In this chapter, some closed sets of the Stone-Čech compactification are obtained to be the fixed point set of a continuous map, or a homeomorphism. This gives interesting results concerning $\beta N$. The corresponding equivariant analogues are also studied. Throughout this chapter, $n \in N$. The following result of Vermeer [30] is frequently used:

"If $X$ is a locally compact extremally disconnected Tychonoff space and $f:X \to X$ is a continuous map, then $\text{Cl}_{\beta n}(\text{fix}(f)) = \text{fix}(\beta f)$."
8.1 $\beta X$ and CIP

In this section, we obtain that under some conditions, certain closed sets of $\beta X$ are the fixed point sets of continuous self-maps on $\beta X$.

**Proposition 8.1.1** Let $X$ be a locally compact extremally disconnected Tychonoff space having CIP. If $F$ is a nonempty closed set of $\beta X$ such that $F = Cl_{\beta X}H$, for some closed set $H$ in $X$, then there exists a continuous map $\varphi : \beta X \to \beta X$ such that $f_{\beta X}(\varphi) = F$.

**Proof.** Since $X$ has CIP, hence there exists a continuous map $f : X \to X$ such that $fix(f) = H$. In [30], it has been obtained that $Cl_{\beta X}(fix(f)) = fix(\beta f)$, where $\beta f : \beta X \to \beta X$ is the Stone extension of $f$. Thus $fix(\beta f) = Cl_{\beta X}(H) = F$. This proves the result.

Recall the following.

**Proposition 8.1.2** [32] Let $F$ be a clopen subset of $\beta N$. Then $F = Cl_{\beta N}A$, for some subset $A$ of $N$.

Martin [15] has obtained that $\beta N$ does not possess CIP. Hence not all closed subsets of $\beta N$ are the fixed point sets of continuous self-maps on $\beta N$. The following proposition gives that the clopen subsets of $\beta N$ enjoy this property.
Proposition 8.1.3 Every clopen subset of $\beta N$ is the fixed point set of a continuous self-map on $\beta N$.

Proof. Let $F$ be a clopen subset of $\beta N$. Then $F=\text{Cl}_{\beta N}A$, for some subset $A$ of $N$ by Proposition 8.1.2. Since $N$ is locally compact extremally disconnected Tychonoff and has CIP, the result follows by Proposition 8.1.1.

8.2 $\beta X$ and CIPH

Besides proving that $\beta X$ does not possess CIPH, in general, it is also proved in this section that the clopen sets of $\beta N$ are precisely the subsets which are the fixed point sets of self-homeomorphisms on $\beta N$. Recall that, if every nonempty closed subset of a space is the fixed point set of a self-homeomorphism, the space is said to have CIPH.

Proposition 8.2.1 Let $X$ be a locally compact extremally disconnected Tychonoff space having CIPH. If $F$ is a nonempty closed set of $\beta X$ such that $F=\text{Cl}_{\beta X}H$, for some closed set $H$ in $X$, then there exists a homeomorphism $\varphi : \beta X \to \beta X$ such that $\text{fix}(\varphi)=F$.

Proof. Since Stone extension of a homeomorphism is a homeomorphism, the result can be proved in the same way as that of Proposition 8.1.1.
Proposition 8.2.2 Let $X$ be an extremally disconnected Tychonoff space and let $F$ be a nonempty closed nonopen subset of $\beta X$. Then there does not exist a one-one continuous map $h: \beta X \rightarrow \beta X$ such that $\text{fix}(h) = F$.

Proof. To the contrary assume that $h: \beta X \rightarrow \beta X$ is a one-one continuous map such that $\text{fix}(h) = F$. Since $\beta X$ is compact Hausdorff and $X$ being extremally disconnected, $\beta X$ is extremally disconnected, hence $\text{fix}(h)$ is clopen [cf. Proposition 2.5.1]. This contradicts that $F$ is not open. Hence the result follows.

The proof that $\beta \mathbb{N}$ does not possess CIP is quite tedious. Since a space possessing CIPH, has CIP, hence $\beta \mathbb{N}$ does not possess CIPH. We give below a simpler proof of the fact that $\beta \mathbb{N}$ does not possess CIPH.

