Iterated function system in this general setting are defined and the existence and uniqueness of attractor of IFS is established. Some of the recent results are derived as special cases of our results. As an application, a collage theorem is also obtained in this new setting.
3.1 INTRODUCTION

Zadeh [42] introduced one of the revolutionary concept of a fuzzy set in 1965 to deal with uncertainty present in any system. Thereafter, the notion of fuzziness has been extended and studied in diverse directions of sciences and engineering (for example, see [123-125]). In the fuzzy theory, fuzzy topology became one of the active areas of research due to its wide range of applicability in different domains. One of the main topics of research in fuzzy topology was to introduce an appropriate notion of fuzzy metric space. Different authors introduced different definitions of fuzzy metric space. Kramosil and Michalek [46] proposed the definition of fuzzy metric space as a modification of probabilistic metric space introduced by Menger [126]. George and Veeramani [49] defined fuzzy metric space with the help of $t$-norm and also explored the notion of topology on the space. Further, these results have been generalized in different settings by various authors (see [41, 44, 49-50, 109] and several references thereof).

There are a number of contractive maps studied in the literature other than the Banach contraction map (for detail, see [26]). Kannan [18] introduced an important contractive condition which is independent of Banach contraction and characterizes metric completeness. Suzuki [37] generalized Banach contraction and established a characterization of the metric completeness for a map satisfying a new condition. Thereafter, a number of papers appeared in literature using the Suzuki’s notion of generalization [22, 35, 40-41, 127]. Sehgal and Reid [128] proposed a Banach type contraction map in fuzzy space. Grabiec [45] obtained fuzzy version of Banach contraction principle in fuzzy metric space due to Kramosil and Michalek. Gregori and Sapena [109] explored Banach contraction principle in fuzzy metric space proposed by George and Veeramani. Rodriguez and Ramaguera [47] obtained some results
for Hausdorff fuzzy metric space which played an important role in extension of Hutchinson Barnsley theory.

Uthayakumar and Easwaroorthy [54] fuzzified the IFS theory using $B$-contraction introduced by Sehgal and Reid [128] and obtained existence and uniqueness theorems for Hutchinson Barnsley operator.

Taking idea from the Suzuki type mapping, we obtain some existence and uniqueness results for a map satisfying a more generalized contraction condition in fuzzy metric spaces. Further, Hutchinson Barnsley theory is generalized in the setting of a fuzzy metric space. A solution to the inverse problem is also proposed in the form of a collage theorem.

3.2 PRELIMINARIES

In this section, we present the basic definitions and results required in the sequel.

**Definition 3.2.1** [107]. A binary operation $'*'$: $[0, 1] \times [0, 1] \rightarrow [0, 1]$ is a continuous $t$-norm if $'*'$ satisfies the following properties:

(a) $*$ is commutative and associative;
(b) $*$ is continuous;
(c) $a*1 = a \ \forall a \in [0, 1]$;
(d) $a*b \leq c*d$ whenever $a \leq c$ and $b \leq d$, and $a, b, c, d \in [0, 1]$.

**Definition 3.2.2** [108]. A $t$-norm is said to be of Hadzic-type if the family $\{*^p\}_{p \in N}$ of its iterates defined for each $s \in (0, 1)$ by

$$*^0(s) = 1, \ *^{p+1}(s) = *(s^p(s), s) \ \forall \ p \geq 0$$

is equi-continuous at $s = 1$, i.e., given $\lambda > 0$, $\exists \eta_i(\lambda) \in (0, 1)$ such that

$$1 \geq s > \eta_i(\lambda) \Rightarrow *^p(s) > 1 - \lambda \ \forall \ p \in N.$$
Definition 3.2.3 [49]. Let $X$ be a non-empty set. A fuzzy metric space (FMS) is a 3-tuple $(X, M, *)$, where $*$ is a continuous $t$-norm and $M$ is a fuzzy set on $X \times X \times (0, \infty)$ satisfying the following conditions, $\forall x, y, z \in X; \ t, s > 0$:

(a) $M(x, y, t) > 0$;
(b) $M(x, y, t) = 1$ iff $x = y$;
(c) $M(x, y, t) = M(y, x, t)$;
(d) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$;
(e) $M(x, y, _) : (0, \infty) \rightarrow (0,1]$ is continuous.

