Chapter 4

On Equivariant Complete Invariance Property with respect to a Homeomorphism

In this chapter, we define E-CIPH and provide examples of spaces possessing E-CIPH. Besides other results, equivariant analogues of some of the results in [10] have been obtained. The equivariant complete invariance property with respect to a homeomorphism for the closed unit interval $I$ acted by a topological group $G$ has also been discussed.

For a metric space $X$ and a subset $A$ of $X$, $d(x, A)$ denotes the distance between $x \in X$ and $A$. It is sometimes also denoted by $d_A(x)$. 
4.1 Definition

A $G$-space $X$ is said to have the equivariant complete invariance property with respect to a homeomorphism (E-CIPH), if every nonempty invariant closed set of $X$ is the fixed point set of a self $G$-homeomorphism on $X$.

4.2 Examples

4.2.1 Let $X$ be an indiscrete $G$-space. Consider the identity map $Id: X \rightarrow X$. Then $\text{fix}(Id) = X$. Since $Id$ is a $G$-homeomorphism and $X$ is the only nonempty invariant closed set of $X$, therefore $X$ has E-CIPH.

4.2.2 Let $I \times S^1$ be the cylinder and let $S^1$ act on $I \times S^1$ by the action defined by $\Theta(q, (x, p)) = (x, p, q)$, where $x \in I$ and $p, q \in S^1$. Then the set of stationary points is $\phi$. Let $F$ be a nonempty invariant closed set of $I \times S^1$ and $d_1$ be the usual metric on $I$ and $d_2$ be the arc length metric on $S^1$. Then there exists a map $f: I \times S^1 \rightarrow I \times S^1$ defined by $f(x,p) = (x, pe^{(1/2)d_1}F(x, p))$ and $d((x, p), (y, q)) = [(d_1(x, y))^2 + (d_2(p, q))^2]^{1/2}$, where $(x, p), (y, q) \in I \times S^1$. It has been proved in [10] that $f$ is a homeomorphism. Also $f$ is equivariant and $\text{fix}(f) = F$ [2]. Hence $I \times S^1$ has E-CIPH.
4.3 Results on E-CIPH

**Proposition 4.3.1** An open invariant subset of a G-space having E-CIPH has E-CIPH.

**Proof.** Let $U$ be an open invariant subset of a G-space $X$ having E-CIPH. Let $B=X-U$. Let $A$ be a nonempty invariant closed subset of $U$. Then $X-(A\cup B)=(U-A)$ and $(U-A)$ is open in $U$. Since $A$ is invariant in $U$ and complement of an invariant set is invariant, hence $(U-A)$ is invariant in $U$ and hence in $X$. Since $U$ is open in $X$ and $(U-A)$ is open in $U$, hence $A\cup B$ is closed in $X$. Thus $A\cup B$ is a nonempty invariant closed set of $X$. Since $X$ has E-CIPH, hence there exists a G-homeomorphism $f:X\rightarrow X$ with fixed point set as $A\cup B$. Since $f$ is a homeomorphism and $f(x)=A\cup B$, therefore $f(U-A)=U-A$. Then the restriction $f|U$ of $f$ on $U$ is a G-homeomorphism from $U$ to $U$ with fixed point set $A$. Hence $U$ has E-CIPH.

**Proposition 4.3.2** Let $f$ be a G-homeomorphism from a G-space $X$ having E-CIPH to a G-space $Y$. Then $Y$ has E-CIPH.

**Proof.** Let $F$ be a nonempty invariant closed subset of $Y$. Since $f$ is onto, hence $f^{-1}(F)$ is nonempty. The map $f$ being continuous, $f^{-1}(F)$ is closed in $X$. Also $f^{-1}(F)$ is invariant. Since $X$ has E-CIPH, hence for the nonempty invariant closed set $f^{-1}(F)$ of $X$, there exists a G-homeomorphism $h:X\rightarrow X$ with fixed point set $f^{-1}(F)$. Since the composition of G-homeomorphisms is a G-homeomorphism, hence $fhf^{-1}:Y\rightarrow Y$ is a G-homeomorphism. Now to prove that $\text{fix}(fhf^{-1})=F$, let $y\in F$ and let $x$ be such that $f(x)=y$. Then $x\in f^{-1}(F)$ and
\[(fhf^{-1})(y)=f(h(x))=f(x)=y.\] For \(z \in F\)', let \(p \in (f^{-1}(F))'\) be such that \(f(p)=z\). Then \((fhf^{-1})(z)=f(h(p))\). Since \(h(p) \neq p\) and \(f\) is one-one, hence \(f(h(p)) \neq z\). This proves that \(\text{fix}(fhf^{-1})=F\). Hence \(Y\) has E-CIPH.

