots X_n = 1 \mod 2$ is satisfied in the non-principal block group, then $X_1 \cdot X_2 \ldots X_n$ is identified as a confounded interaction, otherwise the interaction is unconfounded. Similar modification should be carried out for $2^n$ factorial experiments when there are more than two blocks per replication.

Therefore, the extra conditions added to Kane’s (1994) method to identify the confounded interactions from the non-principal block group in $2^n$ factorial experiments are

(vi) Block I which contains (i) must satisfy $X_1 + X_2 + \ldots X_n = 0 \mod 2$ for the corresponding confounded interactions;

(vii) Other blocks must satisfy $X_1 + X_2 + \ldots X_n = 1 \mod 2$ for the corresponding confounded interactions in $2^n$ factorial experiments.
5.2.3 Procedure for Identifying Confounded main effects and Interaction effects in $3^n$ Factorial Experiments

In the case of confounding $3^n$ factorial experiments into $3^{n-1}$ block sizes, there will be only three blocks per replication for $n = 2, 3, \ldots$

To identify the confounded main effects and interaction effects the following steps should be performed.

(i) The block that contains symbol ‘(i)’ (i.e., all the factors are at a lower level) will be called the principal block group and the remaining two blocks will be called the non-principal block group.

(ii) Collect all those treatment combinations which contain a single letter but which have a power of one or two from any one of the non-principal block group.

(iii) Combine these treatment levels multiplicatively to give the confounded main effects and interaction effects.

(iv) Next, one must check that the geometrical representation of this confounded interaction satisfies $\sum_{i=1}^{n} p_i X_i = 0 \mod 3$, for the corresponding identified confounded main effects and interaction effects in the principal block group where $p_i \in \text{GF}(3)$.

(v) In the remaining two blocks (which happen to be the non-principal block group) $\sum_{i=1}^{n} p_i X_i \equiv j \mod 3$ for the confounded main effects and interaction effects must hold for $j = 1, 2$. 
To identify the confounded main effects and interaction effects in $3^n$ factorial experiments when there are more than three blocks per replication, the following steps should be performed.

(i) The block that contains the symbol ‘(i)’, along with two more blocks, will be considered as the principal block group, and a group of three blocks will be called the non-principal block group.

(ii) Collect all the single letters that have a power of one or two from the non-principal block group.

(iii) Combine these treatment levels multiplicatively and check that steps (iv) and (v) from the case when there are two blocks per replication, are satisfied. Otherwise, select another non-principal block group by combining three other blocks. Continue this process of identifying the confounded main effects and interaction effects until $(3^{n-r}-1)/2$ principal block groups are formed and identified.

In general, in an $s^n$ factorial experiment confounded in $s^r$ block sizes, there will be $(s^{n-r}-1)/(s-1)$ principal block groups and a total of $(s^{n-1}-1)/(s-1)$ main effects and interaction effects will be confounded, out of which $(n-r)$ will be independent.

5.3 Construction of Confounded Experiments and Identification of Confounded effects using Hypercubic Designs

Suppose we have a $t^m$ symmetrical factorial experiment whose treatments are denoted by $(x_1, x_2, \ldots, x_m)$ where $x_i = 1, 2, \ldots, t$
for $i = 1, 2, \ldots m$. Then the hypercubic designs discussed in section 3.2 provide some confounding systems for this factorial experiment. The confounded main effects and interaction effects are identified by the methods given in sections 5.2.2 and 5.2.3 for the hypercubic designs considering it as a confounded factorial experiment which are explained below by examples.

5.3.1 \ 2^n Factorial Experiment Confounded in $2^r$ block size ($r < n$)

5.3.1.1 Confounding in two blocks

Example 5.3.1

Consider the hypercubic design with parameters $v = 2^4$, $r = 4$, $b = 32$, $k = 2$, $\lambda_1 = 1$, $\lambda_2 = 0$, $\lambda_3 = 0$ and $\lambda_4 = 0$ (eg. 3.3.1 (a)).

To construct the confounded experiment and to identify the confounded main effects and interaction effects, the blocks of the design are rearranged as follows.

