Chapter 3

On Equivariant-Complete Invariance Property

In this chapter we redefine the concepts of S-ECIP, weak EQ, ED by considering nonempty invariant closed set instead of nonempty invariant closed set containing the set of stationary points. We find that Propositions 1.8 and 2.4 [2] continue to be true. Several other results have also been obtained. Equivariant analogues of some of the results in [11], [13], [15], [24] and [31] and Proposition 2.4.1, Chapter 2 have been obtained. Throughout this chapter, unless otherwise mentioned, $I$ is considered with the trivial action.
3.1 Definition, Examples and Results

In [2], a $G$-space $X$ is defined to have $S$-equivariant complete invariance property (S-ECIP), if every nonempty invariant closed set $F$ of $X$ containing the set of stationary points is a fixed point set of an equivariant continuous self-map on $X$. The following example shows that the condition that the closed set contains stationary points is redundant.

**Example 3.1.1** Let $Z_2=\{0, 1\}$ be the additive group of integers modulo 2 with the discrete topology. Consider the closed interval $X=[-1, 1]$ also with the discrete topology. Let $\Theta: Z_2 \times X \to X$ be an action of $Z_2$ on $X$ defined by $\Theta(1, 1)=1.1=1$, $\Theta(1,-1)=1.(-1)=-1$ and $\Theta(1, x)=-x$, $x\in(-1, 1)$. The stationary points are 0, 1, and -1. Let $F=\{-1/2, 1/2\}$. Define $f:[-1, 1] \to [-1, 1]$ by $f(x)=x$; $x\in F$, $f(x)=-1$; $x=1$ and $f(x)=1$; $x\not\in F$ and $x\neq 1$. For $x\in F$, $f(1.x)=f(-x)=-x$ and 1. $f(x)=1.x=-x$, for $x=1$, $f(1.1)=f(1)=-1$ and $1.f(1)=1.(-1)=-1$, for $x=-1$, $f(1.-1)=f(-1)=1$ and 1.$f(-1)=1.1=1$, for $x\not\in F$, $x\neq \pm 1$, $f(1.x)=f(-x)=1$ and 1. $f(x)=1.1=1$. Thus $f$ is equivariant and continuous, $\text{fix}(f)=F$ and $F$ does not contain the set of stationary points.

Hence we have the following definition.

**Definition 3.1.2** A $G$-space $X$ is said to have the equivariant complete invariance property (E-CIP), if every nonempty invariant closed set $F$ of $X$ is a fixed point set of an equivariant continuous self-map on $X$. 
**Example 3.1.3** Let $X$ be an indiscrete space. Let a topological group $G$ act on $X$. The only nonempty invariant closed set of $X$ is $X$ itself. Then the identity map $Id: X \to X$ is an equivariant continuous map on $X$ such that $\text{fix}(Id) = X$. Thus $X$ has E-CIP.

The following is Proposition 1.8 [2] with E-CIP in place of S-ECIP.

**Proposition 3.1.4** Let $X$ be a metric space and $S^1$ act on $X \times S^1$ by the action defined by $\Theta(q, (x, p)) = (x, pq)$, where $p, q \in S^1$ and $x \in X$. Then $X \times S^1$ has E-CIP.

**Proof.** Clearly the set of stationary points of $X \times S^1$ is $\emptyset$. Hence the proof of this result is similar to that of Proposition 1.8 [2].

**Proposition 3.1.5** Let $f$ be a $G$-homeomorphism from a $G$-space $X$ having E-CIP to a $G$ space $Y$. Then $Y$ has E-CIP.

**Proof.** It is similar to that of Proposition 2.3.1.

### 3.2 Definitions

Let a topological group $G$ act on a space $X$. The action of $G$ on $X \times I$ is defined by $g.(x, t) = (g \cdot x, t)$, $x \in X$, $g \in G$ and $t \in I$. 
**Definition 3.2.1** A $G$-space $X$ has property E-EQ, if for every nonempty invariant closed subset $F$ of $X$ there is a point $p \in F$, a $G$-retract $T$ of $X$ containing $F$ and a $G$-deformation $H : T \times I \to T$ such that $H(x, t) \neq x$, if $x \neq p$ and $t > 0$.

**Definition 3.2.2** A $G$-space $X$ has property E-EW, if for every point $p \in X$ there is a $G$-deformation $H : X \times I \to X$ such that $H(x, t) \neq x$, if $x \neq p$ and $t > 0$.

