Chapter 6

Reliability performance measures of systems with location-scale
generalized absolutely continuous multivariate exponential
failuretime distribution

6.0 Summary

In the Location-Scale Generalized Absolutely Continuous Multivariate
Exponential failuretime distribution, by considering k unit systems we derive the
reliability performance measures of the systems in Chapter 6. We derive a theorem
which gives the mgf of the sum and maximum of the failure times and lemma
which expresses the mgf as the weighted average of individual exponentials which
help in obtaining the reliability performance measures of the standby and parallel
systems. K component standby, parallel, series and three component relay systems
with multivariate exponential failure times are discussed and their MTBF, MREE
of the MTBF and reliability function are obtained.

In Section 6.1, we derive a theorem which gives the mgf of the sum and
maximum of the failure times and a lemma which expresses the mgf as the
weighted average of individual exponentials. In Section 6.2, we consider a k unit
standby system and get the MTBF, MREE of the MTBF and reliability function of
the system. Section 6.3 gives the MTBF, MREE of the MTBF and reliability
function of a k unit parallel system, Section 6.4 gives the MTBF, MREE of the
MTBF and reliability function of a k unit series system, and Section 6.5 gives the
MTBF, MREE of the MTBF and reliability function of a three unit relay system.
6.1 Distribution of sum and maximim

Consider a \( k \) component system with identical components. Let the component failure times be \( T_1, T_2, \ldots, T_k \) respectively. Assume that \((T_1, T_2, \ldots, T_k)\) follows \(GACMVE(\lambda_1, \lambda_2, \ldots, \lambda_k; \xi, \tau)\) with pdf given in equation (5.1.2).

The following theorem will help us in obtaining the reliability performance measures of standby and parallel systems.

**Theorem 6.1.1**

Let \((T_1, T_2, \ldots, T_k)\) follow \(GACMVE(\lambda_1, \lambda_2, \ldots, \lambda_k; \xi, \tau)\) with pdf given in equation (5.2). Then

(i) \[ \sum_{i=1}^{k} T_i - k \xi \frac{d}{\tau} V_1 + V_2 + \ldots + V_k, \text{ where } V_1, V_2, \ldots, V_k \text{ are independent and} \]

where \(V_i \sim E\left(0, \frac{k - (i - 1)\tau}{B_{i-1}}\right), \text{ for all } l = 1, 2, \ldots, k. \quad \ldots(6.1.1)\)

(ii) \((T_1 \lor T_2 \lor \ldots T_k) - k \xi \frac{d}{\tau} V_1^* + V_2^* + \ldots + V_k^* \), where \(V_1^*, V_2^*, \ldots, V_k^* \) are

independent and \(V_j^* \sim E\left(0, \frac{\tau \sum_{i=1}^{j} \binom{i}{k-1}}{\sum_{i=1}^{j} \binom{i}{k-1}} \lambda_k\right), \text{ for all } l = 1, 2, \ldots, k. \quad \ldots(6.1.2)\)
Proof:

The MGF of \( \sum_{i=1}^{k} T_i, \sum_{i<j=1}^{k} T_i \vee T_j, ..., T_1 \vee T_2 \vee ... T_k \) at \( (U_1, U_2, ..., U_k) \) is

\[
M(u_1, u_2, ..., u_k) = \int_{\xi} \int_{\xi} \frac{1}{\tau^k k!} \prod_{i=0}^{k-1} \prod_{j=0}^{i-1} \left( i \right)^{\lambda_{j+1}}
\exp \left\{ u_1 \sum_{i=1}^{k} t_i + u_2 \sum_{i=1}^{k} \sum_{i < j=1}^{k} \left( t_i \vee t_j \right) + ... + u_k \left( t_1 \vee t_2 \vee ... \vee t_k \right) \right\}
\exp \left\{-\frac{1}{\tau} \left( \lambda_1 u_1 \sum_{i=1}^{k} t_i + \lambda_2 \sum_{i=1}^{k} \sum_{i < j=1}^{k} \left( t_i \vee t_j \right) + ... + \lambda_k \left( t_1 \vee t_2 \vee ... \vee t_k \right) \right. \right. \\
\left. \left. - \sum_{p=1}^{k} \left( \frac{k}{p} \right) \lambda_p u_p \right\} \right\} dt_1 dt_2 ... dt_k
\]