Consider the first replication of the hypercubic design. The first elements of all the blocks are taken together and kept as Block I of the new design and the second elements of all the blocks are kept as Block II of the new design. We get the following two blocks corresponding to the first replication. For convenience, we replace the symbols '1' and '2' by '0' and '1' respectively.

Block I 0000 0010 0100 0110 1000 1010 1100 1110

Block II 0001 0011 0101 0111 1001 1011 1101 1111
This is equivalent to the layout of a $2^4$ factorial experiment confounded in $2^3$ block size.

Here Block I is the principal block group as one treatment combination consists of all the factors at the lower level and Block II is the non-principal block group. Now collecting the treatment combinations, which contains a single higher level ‘1’ from the non-principal block group and combine these treatment combinations multiplicatively to give the confounded interaction. Let the factors be denoted by A, B, C and D as there are four factors in the design.

Again A, B, C and D will consists of treatment combinations 1000, 0100, 0010 and 0001 respectively as those treatment combinations are considered which are having only one higher level in the non-principal block group. In this case, the non-principal block group, say Block II has only one treatment combination which has single higher level ‘1’, that is, this treatment combination is 0001. This shows that $A^0B^0C^0D^1 = D$ is confounded.

It can easily be seen that its geometrical representation $X_4 = 0$ mode 2 is satisfied in the principal block group, $X_4 = 1$ mod 2 is satisfied in the non-principal block group.

In the similar way, it can be verified that the main effect C is confounded in Replication II, B is confounded in Replication III and A is confounded in Replication IV. All the main effects are confounded same number of times. So the resulting design is balanced confounding experiment.
Here $2^4$ factorial experiment is confounded in $2^3$ block sizes, the total number of main effects or interaction effects confounded is 

$\frac{(2^{4-3} - 1)}{(2 - 1)} = 1$ in each replication.

### 5.3.1.2 Confounding in more than two blocks

**Example 5.3.2**

Consider the hypercubic design discussed in eg. 5.3.1. To obtain the confounded experiment and to identify the confounded main effects and confounded interaction effects using this hypercubic design, we proceed as follows.

Consider the first replication. Block I of the new design is obtained by taking the first element of each of the initial 4 blocks of the first replication, Block II is obtained by taking the first element of each of the last 4 blocks of the first replication. Similarly Block III is formed by the second element of each of the initial 4 blocks and Block IV is formed by the second element of each of the last 4 blocks. We obtain the following four blocks corresponding to Replication I.

<table>
<thead>
<tr>
<th>Block I</th>
<th>Block II</th>
<th>Block III</th>
<th>Block IV</th>
</tr>
</thead>
<tbody>
<tr>
<td>0000</td>
<td>1000</td>
<td>0001</td>
<td>1001</td>
</tr>
<tr>
<td>0010</td>
<td>1010</td>
<td>0011</td>
<td>1011</td>
</tr>
<tr>
<td>0100</td>
<td>1100</td>
<td>0101</td>
<td>1101</td>
</tr>
<tr>
<td>0110</td>
<td>1110</td>
<td>0111</td>
<td>1111</td>
</tr>
</tbody>
</table>
This is the layout of a $2^4$ factorial experiment confounded in $2^2$ block sizes. Let the main effects be denoted as A, B, C and D. To identify the confounded main effects and interaction effects we proceed as follows. Here, Block I contains one treatment combination whose all levels are lower. Consider Block I together with Block II as the principal block group. Obviously Block III together with Block IV constitute the non-principal block group. Collecting the treatment combinations which contains a single higher level '1' from the non-principal block group and combine these treatment combinations multiplicatively to give the confounded interaction effects. Here A, B, C and D will consists of the treatment combination 1000, 0100, 0010 and 0001 respectively as those treatment combinations are considered which are having only one higher level in the non-principal block group. In this case, the non-principal block group has only one treatment combination which has single higher level '1' that is, the treatment combination is 0001. This shows that $A^0B^0C^0D^1 = D$ is confounded.