Proposition 8.2.3 Let $X$ be an extremally disconnected Tychonoff noncompact space. Then $\beta X$ does not possess CIPH. In particular, $\beta \mathbb{N}$ does not possess CIPH.

Proof. Since Stone-Čech compactification of an extremally disconnected space is extremally disconnected, hence $\beta X$ is extremally disconnected. Also, $X$ being dense in $\beta X$, $\beta X$ is nondiscrete. Hence by Proposition 2.5.3, $\beta X$ does not possess CIPH.

Proposition 8.2.4 Every clopen subset of $\beta \mathbb{N}$ is the fixed point set of a self-homeomorphism on $\beta \mathbb{N}$.
Proof. Since $N$ has CIPH [cf. Example 2.5.7] and Stone extension of a homeomorphism is a homeomorphism, hence the result can be proved in the same way as Proposition 8.1.3.

**Proposition 8.2.5** The subsets of $\beta N$ which are the fixed point sets of self-homeomorphisms on $\beta N$ are precisely the clopen sets.

Proof. Since the fixed point set of a one-one continuous self-map on an extremally disconnected compact Hausdorff space is clopen [cf. Proposition 2.5.1], the result follows by Proposition 8.2.4.

### 8.3 $\beta X$ and $n$-CIP

This section is devoted to the study of preservation of $n$-CIP by $\beta X$.

**Proposition 8.3.1** Let $X$ be a locally compact extremally disconnected Tychonoff space having $n$-CIP and $F$ be a closed set of $\beta X$ such that the number of elements in $F$ is greater than or equal to $n$ and $F = \text{cl}_{\beta X} A$, for some closed set $A$ of $X$. Then there exists a continuous map $\phi : \beta X \to \beta X$ such that $\text{fix}(\phi^n) = F$.

Proof. Notice that the number of elements in $A$ is greater than or equal to $n$, for otherwise, $A$ would be closed in $\beta X$ and $A = F$ and hence number of elements in $F$ would also be less than $n$, a contradiction. Since $X$ has $n$-CIP, there exists a continuous map $f : X \to X$ such that $\text{fix}(f^n) = A$. As in the proof of Proposition 8.1.1,
it can be obtained that $\text{fix}(f^n) = F$. The fact that $f^n = (f^n)^n$, for every $n \in \mathbb{N}$, proves that $\text{fix}((f^n)^n) = F$.

**Proposition 8.3.2** Let $F$ be a clopen subset of $\beta \mathbb{N}$ containing at least $n$ elements. Then there exists a continuous map $f: \beta \mathbb{N} \to \beta \mathbb{N}$ such that $\text{fix}(f^n) = F$.

**Proof.** Since $\mathbb{N}$ has $n$-CIP, the result follows by Propositions 8.1.2 and 8.3.1.

### 8.4 $\beta X$ and $n$-CIPH

In this section, we obtain results corresponding to Propositions 8.2.2 and 8.2.3.

**Proposition 8.4.1** Let $X$ be an extremally disconnected Tychonoff space and let $F$ be a nonempty nonopen closed subset of $\beta X$ containing at least $n$ elements, where $n \in \mathbb{N}$. Then there does not exist a one-one continuous map $h: \beta X \to \beta X$ such that $\text{fix}(h^n) = F$.

**Proof.** To the contrary assume that $h: \beta X \to \beta X$ be a one-one continuous map such that $\text{fix}(h^n) = F$, for some $n \in \mathbb{N}$. Notice that $h^n$ is one-one and continuous. Since $\beta X$ is compact Hausdorff and $X$ being extremally disconnected, $\beta X$ is extremally disconnected, hence $\text{fix}(h^n)$ is clopen by Proposition 2.5.1. This contradicts that $F$ is not open. Hence the result follows.
**Proposition 8.4.2** Let $X$ be an extremally disconnected Tychonoff noncompact space. Then $\beta X$ does not have $n$-CIPH. In particular, $\beta N$ does not possess $n$-CIPH.

**Proof.** Let $F$ be a closed subset of $\beta X$ containing at least $n$ elements and $F \subseteq \beta X - X$. Since $X$ is dense in $\beta X$, $F$ is not open. The space $X$ being extremally disconnected, $\beta X$ is extremally disconnected. The result follows by Proposition 8.4.1.