\[
M(u_1, u_2, ..., u_k) = \int_{\xi} \int_{\xi} \frac{1}{\tau^k k!} \prod_{i=0}^{k-1} \prod_{j=0}^{i-1} \left( i \right)^{\lambda_{j+1}} \exp \left\{-\frac{1}{\tau} \left( \lambda_1 - \tau u_1 \right) \sum_{i=1}^{k} t_i + \right. \\
\left. \left( \lambda_2 - \tau u_2 \right) \sum_{i=1}^{k} \sum_{i < j=1}^{k} \left( t_i \vee t_j \right) + ... + \left( \lambda_k - \tau u_k \right) \left( t_1 \vee t_2 \vee ... \vee t_k \right) \right. \\
\left. \left. \right. - \sum_{p=1}^{k} \left( \frac{k}{p} \right) \lambda_p u_p \right\} \right\} dt_1 dt_2 ... dt_k
\]

\[
= \prod_{i=0}^{k-1} \left( \frac{\tau \sum_{i=1}^{k} \sum_{j=0}^{i} \left( \frac{i}{j} \right) u_{j+1}}{1 - \left( \frac{\sum_{p=1}^{k} \left( \frac{k}{p} \right) u_p \xi}{B_1} \right)} \right)
\]

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\[(i) \quad M(u_1,0,...,0) = \prod_{l=0}^{k-1} \exp\left(-k u_l \tilde{\xi}\right) \left(1 - \frac{(k-l)\tau u_l}{B_l}\right)\]

\[\therefore \sum_{j=1}^{k} T_j - k \tilde{\xi} d \sum_{j=1}^{k} V_j,\]

where \(V_j\)'s are independent and

\[V_j \sim E \left[0, \frac{(k-l)\tau}{B_l}\right], \quad l = 1,2,...,k\]

\[(ii) \quad M(0,0,...,u_k) = \prod_{l=0}^{k-1} \exp\left(-u_k \tilde{\xi}\right) \left(1 - \frac{\tau \sum_{i=1}^{k-1} \left(\begin{array}{c} i \\ k-1 \end{array}\right) u_k}{B_l}\right)\]

\[\therefore (T_1 \lor T_2 \lor ... T_k) - k \tilde{\xi} d V_1^* + V_2^* + ... V_k^*,\]

where \(V_j^*\)'s are independent and

\[V_j^* \sim E \left[0, \frac{\tau \sum_{i=1}^{k-1} \left(\begin{array}{c} i \\ k-1 \end{array}\right) u_k}{B_l}\right], \quad l = 0,1,...,k-1.\]

6.2 Standby system

Consider a k unit standby system with component failure times \(T_1, T_2, ..., T_k\) respectively. Then the system failure time is \(T = \sum_{i=1}^{k} T_i\).
The MTBF of the system is

$$\text{MTBF} = \mathbb{E}(T)$$

$$= E(V_1 + V_2 + ... + V_k) + k \xi$$

$$= \sum_{j=1}^{k} \left\{ \frac{(k-l)\tau}{B_{l-1}} \right\} + k \xi, \text{ in view of (6.1.1)}$$

When $\eta = \alpha \xi + \beta \tau$, $\alpha, \beta \in \mathbb{R}$, the MREE of $\eta$ is given by

$$\delta^* = \alpha \delta_{01} + \frac{1}{kn} \left[ \beta - \frac{\alpha}{n B_0} \right] \delta_{02} \quad \text{from (5.4.1)}$$

By taking $\alpha = k$ and $\beta = \sum_{j=1}^{k} \left( \frac{k-1}{B_{l-1}} \right)$,

the MREE of the MTBF is given by

$$k \delta_{01} + \frac{1}{kn} \left[ \sum_{j=1}^{k} \frac{(k-1)}{B_{l-1}} + \frac{\alpha}{n B_0} \right] \delta_{02}$$

Reliability function of standby system

$$R(t) = P(T > t)$$

$$= P \left( \sum_{i=1}^{k} T_i - k \xi > t \right), \quad t > 0$$

$$= P \left( \sum_{i=1}^{k} V_i > t \right), \quad t > 0$$

$$= \sum_{l=1}^{k} \beta_l \exp \left( -\frac{1}{\alpha_l} t \right) \quad \text{in view of Lemma 4.2.1.}$$

Here $\alpha_l = \frac{(k-l)\tau}{B_{l-1}}, \quad \forall l = 1, 2, ..., k$, and
\[ \beta_j = \frac{\alpha_i^{k-1}}{\prod_{r=1}^{\ell} (\alpha_i - \alpha_r)} \]

and \( \sum_{j=1}^{k} \beta_j = 1 \)

Therefore,

\[
R(t) = \sum_{i=1}^{k} \left( \frac{(k-l)\tau}{B_{l-1}} \right)^{k-1} \prod_{r=1}^{\ell} \left( \frac{(k-l)\tau}{B_{l-1}} - \frac{(k-r)\tau}{B_{l-1}} \right) \exp \left\{ - \frac{1}{(k-l)\tau} t \right\}
\]

6.3 Parallel system

Consider a \( k \) unit parallel system with component failure times \( T_1, T_2, \ldots, T_k \) respectively. Then the system failure time is \( T = \text{Max}_{1 \leq i \leq k} T_i \).