Definition 3.2.4 [49]. Let $(X, M, *)$ be an FMS. Then

(i) a sequence $\{x_n\}$ is called a Cauchy sequence if for each $\varepsilon \in (0, 1), \ t > 0$ there exists $n_0 \in N$ such that $M(x_n, x_m, t) > 1 - \varepsilon$ for all $n, m > n_0$;
(ii) a sequence $\{x_n\}$ converges to a point $x \in X$ iff $\lim_{n \to \infty} M(x, x_n, t) = 1$, for all $t > 0$;
(iii) the space $(X, M, *)$ is said to be complete, if every Cauchy sequence is convergent in the space $X$.

Definition 3.2.5 [47]. Let $(X, M, *)$ be an FMS and $H(X)$ be the collection of non-empty compact subsets of $X$. Then Hausdorff fuzzy metric in the space $H(X)$ may be defined as

$$H_M(A, B, t) = \min\left\{\inf_{a \in A} M(a, B, t), \inf_{b \in B} M(A, b, t)\right\} \forall A, B \in H(X),$$

where $M(a, B, t) = \sup_{b \in B} M(a, b, t)$.

Here, $H_M$ also satisfies properties (a) to (e) of Definition 3.2.3. So, $(H(X), H_M, *)$ is also an FMS. The space $(H(X), H_M, *)$ is called Hausdorff-fuzzy metric space (H-FMS) induced by the FMS $(X, M, *)$.

Lemma 3.2.1 [54]. Let $(X, M, *)$ be an FMS and $(H(X), H_M, *)$ the corresponding H-FMS. If $A, B, C, D \subseteq X$, then

$$H_M(A \cup B, C \cup D, t) \leq \min\{ H_M(A, C, t), H_M(B, D, t)\} \text{ for every } t > 0. \quad (3.1)$$
**Theorem 3.2.2** [47]. Let \((X, M, *)\) be an FMS. Then the H-FMS \((H(X), H_M, *)\) corresponding to FMS \((X, M, *)\) is complete if and only if \((X, M, *)\) is complete.

**Theorem 3.2.3** [56]. Let \((X, M, *)\) be a complete fuzzy metric space. Let \(T : X \to X\) be a Kannan type mapping (\(K\)-contraction) such that

\[
M(Tx, Ty, kt/2) \geq \min\{M(x, Tx, t), M(y, Ty, t)\}
\]

for all \(x, y \in X\) and \(0 < k < 1\). Then \(T\) has a unique fixed point.

### 3.3 Attractor of Suzuki Type IFS in Fuzzy Metric Spaces

Subrahmanyam [19] established the metric completeness characterization for Kannan contraction, which is independent of Banach contraction. Suzuki [37] provided a generalized version of some usual contractions like Banach contraction, Edelstien contraction etc, which also characterizes metric completeness. Motivating from the notion of Suzuki contraction, we first define a generalized contraction in fuzzy metric space and then obtain the existence and uniqueness results for the attractor of iterated function systems.

**Definition 3.3.1.** Let \((X, M, *)\) be a complete fuzzy metric space and \(T\) a self map on \(X\). Define a non-increasing function \(\theta (r)\) from \((0, 1)\) onto \([0.5, 1)\) by

\[
\theta (r) = \begin{cases} 
1 - r & \text{if } 0 < r < 0.5 \\
0.5 & \text{if } 0.5 \leq r < 1.
\end{cases}
\]

Assume that there exists \(r \in (0, 1)\), such that

\[
M(x, y, \theta(r)t) \leq M(x, Tx, t) \implies M(Tx, Ty, rt) \geq M(x, y, t)
\]

for all \(x, y \in X\).

Then, the map \(T\) satisfying (3.3) is called an \(S\)-contraction.