**Definition 3.2.3** A $G$-space $X$ has property weak E-EQ, if for every nonempty invariant closed subset $F$ of $X$ there is a $G$-retract $T$ of $X$ containing $F$ and a $G$-deformation $H : T \times I \to T$ such that $H(x, t) \neq x$, if $x \notin F$ and $t > 0$.

**Definition 3.2.4** A $G$-space $X$ has property strong E-EW, if there is a $G$-deformation $H : X \times I \to X$ such that $H(x, t) \neq x$, whenever $t > 0$.

**Definition 3.2.5** Let $(X, d)$ be a metric $G$-space. Then $X$ is said to satisfy the property E-ED if given a nonempty invariant closed subset $F$ of $X$ and a point $x$ in $X$, the distance between any point lying in the orbit $G_x$ of $x$ and $F$ is constant.

**Definition 3.2.6** Let $A$ be an invariant subspace of a $G$-space $X$. Let $j : A \to X$ be the inclusion map and let $f : X \to A$ be an equivariant continuous map. Suppose
that the map $j \circ f: X \to X$ is $G$-homotopic to the identity map $Id: X \to X$ under a $G$-homotopy $H: X \times I \to X$. Then $H$ is called a $G$-deformation retraction.

**Definition 3.2.7** A $T_1$ $G$-space $X$ is called $G$-perfectly normal if every nonempty invariant closed subset $A$ of $X$ is such that $A = h^{-1}(\{0\})$ for some continuous equivariant mapping $h: X \to I$. This definition is motivated by the following remark.

**Remark 3.2.8** For a $G$-space $X$ and a continuous equivariant map $h: X \to I$, $h^{-1}(\{0\})$ is invariant and closed.

### 3.3 Results on E-CIP (continued)

We obtain below some of the results in [11], [13], [15], [24] and [31] in equivariant setting and thus obtain conditions for a $G$-space to possess E-CIP.

**Proposition 3.3.1** Any perfectly normal $G$-space $X$ having property $E$-EW has E-CIP.

**Proof.** Let $F$ be a nonempty invariant closed set of $X$ and let $p \in F$. Then there is a $G$-deformation $H: X \times I \to X$ such that $H(x, t) \neq x$ if $x \neq p$ and $t > 0$. Since $X$ is perfectly normal, there is an equivariant continuous map $h: X \to I$ such that $h(x) = 0$
iff \( x \in F \). Let \( f: X \to X \) be a map defined by \( f(x) = H(x, h(x)) \), \( x \in X \). It has been given in [15] that \( f \) is continuous and \( \text{fix}(f) = F \). We prove that \( f \) is equivariant. For \( g \in G \) and \( x \in X \), \( f(g.x) = \text{H}(g.x, h(g.x)) = \text{H}(g.x, g.h(x)) = g.H(x, h(x)) = g.f(x) \). Hence \( X \) has E-CIP.

The following is Proposition 2.4 [2] with E-CIP in place of S-ECIP.

**Proposition 3.3.2** Any metric \( G \)-space \( (X, \rho) \) having property weak E-EQ and E-ED has E-CIP.

**Proof.** We assume that \( \rho \leq 1 \). Let \( F \) be a nonempty invariant closed set of \( X \). Let \( T \) be a \( G \)-retract such that \( T \) contains \( F \) and \( H: T \times I \to T \) be a \( G \)-deformation such that \( H(x, t) \neq x \), for \( x \notin F \) and \( t > 0 \). Let \( f: X \to X \) be defined by \( f(x) = H(r(x), \rho(r(x), F)) \), \( x \in X \), where \( r: X \to T \) is a \( G \)-retraction. We first prove that \( \text{fix}(f) = F \). Since for \( x \in F \), \( r(x) = x \), therefore for \( x \in F \), \( \rho(r(x), F) = \rho(x, F) = 0 \). Hence \( f(x) = H(x, 0) = x \). For \( x \in F' \), there are two cases:

Case 1: Let \( x \in T - F \), then \( f(x) = H(x, \rho(x, F)) \neq x \), because \( \rho(x, F) \neq 0 \).

Case 2: Let \( x \in T' \), then \( f(x) = H(r(x), \rho(r(x), F)) \neq x \), because \( r(x) \in T \) and \( H(x, t) \in T \), for every \( x \in T \).