MTBF = \( E(T) \)

\[
= E\left(V_1^* + V_2^* + \ldots + V_k^*\right) + k\xi \\
= \tau \sum_{l=0}^{k-1} \left[ \sum_{j=0}^{l-1} \frac{(k-l-1)!}{B_l} \right] + k\xi, \text{ in view of (6.1.2)}
\]

When \( \eta = \alpha \xi + \beta \tau, \alpha, \beta \in \mathbb{R} \), the MREE of \( \eta \) is given by

\[
\delta^* = \alpha \delta_{01} + \frac{1}{kn} \left[ \beta - \frac{\alpha}{nB_0} \right] \delta_{02} \text{ from (5.4.1)}
\]
By taking $\alpha = k$ and $\beta = \sum_{i=0}^{k-1} \left( \frac{i}{B_i} \right)$, the MREE of the MTBF is given by

$$\delta^* = k \delta_{01} + \frac{1}{kn} \left[ \sum_{i=0}^{k-1} \left( \frac{i}{B_i} \right) \right] - k \frac{1}{nb_0} \delta_{02}$$

Reliability function

$R(t) = P(T > t)$

$$= P \left( \sum_{i=1}^{k} V_i > t \right), \ t > 0$$

$$= \sum_{i=1}^{k} w_i \exp \left( -\frac{1}{\alpha_i} t \right)$$

Here $\alpha_i = k \frac{\sum_{r=1}^{k-1} \frac{k-1}{B_{i-1}}} \forall i = 0, 1, ..., k - 1$, and

$$w_i = k \frac{\alpha_i^{-1}}{\prod_{r=1, r \neq i}^{k} (\alpha_i - \alpha_r)} \forall i = 0, 1, ..., k - 1.$$

### 6.4 Series system

Consider a $k$ unit series system with component failure times $T_1, T_2, ..., T_k$ respectively. Then the system failure time is $T = \text{Min}_{1 \leq i \leq k} T_i$.

From Theorem 5.4.1,
Min\(T_i \sim E\left[\xi, \frac{\tau}{B_0}\right]\)

Thus,

\[
\text{MTBF} = \frac{\tau}{B_0} + \xi
\]

When \(\eta = \alpha \xi + \beta \tau\), \(\alpha, \beta \in \mathbb{R}\), the MREE of \(\eta\) is given by

\[
\delta^* = \alpha \delta_{01} + \frac{1}{k\eta} \left[ \beta - \frac{\alpha}{n B_0} \right] \delta_{02} \quad \text{from (5.4.1)}
\]

By taking \(\alpha = 1\) and \(\beta = \frac{1}{B_1}\), the MREE of the MTBF is given by

\[
\delta^* = \delta_{01} + \frac{1}{k\eta} \left[ \frac{1}{B_1} - \frac{1}{n B_0} \right] \delta_{02}
\]

Reliability function

\[
R(t) = P(T > t)
\]

\[
= \exp \left[ \frac{B_0}{\tau} (t - \xi) \right], \quad t > \xi
\]

6.5 Relay system

Consider a k unit relay system with component failure times \(T_1, T_2, \ldots, T_k\), respectively. A relay system of order k operates if the first component and anyone of the remaining (k-1) components operate. Therefore, the failure time of the system is

\[
T = T_1 \wedge (T_2 \vee T_3 \vee \ldots \vee T_k).
\]

The reliability function of the system is

\[
R(t) = P(T > t)
\]
\[
= \sum_{r=2}^{k} (-1)^{r-1} \binom{k-1}{r-1} \overline{F}_r (t, t, \ldots, t, 0, \ldots, 0),
\]
using distributive law and routine arguments.

Here \( \overline{F}_r (t, t, \ldots, t, 0, \ldots, 0) \) represents \( P \left( X_1 > t, X_2 > t, \ldots, X_r > t, X_{r+1} > 0, \ldots, X_k > 0 \right) \).