Here it can be verified that $X_4 = 0 \mod 2$ holds in the principal block group and $X_4 = 1 \mod 2$ holds in the non-principal block group.

Next consider Block I together with Block III as the principal block group. Hence, from Blocks II and IV, being the non-principal block group, we see that A is the confounded main effect corresponding to the single higher level treatment combination. It can be seen that $X_1 = 0 \mod 2$ in the principal block group and $X_1 = 1 \mod 2$ in the non-principal block group, are satisfied.

Finally, if Blocks I and IV constitute the principal block group, then, Blocks II and III, will constitute the non-principal block group.
Collecting the treatment combinations, which contains a single higher level ‘1’ from the non-principal block group. In this case, in the non-principal block group, the treatment combinations of a single higher level ‘1’ are 1000 and 0001, that is, A and D, and their multiplication is AD. Hence interaction AD is confounded.

Obviously, \( X_1 + X_4 = 0 \mod 2 \) holds in the principal block group and \( X_1 + X_4 = 1 \mod 2 \) holds in the non-principal block group. In fact, it can be seen that the third confounded interaction is the generalized confounded interaction. This shows that main effects A and D are confounded in Replication I along with its generalized interaction AD. Finally Block I will be considered as a key block.

In the similar way it can be verified that the main effects A and C and their generalized interaction AC are confounded in Replication II, main effects A and B and their generalized interaction AB are confounded in Replication III and the main effects A and B and their generalized interaction AB are confounded in Replication IV.

Here the total number of main effects or interaction effects confounded in each replication is \( \frac{2^{n-2}-1}{2-1} = 3 \), out of which 4 - 2 (= 2) are independent confounded effects.

Hence, for this design, the experiment is partially confounded design as different main effects and interaction effects are confounded in different replications.

If the order of the blocks are changed in each replication of the hypercubic design the confounded main effects and confounded
interaction effects will also be changed, but the total number of main
effects and interaction effects confounded will remains the same in
each replication.

Example 5.3.3

Consider the hypercubic design in example 3.3.1 (b) with
parameters \( v = 2^4,\ r = 6,\ b = 24,\ k = 4,\ \lambda_1 = 3,\ \lambda_2 = 1,\ \lambda_3 = 0\) and \( \lambda_4 = 0 \).

To construct the confounded factorial experiment and identify the
confounded main effects and confounded interaction effects we proceed
as follows.

In each replication of the design, the first elements of each block
form the Block I of the confounded factorial experiment. Similarly the
second elements of each block form Block II of the confounded factorial
experiment and so on. In this way we get four blocks each in each
replication. As in example 5.3.2 we observed that the total number of
main effects and interaction effects confounded in each replication are
\((2^{4-2}-1) / (2 - 1) = 3\) and out of which \( 4 - 2 (= 2) \) are independent. In the
first replication the main effects C, D and their generalized interaction
effect CD are confounded, in the second replication the main effects B,C
and their generalized interaction effect BC are confounded, in the third
replication the main effects A, B and their generalized interaction effect
AB are confounded, in the fourth replication the main effects B,D and
their generalized interaction effect BD are confounded, in the fifth
replication the main effects A,D and their generalized interaction effect
AD are confounded and in the sixth replication the main effects A,C and
their generalized interaction effect AC are confounded.
Here, all the main effects A, B, C and D are confounded three times each in all the six replications and their generalized interaction of two factors viz., AB, AC, AD, BC, BD and CD are also confounded once and only once automatically in six replications and hence the given design is balanced confounding factorial experiment.

5.3.2 3ⁿ Factorial Experiment Confounded in 3ⁿ Block sizes (r < n)

5.3.2.1 Confounding in three blocks

Example 5.3.4

Consider the hypercubic design, given in Example 3.3.5, with parameters \( v = 3^2 \), \( r = 2 \), \( b = 6 \), \( k = 3 \), \( \lambda_1 = 0 \) and \( \lambda_2 = 1 \).

In the design we replace 1 by 0, 2 by 1 and 3 by 2 to obtain the confounded factorial experiment. To identify the confounded main effects and interaction effects we proceed as follows.