To prove that \( f \) is equivariant let \( g \in G \) and \( x \in X \). Then \( f(g.x) = H(r(g.x), \rho(g.r(x), F)) = H(g.r(x), \rho(g.r(x), F)) = g.H(r(x), \rho(g.r(x), F)) \). Since \( X \) has property E-ED, hence \( \rho(r(x), F) = \rho(g.r(x), F) \). Thus \( f(g.x) = g.H(r(x), \rho(r(x), F)) = g.f(x) \). This implies that \( f \) is equivariant. Hence \( X \) has E-CIP.
Proposition 3.3.3 Let $X$, $Y$ be compact Hausdorff $G$-spaces such that $X$ has property $E$-$EQ$ and $Y$ has property $E$-$EW$. Then $X \times Y$ has property $E$-$EQ$.

Proof. Let $F$ be a nonempty invariant closed set of $X \times Y$. Let $\pi$ denote the natural projection of $X \times Y$ onto $X$. Then $\pi(F)$ is closed. Since $X$ has property $E$-$EQ$, there is a point $p \in \pi(F)$, a $G$-retract $T$ of $X$ containing $\pi(F)$ and a $G$-deformation $f_1 : T \times I \rightarrow T$ such that $f_1(x, 0) = x$ and $f_1(x, t) \neq x$ if $x \neq p$ and $t > 0$, $x \in X$. Let $q$ be a point in $Y$ such that $(p, q) \in F$. Then there is a $G$-deformation $f_2 : Y \times I \rightarrow Y$ such that $f_2(y, 0) = y$ and $f_2(y, t) \neq y$ if $y \neq q$ and $t > 0$, where $y \in Y$. Clearly $(p, q) \in F \subset T \times Y$. Let $r : X \times Y \rightarrow T \times Y$ be defined by $r(x, y) = (r_1(x), y)$, where $r_1 : X \rightarrow T$ is a $G$-retraction. Then $r$ is a retraction. For $g \in G$ and $(x, y) \in X \times Y$, $r(g(x, y)) = r(g, x, g, y) = (r_1(g, x), g, y) = (g, r_1(x), g, y) = g, (r_1(x), y) = g, r(x, y)$ which implies that $r$ is equivariant. This proves that $T \times Y$ is a $G$-retract of $X \times Y$. Let $H : (T \times Y) \times I \rightarrow T \times Y$ be defined by $H((x, y), t) = (f_1(x, t), f_2(y, t))$, $(x, y) \in T \times Y$ and $t \in I$. Then $H((x, y), 0) = (x, y)$ and $H((x, y), t) \neq (x, y)$, if $(x, y) \neq (p, q)$ and $t > 0$. We now prove that $H$ is equivariant. For $g \in G$ and $(x, y) \in T \times Y$, $H(g, ((x, y), t)) = H((g, x, g, y), t) = (f_1(g, x, t), f_2(g, y, t)) = (g, f_1(x, t), g, f_2(y, t)) = g, (f_1(x, t), f_2(y, t)) = g, H((x, y), t)$. Hence $X \times Y$ has property $E$-$EQ$.

Remark 3.3.4 The following implications are easy to prove:

Strong $E$-$EW \Rightarrow E$-$EW \Rightarrow E$-$EQ \Rightarrow$ Weak $E$-$EQ$. 

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Hence any metric $G$-space $X$ satisfying one of the above four properties with property E-ED has E-CIP.

**Theorem 3.3.5** Let $X$, $Y$ be metric compact Hausdorff $G$-spaces such that $X$ has property E-EQ and $Y$ has property E-EW and let $X \times Y$ has property E-ED. Then $X \times Y$ has E-CIP.

**Proof.** By Proposition 3.3.3, $X \times Y$ has property E-EQ. Since a space having property E-EQ has property weak E-EQ, hence by Proposition 3.3.2, $X \times Y$ has E-CIP.

**Corollary 3.3.6** Let $(X, \rho)$ be a metric real linear $G$-space and let $X$ have a stationary point $p$. We assume that $X$ is star shaped with respect to $p$. If $F$ is a nonempty invariant closed subset of $X$ which contains $p$ and if $X$ has property E-ED, then there exists an equivariant continuous map $f:X \to X$ such that $\text{fix}(f)=F$.

