In this chapter, we show that if $N$ is acted upon by a topological group $G$ such that the set of stationary points is nonempty and $G$ acts on $\overline{N}$ such that $N$ is invariant, then $\overline{N}$ has E-CIP. A study regarding the preservation of complete invariance property with respect to homeomorphisms (CIPH) by one-point compactification has also been made.
7.1 One-Point Compactification and E-CIP

Let \((X, G, \Theta)\) be a \(G\)-space and \(\overline{X} = X \cup \{\infty\}\) be the one-point compactification of \(X\) acted upon by \(G\) such that \(X\) is invariant. Unless otherwise stated, \(A'\) denotes \(\overline{X} - A\) for a subset \(A\) of \(X\).

**Proposition 7.1.1** Let \(\overline{X} = X \cup \{\infty\}\) be the one-point compactification of \(X\) acted upon by a topological group \(G\) such that \(X\) is invariant. Then \(g . \infty = \infty\), for each \(g \in G\).

**Proof.** Since complement of an invariant set is invariant, hence the result follows.

**Example 7.1.2** Let \(G\) act on the set \(N\) of natural numbers such that the set of stationary points is nonempty. Let \(\overline{N} = N \cup \{\infty\}\) be the one-point compactification of \(N\). Let \(G\) act on \(\overline{N}\) such that \(N\) is invariant in \(\overline{N}\). Then \(\overline{N} - N\) being invariant, \(g . \infty = \infty\), for \(g \in G\). We show that \(\overline{N}\) has E-CIP. Notice that the open sets of \(\overline{N}\) are precisely the subsets of \(N\) and complements in \(\overline{N}\) of finite sets of \(N\) and the closed sets of \(\overline{N}\) are all finite sets of \(N\) and complements in \(\overline{N}\) of subsets of \(N\).

To show that \(\overline{N}\) has E-CIP consider a non empty invariant closed set \(F\) of \(\overline{N}\). The following cases arises:
Case(1) \( F=\{\infty\} \): Define \( f: \overline{N} \to \overline{N} \) by \( f(x)=\infty \), \( \forall x \in \overline{N} \). For an open set \( G \) of \( \overline{N} \), \( f^{-1}(G)=\overline{N} \) or \( \phi \) according as \( \infty \in G \) or \( \infty \notin G \). Hence \( f \) is continuous. Since \( f(g \cdot x)=\infty \) and \( g \cdot f(x)=g \cdot \infty=\infty \), for \( x \in \overline{N} \), hence \( f \) is equivariant.

Case(2) \( F=A \cup \{\infty\} \), where \( A \) is a nonempty invariant subset of \( N \): Define \( f: \overline{N} \to \overline{N} \) by \( f(x)=x \), for \( x \in A \) and \( f(x)=\infty \), for \( x \in A' \). For \( G \subseteq N \), \( f^{-1}(G) \subseteq N \). If \( G \) is an open set of \( \overline{N} \) such that \( \infty \in G \), then \( (\overline{N}-G) \) is finite and \( f^{-1}(G) \) is \( \overline{N}-[(\overline{N}-G) \cap A] \), which is open. This shows that \( f \) is continuous. To show that \( f \) is equivariant, let \( x \in A \). Then \( g \cdot x \in A \) and \( f(g \cdot x)=g \cdot x=g \cdot f(x) \). For \( x \in (\overline{N}-A) \), \( f(g \cdot x)=\infty \) and \( g \cdot f(x)=g \cdot \infty=\infty \). Thus \( f \) is equivariant.

Case(3) \( F \) is a finite subset of \( N \): Let \( x_0 \) be a stationary point of \( N \). If \( x_0 \in F \), define \( f: \overline{N} \to \overline{N} \) by \( f(x)=x \), for \( x \in F \) and \( f(x)=x_0 \), otherwise. To show that \( f \) is continuous, let \( G \) be open in \( \overline{N} \). If \( \infty \notin G \), then \( x_0 \in G \) or \( x_0 \notin G \). If \( x_0 \in G \), then \( f^{-1}(G) \) is \( (\overline{N}-(G \cap F)) \cup \{\infty\} \), which is open in \( \overline{N} \) and if \( x_0 \notin G \), then \( f^{-1}(G) \) is \( G \cap F \) which is also open. The case when \( \infty \in G \) is dealt with similarly. This shows that \( f \) is continuous. For \( x \in F \), \( g \cdot x \in F \), \( \forall g \in G \) and hence \( f(g \cdot x)=g \cdot x=g \cdot f(x) \). For \( x \in \overline{N}-F \), \( f(g \cdot x)=x_0 \) and \( g \cdot f(x)=g \cdot x_0=x_0 \). This shows that \( f \) is equivariant.