**Proposition 4.3.3** Let \(X\) be a \(G\)-space, where \(G=\text{Homeo}(X)\) with the discrete topology and the action of \(G\) on \(X\) be defined by \(f.x=f(x)\), for \(f\in G\) and \(x \in X\). If \(F\) is a nonempty invariant closed subset of \(X\) and there exists a \(G\)-homeomorphism \(f:X \rightarrow X\) such that \(\text{fix}(f)=F\), then the stationary points of \(X\) belong to \(F\).

**Proof.** Let \(a\) be a stationary point of \(X\). Since \(f\) is a self-homeomorphism on \(X\), hence \(f \in G\). Since \(a\) is a stationary point of \(X\), hence \(f(a)=a\). This implies that \(a \in \text{fix}(f)=F\). Thus \(F\) contains the stationary points of \(X\).

**Proposition 4.3.4** Let \(X\) be a \(G\)-space and \(F\) be a nonempty invariant closed set of \(X\). Let \(f:X \rightarrow X\) be a \(G\)-homeomorphism such that \(\text{fix}(f)=F\). If \(F'\) contains one stationary point, then \(F'\) must contain at least two stationary points.

**Proof.** Let \(a\) be a stationary point of \(X\) and \(a \in F'\). Since \(\text{fix}(f)=F\) and \(f\) is one-one, therefore \(f(a) \neq a\) and \(f(a) \in F'\). For \(g \in G\), \(f(a)=f(g.a)=g.f(a)\). Thus \(f(a)\) is a stationary point and both \(a\) and \(f(a)\) belong to \(F'\). This proves the result.
4.4 I and E-CIPH

Proposition 4.4.1 Let a topological group $G$ act on the closed unit interval $I$ and let there exist an invariant closed subset $F$ of $I$ such that $0 \in F$ and $1 \notin F$. Then there does not exist a $G$-homeomorphism $h: I \to I$ such that $\text{fix}(h) = F$.

Proof. Suppose to the contrary that there exists a $G$-homeomorphism $h: I \to I$ such that $\text{fix}(h) = F$. Since $0 \in F$, hence $h(0) = 0$. Because $h$ is one-one, $h(1) \neq 0$. The fact that a homeomorphism on $I$ is strictly increasing or decreasing implies that $h(1) = 1$. This contradicts that $\text{fix}(h) = F$.

Proposition 4.4.2 Let a topological group $G$ act on the closed unit interval $I$ and let there exist an invariant closed subset $F$ of $I$ such that $1 \in F$ and $0 \notin F$. Then there does not exist a $G$-homeomorphism $h: I \to I$ such that $\text{fix}(h) = F$.

Proof. It is similar to that of Proposition 4.4.1.

Proposition 4.4.3 Let a topological group $G$ act on the closed unit interval $I$ and let there exist an invariant closed subset $F$ of $I$ such that $F$ contains at least two elements and $0, 1 \notin F$. Then there does not exist a $G$-homeomorphism $h: I \to I$ such that $\text{fix}(h) = F$.

Proof. Let $a < b$ and $a, b \in F$. If there exists a $G$-homeomorphism $h: I \to I$ such that $\text{fix}(h) = F$, then $h(a) = a, h(b) = b, h(0) = 1$ and $h(1) = 0$. Hence $h$ is strictly
decreasing. Since \(a < b\), hence \(h(a) < h(b)\). This contradicts that \(h\) is strictly decreasing. This proves the result.

In view of Propositions 4.4.1, 4.4.2 and 4.4.3 we have the following remark.

**Remark 4.4.4** Let \(G\) act on \(I\). Then \(I\) can possess E-CIPH only if every nonempty nonsingleton closed invariant set of \(I\) contains both 0 and 1.

**Remark 4.4.5** It is also clear from the proofs of Propositions 4.4.1, 4.4.2 and 4.4.3. that only singletons or closed set possessing both 0 and 1 can be the fixed point set of a self-homeomorphism on \(I\).