**Proof.** We assume that $\rho \leq 1$. Since $X$ is star shaped, the line segment joining $x \in X$ and $p$ is contained in $X$. Hence the map $h:X \times I \to X$ defined by $h(x, t) = (1-t)x + tp$ is a $G$-homotopy such that $h(x, 0) = x$, $x \in X$ and $h(x, t) \neq x$ if $x \neq p$ and $t>0$. Let $f$ be defined by $f(x) = h(x, t) = (1-t)x + tp$, where $t = \rho(x, F)$. That $f$ is continuous and $\text{fix}(f)=F$ has been given in [31]. We prove that $f$ is equivariant. Since $X$ has property E-ED, therefore for $g \in G$ and $x \in X$, $\rho(g.x, F) = \rho(x, F) = t$. Hence $f(g.x) = h(g.x, t) = g.h(x, t) = g.f(x)$. Thus $f$ is equivariant. Hence the result.
Lemma 3.3.7 Let \((X, \rho)\) be a metric \(G\)-space. Let \(h: X \times I \to X\) be a \(G\)-homotopy satisfying \(h(x, 0) = x\) and \(h(x, t) \neq x\), for \(x \in X\) and \(t > 0\). Then for any metric \(G\)-space \(Y, X \times Y\) with property E-ED has E-CIP.

Proof. We assume that \(\rho \leq 1\). Let \(F\) be a nonempty invariant closed set of \(X \times Y\). Let \(f: X \times Y \to X \times Y\) be a map defined by \(f(x, y) = (h(x, \rho((x, y), F)), y)\), for \((x, y) \in X \times Y\). It has been given in [Lemma 2.2; 11] that \(f\) is continuous with fixed point set \(F\).

We prove that \(f\) is equivariant. For \(g \in G\) and \((x, y) \in X \times Y, f(g.(x, y)) = f(g.x, g.y) = (h(g.x, \rho((g.x, g.y), F)), g.y)\) and \(g.f(x, y) = g.(h(x, \rho((x, y), F)), y) = (g.h(x, \rho((x, y), F)), g.y) = (h(g.x, \rho((x, y), F)), g.y)\). Since \(X \times Y\) has property E-ED, \(\rho((g.x, g.y), F) = \rho((x, y), F)\). Hence \(f(g.(x, y)) = g.f(x, y)\). Hence \(X \times Y\) has E-CIP.

Theorem 3.3.8 Let \((G, .)\) be a metrizable compact Hausdorff topological abelian group which contains an arc, with the usual action of \(G\) on \(G\) and let for any metric \(G\)-space \(X, G \times X\) has property E-ED. Then \(G \times X\) has E-CIP.

Proof. Let \(e\) be the identity element of \(G\). Using the translation map, we can assume that \(\gamma\) is an arc in \(G\) with \(e\) as one of its end points: arc is a space homeomorphic to \([0,1]\). Let \(\alpha: [0,1] \to \gamma\) be a homeomorphism on \(\gamma\) such that \(\alpha(0) = e\). Consider \(h: G \times I \to G\) defined by \(h(x, t) = \alpha(t).x\), where \((x, t) \in G \times I\). Then \(h(x, 0) = x\) and for \(t > 0\), \(h(x, t) = \alpha(t).x \neq x\), because \(\alpha\) is one-one implies \(\alpha(t) \neq e\). We prove that \(h\) is equivariant. For \(g \in G\) and \(x \in G, h(g.(x, t)) = h(g.x, t) = \alpha(t).(g.x)\) and \(g.h(x, t) = g.(\alpha(t).x) = \alpha(t).g.x\), because \(G\) is abelian. By Lemma 3.3.7, \(G \times X\) has E-CIP.
**Proposition 3.3.9** Let $X$ be a $G$-space, $F$ be a nonempty invariant closed subset of $X$, $B$ be a $G$-retract having E-CIP and $F \subseteq B$. Then there exists an equivariant continuous map $f:X \to X$ such that $\text{fix}(f) = F$.

**Proof.** Let $r:X \to B$ be a $G$-retraction and let $h:B \to B$ be an equivariant continuous map such that $\text{fix}(h) = F$. Then $f = i \circ h \circ r:X \to X$ is the required map, where $i:B \to X$ is the inclusion. For $x \in F$, $f(x) = x$ and for $x \in (B-F)$, $f(x) = h(r(x)) = h(x) \neq x$, because otherwise $x \in \text{fix}(h)$, a contradiction. If $x \in B'$, then $f(x) = x$, because $f(x) \in B$. Thus $\text{fix}(f) = F$.