Case (i)

If the blocks of the hypercubic design are taken as the blocks of the confounded factorial experiment in each replication. We get the following blocks.

<table>
<thead>
<tr>
<th>Replication I</th>
<th>Replication II</th>
</tr>
</thead>
<tbody>
<tr>
<td>Block I</td>
<td>Block II</td>
</tr>
<tr>
<td>00</td>
<td>01</td>
</tr>
<tr>
<td>11</td>
<td>12</td>
</tr>
<tr>
<td>22</td>
<td>20</td>
</tr>
</tbody>
</table>
Here there are three blocks, each of three treatment combinations in each replication.

Consider the first replication, Block I is the principal block group as it contains the treatment combination with all the factors at the lower level and Blocks II and III are the non-principal block group. Now, take Block II and collect all the treatment combinations which contain a single level and combine these treatment combinations multiplicatively to get the confounded main effects or interaction effects. In this case, in block II, the treatment combination which have single level are 01 and 20, which correspond to $B (=A^0B^1)$ and $A^2 (=A^2B^0)$ and their multiplication is $A^2B$. But $A^2B$ is the same as $AB^2$. So $AB^2$ is confounded. In this case its geometrical representation $X_1 + 2X_2 = 0$ mod 3 is satisfied in the principal block group and $X_1 + 2X_2 = j$ mod 3 is satisfied in the other two blocks for $j = 1, 2$.

If we take Block III as a non-principal block group, we can see that $AB^2$ is the confounded interaction. This shows that the same interaction $AB^2$ is confounded. Therefore the confounded interaction can be identified from any block in the non-principal block group.

In a similar way, we can see that $AB$ or $A^2B^2$ is confounded in Replication II.

All the two factor interactions are confounded the same number of times, so the resulting design is balanced confounding experiment.
Case (ii)

If columns are taken as blocks, i.e. the first elements of the blocks in the design forming Block I of the confounded factorial experiment. Second elements of the blocks in the design forming Block II of the confounded factorial experiment and third elements of the blocks in the design forming Block III of the confounded factorial experiment in each replication, we get the following.

<table>
<thead>
<tr>
<th>Replication I</th>
<th>Replication II</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Block I</strong></td>
<td><strong>Block I</strong></td>
</tr>
<tr>
<td><strong>Block II</strong></td>
<td><strong>Block II</strong></td>
</tr>
<tr>
<td><strong>Block III</strong></td>
<td><strong>Block III</strong></td>
</tr>
<tr>
<td>00</td>
<td>12</td>
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<tr>
<td>01</td>
<td>12</td>
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<tr>
<td>02</td>
<td>10</td>
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<tr>
<td>11</td>
<td>22</td>
</tr>
<tr>
<td>12</td>
<td>20</td>
</tr>
<tr>
<td>10</td>
<td>21</td>
</tr>
<tr>
<td>22</td>
<td>20</td>
</tr>
</tbody>
</table>

Proceeding as in case (i), we can see that the main effect $A$ is confounded in both the replications, and its geometrical representation $X_1 = 0 \mod 3$ is satisfied in the principal block group and $X_1 = j \mod 3$ were $j = 1, 2$, is satisfied in each block of the non-principal block group in each replication. Here $A$ is confounded completely.

We can conclude that the confounding is balanced partial or complete which depends on the way in which the blocks of the confounding design are formed. Another property of hypercubic design used as a confounding experiment for a symmetrical factorial is that
complete balance is achieved over each of the main effects and any order of interactions, i.e., all normalized contrasts belonging to the same main effect or interaction effect are estimated with the same variance, due to the fact that a hypercubic design is also a Balanced Factorial Experiment.

Cheng (1989) has discussed to obtain partial and balanced confounding factorial experiments only. While in this method one can obtain balanced, partial or complete confounding factorial experiment that depends on the way in which the blocks are taken.

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A paper based on this chapter and chapter 3 entitled "Hypercubic Designs and Applications" is communicated for publication in Statistical Papers.