**Remark 8.4.3** Let $X$ have $n$-CIPH. If $f : X \to X$ is a homeomorphism, then $\beta f$ is a homeomorphism. Hence, the results corresponding to Propositions 8.3.1 and 8.3.2 are true.

**8.5 $\beta X$ and E-CIP**

In this section we obtain the equivariant analogues of Propositions 8.1.1, 8.1.2 and 8.1.3. First we recall from [27] the action induced on $\beta X$ by an action of a discrete group $G$ on a Tychonoff space $X$. 
Induced action on \( \beta X \) [27]

Let \( X \) be a Tychonoff G-space, where \( G \) is a discrete topological group. Let \( A^p \) be the \( z \)-ultrafilter on \( X \) converging to \( p \in \beta X \). For \( g \in G \) and \( Z \subseteq X \), \( g.Z = \{ g.x : x \in Z \} \) and \( g \cdot A^p = \{ g.Z : Z \in A^p \} \). Then \( g \cdot A^p \) is a \( z \)-ultrafilter on \( X \). It corresponds to a point, say \( g.p \), in \( \beta X \). Then \( G \) acts on \( \beta X \) by the action \( \Theta : G \times \beta X \to \beta X \) defined by \( \Theta(g, p) = g.p \), where \( g \in G \) and \( p \in \beta X \). This action on \( \beta X \) is called the induced action of \( G \) on \( X \). The Stone extention \( \beta f : \beta X \to \beta X \) of an equivariant map \( f : X \to X \) is equivariant.

**Proposition 8.5.1** Let \( X \) be a locally compact extremally disconnected Tychonoff G-space having E-CIP and \( \beta X \) be the Stone-Čech compactification of \( X \) with the induced action of \( G \) on \( X \). If \( F \) is a nonempty invariant closed set of \( \beta X \) such that \( F = \text{Cl}_{\beta X} H \), for some invariant closed set \( H \) in \( X \), then there exists an equivariant continuous map \( \phi : \beta X \to \beta X \) such that \( \text{fix}(\phi) = F \).

**Proof.** Since \( X \) has E-CIP, hence there exists an equivariant continuous map \( f : X \to X \) such that \( \text{fix}(f) = H \). In [30], it has been obtained that \( \text{Cl}_{\beta X}(\text{fix}(f)) = \text{fix}(\beta f) \), where \( \beta f : \beta X \to \beta X \) is the Stone extension of \( f \). Also \( \beta f \) is equivariant [27]. Thus \( \text{fix}(\beta f) = \text{Cl}_{\beta X}(H) = F \). This proves the result.

We now obtain the equivariant analogues of Propositions 8.1.2 and 8.1.3.

**Lemma 8.5.2** Let \( N \) be acted upon by a topological group \( G \) and let \( F \) be an invariant clopen subset of \( \beta N \). Then \( F = \text{Cl}_{\beta N} A \), for some invariant subset \( A \) of \( N \).
Proof. By Proposition 8.1.2, there exists a subset $A$ of $N$ such that $F = \text{Cl}_{\beta N} A$. We now prove that $A$ is invariant. Singletons are open in $N$ and hence in $\beta N$. This proves that $x \notin F$, if $x \in N - A$. This proves that $F - A \subseteq \beta N - N$. For $g \in G$ and $a \in A$, $F$ being invariant, $g.a \in F$. If $g.a \notin A$, then $g.a \in \beta N - N$, which is not possible because $N$ is invariant and $A$ is a subset of $N$. Hence $g.a \in A$.

**Proposition 8.5.3** Let $N$ be acted upon by a topological group $G$. Then every clopen subset of $\beta N$ is the fixed point set of an equivariant continuous self-map on $\beta N$.

Proof. Since Stone extension of an equivariant map is equivariant, the result can be proved in the same way as that of Proposition 8.1.3 using Lemma 8.5.2.

### 8.6 $\beta X$ and E-CIPH, E-n-CIP, E-n-CIPH

Using Lemma 8.5.2 and the fact that $\beta f$ is equivariant, if $f:X \to X$ is equivariant, the corresponding equivariant analogues of Propositions 8.2.1 and 8.3.1 can be proved.