**Proposition 3.3.10** Let $X$ be a locally connected $G$-space having a nonempty set of stationary points and let $X$ have invariant components having E-CIP. Then for a nonempty invariant closed set $F$ of $X$ containing a stationary point, there exists a continuous map $f:X \to X$ such that $\text{fix}(f) = F$.

**Proof.** Let $C_\alpha$ be a nonempty invariant component of $X$ and $p$ be a stationary point of $X$ such that $p \in F$. Then $F \cap C_\alpha$ is invariant and closed in $C_\alpha$. Since $C_\alpha$ has E-CIP, hence there exists an equivariant continuous map $f_\alpha:C_\alpha \to C_\alpha$ such that $\text{fix}(f_\alpha) = F \cap C_\alpha$ if $F \cap C_\alpha$ is nonempty. Let $f:X \to X$ be defined by $f(x) = f_\alpha(x)$, if $x \in C_\alpha$ for which $F \cap C_\alpha \neq \emptyset$ and $f(x) = p$, if $x \in C_\alpha$ for which $F \cap C_\alpha = \emptyset$. That such a map $f$ is continuous and $\text{fix}(f) = F$ has been given in [Theorem 3, 24]. We only prove that $f$ is equivariant. Let $x \in C_\alpha$ and $F \cap C_\alpha \neq \emptyset$. Since $C_\alpha$ is invariant, therefore $g \cdot x \in C_\alpha$, for $g \in G$. Hence for $g \in G$, $f(g \cdot x) = f_\alpha(g \cdot x) = g \cdot f_\alpha(x) = g \cdot f(x)$. If $x \in C_\alpha$ and $F \cap C_\alpha = \emptyset$, then for $g \in G$, $g \cdot x \in C_\alpha$, because $C_\alpha$ is invariant. Hence $f(g \cdot x) = p$ and $g \cdot f(x) = g \cdot p = p$. Thus $f$ is equivariant. Hence the result follows.
Proposition 3.3.11 Let $X$ be a $G$-space and $X$ have only one stationary point $p$. If for every nonempty invariant closed set $F$ of $X$ there exists an equivariant continuous map $f:X \to X$ such that $\text{fix}(f)=F$, then $p \in F$.

**Proof.** To the contrary, let $p \in F'$. Since $\text{fix}(f)=F$, hence $f(p) \neq p$. Let $f(p)=q$, where $q \neq p$. Since $p$ is stationary, hence for $g \in G$, $f(g.p)=f(p)=q$. The map $f$ being equivariant, $q$ is a stationary point. This contradicts the fact that $p$ is the only stationary point of $X$. Hence $p \in F$.

**Example 3.3.12** Consider the $G$-space $X=(I, Z_2, \Theta)$, where $\Theta$ is defined as follows: $0.t=t$ and $1.t=1-t$. Since $1.t=t$ implies that $t=1/2$, hence $1/2$ is the only stationary point. By Proposition 3.3.11, $X$ does not possess E-CIP.

Proposition 3.3.13 Let $X$ be a $G$-space having nonempty set of stationary points and $F$ be a nonempty invariant closed set of $X$. Let $f:X \to X$ be an equivariant continuous map such that $\text{fix}(f)=F$. If $F$ does not contain all the stationary points, then $X$ must have at least two stationary points.

**Proof.** Let $p$ be a stationary point of $X$ such that $p \in F'$. As it has been proved in the previous proposition, $f(p)$ is stationary and $f(p) \neq p$. Hence the result.

**Remark 3.3.14** If $F$ is a singleton in a $G$-space $X$ and there exists an equivariant continuous map $f:X \to X$ such that $\text{fix}(f)=F$, then the element in $F$ is stationary.
because for $a \in F$ and $g \in G$, $f(a) = a$ and $g.a = g.f(a) = f(g.a)$ implies that $g.a \in F$. Thus for $g \in G$, $g.a = a$. This statement also follows from the fact that $F$ is invariant.

**Proposition 3.3.15** Let $X$ be a $G$-space and let $a$ be a stationary point of $X$. Then there exists an equivariant continuous map $f : X \to X$ such that $\text{fix}(f) = \{a\}$.

**Proof.** Define $f : X \to X$ by $f(x) = a$. Then $f$ is equivariant, continuous and $\text{fix}(f) = \{a\}$. The converse follows from the previous remark.

Combining Remark 3.3.14 and Proposition 3.3.15, we have the following proposition.

**Proposition 3.3.16** Let $X$ be a $G$-space and let $a \in X$. Then there exists an equivariant continuous map $f : X \to X$ such that $\text{fix}(f) = \{a\}$ iff $a$ is stationary.