Let us discuss in detail the case when \( k = 3 \).

Consider a three unit relay system with component failure times \( T_1, T_2, T_3 \), respectively. A relay system of order three operates if the first component and anyone of the remaining two components operate. Therefore, the failure time of the system is
\[
T = T_1 \land (T_2 \lor T_3).
\]

The reliability function of the system is
\[
R(t) = P(T > t)
\]
\[
= \sum_{r=3}^{3} (-1)^{r} \binom{3-1}{r-1} \overline{F}_r (t, t, 0)
\]
\[
= 2 \exp \left\{ - \frac{2\lambda_1 + \lambda_2}{\tau}(t - \xi) \right\} - \exp \left\{ - \frac{3\lambda_1 + 3\lambda_2 + \lambda_3}{\tau}(t - \xi) \right\}
\]
where \( \overline{F}_3 (t, t, 0) \) represents \( P \left( X_1 > t, X_2 > t, X_3 > 0 \right) \).

\[
\overline{F}_3 (t, t, t) = P \left( X_1 > t, X_2 > t, X_3 > t \right)
\]
\[
= 3! \int \int \ldots \int f(x_1, x_2, x_3) dx_1 dx_2 dx_3
\]
\[
= 3! \frac{(\lambda_1 + 2\lambda_2 + \lambda_3)(2\lambda_1 + 3\lambda_2 + \lambda_3)(3\lambda_1 + 3\lambda_2 + \lambda_3)}{6r^3}
\]

\[
\int_{x_1 < x_2} \int_{x_1 < x_2} \int_{x_1 < x_2} \exp \left[ -\frac{1}{r} \left\{ \lambda_1 \sum_{i=1}^{3} x_i + \lambda_2 \sum_{i=1}^{3} \sum_{j<i}^{3} (x_i \lor x_j) + \lambda_3 (x_1 \lor x_2 \lor x_3) - (3\lambda_1 + 3\lambda_2 + \lambda_3) \xi \right\} \right]
\]

\[dx_1 \, dx_2 \, dx_3\]

\[
= \exp \left\{ -\frac{(3\lambda_1 + 3\lambda_2 + \lambda_3)}{r} (t - \xi) \right\}
\]

\[\overline{F}_2(t, t, 0) = \exp \left\{ -\frac{(2\lambda_1 + \lambda_2)}{r} (t - \xi) \right\}
\]

The MTBF is given by

\[
\text{MTBF} = \frac{2r}{(2\lambda_1 + \lambda_2)} - \frac{r}{(3\lambda_1 + 3\lambda_2 + \lambda_3)} + \xi
\]

\[
= \left[ \frac{(4\lambda_1 + 5\lambda_2 + 2\lambda_3)}{(2\lambda_1 + \lambda_2)(3\lambda_1 + 3\lambda_2 + \lambda_3)} \right] r + \xi
\]

When \(\eta = \alpha \xi + \beta \tau\), \(\alpha, \beta \in \mathbb{R}\), the MREE of \(\eta\) is given by

\[
\delta^* = \alpha \delta_{\alpha 0} + \frac{1}{kn} \beta - \frac{\alpha}{n B_0} \delta_{\alpha 2}, \text{ from } (5.4.1)
\]

By taking \(\alpha = 1\) and \(\beta = \frac{(4\lambda_1 + 5\lambda_2 + 2\lambda_3)}{(2\lambda_1 + \lambda_2)(3\lambda_1 + 3\lambda_2 + \lambda_3)}\), in the above equation,

we get the MREE of the MTBF.

Therefore, MREE of the MTBF is

\[
\delta^* = \delta_{01} + \frac{1}{kn} \left[ \frac{(4\lambda_1 + 5\lambda_2 + 2\lambda_3)}{(2\lambda_1 + \lambda_2)(3\lambda_1 + 3\lambda_2 + \lambda_3)} - \frac{1}{n B_0} \right] \delta_{02}
\]
Remark 6.5.1

From Theorem 5.4.1, we can obtain the UMVUE’s of $\xi$ and $\tau$, and hence obtain the UMVUE of $\alpha \xi + \beta \tau$:

$$
\hat{\delta}^{**} = \alpha \hat{\delta}_0 + \frac{1}{kn-1} \left[ \beta - \frac{\alpha}{nB_0} \right] \delta_{02}
$$

Hence one can obtain the UMVUE’s of the MTBF in each of the four systems discussed in this chapter.