If \( F \) does not contain any stationary point, define \( f: \overline{N} \to \overline{N} \) by \( f(x)=x \), for \( x \in F \), \( f(x_0)=\infty \) and \( f(x)=x_0 \), otherwise. To prove that \( f \) is continuous, let \( G \) be open in \( \overline{N} \). If \( x_0 \notin G \), then \( f^{-1}(G) \) is \( G \cap F \) or \( (G \cap F) \cup \{x_0\} \), according as \( \infty \notin G \) or \( \infty \in G \). If \( x_0 \in G \) and \( \infty \notin G \), then \( f^{-1}(G)=(F' \cup (G \cap F))-\{x_0\} \) and if \( \infty \in G \), then \( f^{-1}(G)=(F' \cup (G \cap F)) \). In any case, \( f^{-1}(G) \) is open. Hence \( f \) is continuous. To prove that \( f \) is equivariant, let \( x \in F \). Then \( g \cdot x \in F \). Now \( f(g \cdot x)=g \cdot x=g \cdot f(x) \), \( f(g \cdot x)=f(x_0)=\infty \) and \( g \cdot f(x_0)=g \cdot \infty=\infty \). For \( x \in \overline{N}-(F \cup \{x_0\}) \), \( f(g \cdot x)=x_0 \) and
Thus $f$ is equivariant. Clearly in both the cases $\text{fix}(f)=F$. Hence $\bar{N}$ has E-CIP, if $N$ has a stationary point.

**Proposition 7.1.3** Let $X$ be a $G$-space and let $\bar{X}$ be the one-point compactification of $X$ such that $X$ is invariant. If $\bar{X}$ has E-CIP and has a nonempty invariant closed set $F$ such that $\infty \not\in F$, then $X$ has a stationary point.

**Proof.** Let $F$ be a nonempty invariant closed set of $\bar{X}$ such that $\infty \not\in F$. Let $f: \bar{X} \to \bar{X}$ be an equivariant continuous map such that $\text{fix}(f)=F$. Then $f(\infty) \neq \infty$. Hence $f(\infty) \in X$. By Proposition 7.1.1, $g.\infty = \infty$, for each $g \in G$. Since $f$ is equivariant, $g.f(\infty) = f(g.\infty) = f(\infty)$, where $g \in G$, thus proving that $f(\infty)$ is stationary. This proves the result.

7.2 One-Point Compactification, CIPH and E-CIPH

**Proposition 7.2.1** Let $\bar{X} = X \cup \{\infty\}$ be the one-point compactification of a compact space $X$. Then $\bar{X}$ does not possess CIPH.

**Proof.** Since $X$ is compact, hence it is closed in $\bar{X}$. Let $f: \bar{X} \to \bar{X}$ be a homeomorphism such that $\text{fix}(f)=X$. Since $\text{fix}(f)=X$, $f(\infty) \neq \infty$. Let $f(\infty) = x$, $x \in X$. Since $\text{fix}(f)=X$, hence $f(x)=x$. This contradicts that $f$ is one-one. Hence there does not exist a homeomorphism $f: \bar{X} \to \bar{X}$ such that $\text{fix}(f)=F$. This proves that $\bar{X}$ does not have CIPH.
Proposition 7.2.2 Let $\bar{X} = X \cup \{\infty\}$ be the one point compactification of a space $X$ and let $\bar{X}$ have CIPH. Then $X$ has CIPH.

Proof. Let $F$ be a nonempty closed set of $X$. Then $F \cup \{\infty\}$ is closed in $\bar{X}$. Since $\bar{X}$ has CIPH, hence there exists a homeomorphism $f: \bar{X} \to \bar{X}$ such that $\text{fix}(f) = F \cup \{\infty\}$. This shows that the restriction $f/|X: X \to X$ of $f$ to $X$ is a homeomorphism from $X$ to $X$ such that $\text{fix}(f|X) = F$. Hence $X$ has CIPH.

Proposition 7.2.3 Let $\bar{X} = X \cup \{\infty\}$ be the one-point compactification of a $G$-space $X$ and let $G$ act on $\bar{X}$ such that $X$ is invariant. If $\bar{X}$ has E-CIPH, then $X$ has E-CIPH.

Proof. Let $F$ be a nonempty invariant closed set of $X$. Then $F \cup \{\infty\}$ is closed in $\bar{X}$ and by Proposition 7.1.1, it is invariant. Since $\bar{X}$ has E-CIPH, hence there exists a $G$-homeomorphism $f: \bar{X} \to \bar{X}$ such that $\text{fix}(f) = F \cup \{\infty\}$. Then $f/|X: X \to X$, the restriction of $f$ to $X$ is a $G$-homeomorphism such that $\text{fix}(f|X) = F$. Hence $X$ has E-CIPH.