### 3.4 Hilbert Space and E-CIP

**Proposition 3.4.1** Let $H$ be a $G$-Hilbert space and let $M$ be a nonempty invariant closed subspace of $H$. If $M^\perp$ is invariant, then there exists an equivariant continuous linear map $f : H \to H$ such that $\text{fix}(f) = M$.

**Proof.** Notice that for $g \in G$ and $x \in H$, $g.0 = g.(x + (-x)) = g.x + g.(-x) = g.x + (-1)g.x = 0$. We show that the projection $P$ discussed in the proof of
Proposition 2.4.1 is equivariant. For \( g \in G \) and \( x \in M \), \( g.x \in M \) and hence \( P(g.x) = g.x \) and \( g.P(x) = g.x \). Let \( x \in M^1 \). Then for \( g \in G \), \( P(g.x) = 0 \) and \( g.P(x) = g.0 = 0 \). For \( x \in H \), there exist \( m \in M \) and \( n \in M^1 \) such that \( x = m + n \). Thus \( P(g.(m+n)) = P(g.m + g.n) = g.m = g.P(m+n) \). This shows that \( P \) is equivariant. Hence the result follows.

### 3.5 Orbit Space and E-CIP

**Lemma 3.5.1** Let \( X \) be a \( G \)-space and \( f: X \to X \) be an equivariant continuous map. Then \( h: X/G \to X/G \) defined by \( h(G_x) = G_{f(x)} \) is continuous.

**Proof.** We first prove that \( h \) is well-defined. For \( x, y \in X \), \( G_x = G_y \) implies that \( y = g.x \), for some \( g \in G \). Since \( f \) is equivariant, \( f(g.x) = g.f(x) \). Hence \( f(y) = g.f(x) \). This proves that \( G_{f(y)} = G_{f(x)} \). Hence \( h \) is well-defined.

Let \( q: X \to X/G \) be the orbit map. To prove that \( h \) is continuous, let \( H \) be open in \( X/G \). Then \( q^{-1}(H) \) is open in \( X \) and \( f \) being continuous, \( f^{-1}(q^{-1}(H)) \) is open in \( X \). Since \( f^{-1}(q^{-1}(H)) = q^{-1}(h^{-1}(H)) \), hence \( q^{-1}(h^{-1}(H)) \) is open in \( X \). The map \( q \) being a quotient map, \( h^{-1}(H) \) is open in \( X/G \). This proves that \( h \) is continuous.

**Proposition 3.5.2** Let \( X \) be a \( G \)-space such that for each nonempty invariant closed set \( P \) of \( X \), there exists an equivariant continuous map \( f: X \to X \) such that \( \text{fix}(f) = P \) and for \( x \in X \) and \( g \in G \), \( f(x) = g.x \) implies that \( g = e \). Then \( X/G \) has CIP.
Proof. Let $F$ be a nonempty closed set of $X/G$. Then $q^{-1}(F)$ is nonempty and closed, where $q: X \rightarrow X/G$ is the orbit map. For $g \in G$ and $x \in q^{-1}(F)$, $q(gx) = G_{gx} = G_x \in F$. Hence $gx \in q^{-1}(F)$. Thus $q^{-1}(F)$ is invariant. By hypothesis, there exists $f: X \rightarrow X$ such that $\text{fix}(f) = q^{-1}(F)$ and for $x \notin q^{-1}(F)$, $f(x) \neq gx$, for any $g \in G$.

Define $h: X/G \rightarrow X/G$ by $h(Gx) = G_j(x)$. By Lemma 3.5.1, $h$ is continuous. We now prove that $\text{fix}(h) = F$. Let $Gx \in F$. Then $h(Gx) = G_j(x)$ and $x \in q^{-1}(F)$. Since $\text{fix}(f) = q^{-1}(F)$, hence $f(x) = x$. Thus $h(Gx) = Gx$ and hence $Gx \in \text{fix}(h)$. To prove that $\text{fix}(h) \subseteq F$, let $Gx \in \text{fix}(h)$. Then $h(Gx) = Gx$. Hence $G_j(x) = Gx$. This implies that $f(x) = gx$, for some $g \in G$. Hence $g = e$, which proves that $f(x) = x$, that is, $x \in q^{-1}(F)$. Hence $Gx \in F$. Thus we have $\text{fix}(h) = F$. This completes the proof.