**Theorem 3.3.1.** Let \((X, M, *)\) be a complete fuzzy metric space and \(T\) be an \(S\)-contraction on \(X\). Then, there exists a unique fixed point \(z\) of \(T\). Moreover,

\[
\lim_{n \to \infty} T^n(x) = z \quad \forall \ x \in X.
\]
**Proof.** Let \( x \in X \) and \( x_n = T^n x \).

Here,

\[
M(x, Tx, \theta(r)t) \leq M(x, Tx, t) \quad \text{implies} \quad M(Tx, T^2x, rt) \geq M(x, Tx, t) \quad \text{(using (3.3))}
\]

Similarly,

\[
M(x_{n+1}, x_n, rt) \geq M(x_n, x_{n-1}, t) \\
\geq M(x_{n-1}, x_{n-2}, t) \\
\vdots \\
\geq M(x_1, x, \frac{t}{r^{n-1}}),
\]

Thus, \( \{x_n\} \) is a Cauchy sequence in a complete space \( X \). So, it will converge in the space \( X \).

Let \( \{x_n\} \) converges to \( z \in X \).

To prove that \( z \) is a fixed point of \( T \), let for \( u \in X \setminus \{z\}, \exists m \in N \) such that

\[
M\left( x_n, z, \frac{t}{4} \right) \geq M(u, z, t) \forall n \in N \text{ and } n \geq m.
\]

Here,

\[
M(x_n, x_{n+1}, t) \geq M(x_n, x_{n+1}, \theta(r)t) \\
\geq M\left( x_n, z, \frac{\theta(r)t}{2} \right) * M\left( x_{n+1}, z, \frac{\theta(r)t}{2} \right) \\
\geq M(u, z, 2\theta(r)t) * M(u, z, 2\theta(r)t) \\
\geq M(u, z, 2\theta(r)t) \\
\geq M(x_n, z, \theta(r)t) * M(x_n, u, \theta(r)t)
\]

Taking limit as \( n \to \infty \) both sides, we have

\[
\lim_{n \to \infty} M(x_n, x_{n+1}, t) \geq \lim_{n \to \infty} M(x_n, z, \theta(r)t) * \lim_{n \to \infty} M(x_n, u, \theta(r)t)
\]

\[
\geq \lim_{n \to \infty} M(x_n, u, \theta(r)t).
\]

Which implies,
\[
\lim_{n \to \infty} M(x_{n+1}, Tu, rt) \geq \lim_{n \to \infty} M(x_n, u, t).
\]

So,

\[
M(z, Tu, rt) \geq M(z, u, t).
\]

Now, we assume that

\[
T^j(z) \neq z \forall j \in N.
\]

Here,

\[
M(T^{j+1}z, z, r^{j+1}t) \geq M(T^jz, z, r^j t)
\]

\[
\quad \geq M(T^{j-1}z, z, r^{j-2} t)
\]

\[
\quad \quad \vdots
\]

\[
\quad \geq M(Tz, z, t).
\]

Now, we have two cases:

(1) When \(0 < r < 0.5\) and
(2) \(0.5 \leq r < 1\).

Case 1. When \(0 < r < 0.5\)

Let if possible,

\[
M(T^2z, z, t) > M(T^2z, T^3z, t).
\]

Then

\[
M(z, Tz, t) \geq M \left( z, T^2z, \frac{r}{2} \right) * M \left( Tz, T^2z, \frac{r}{2} \right)
\]

\[
\quad > M \left( T^2z, T^3z, \frac{r}{2} \right) * M \left( z, Tz, \frac{r}{2} \right)
\]

\[
\quad \geq M \left( z, Tz, \frac{r}{2r^2} \right) * M \left( z, Tz, \frac{r}{2r} \right)
\]

\[
\quad \geq M(z, Tz, t).
\]

Which is a contradiction.

So,
Now,

\[ M(z, Tz, t) \geq M \left( z, T^2 z, \frac{t}{2} \right)^* M \left( Tz, T^2 z, \frac{t}{2} \right) \]
\[ \geq M \left( T^2 z, T^3 z, \frac{t}{2} \right)^* M \left( z, Tz, \frac{t}{2} \right) \]
\[ \geq M \left( z, Tz, \frac{t}{2r} \right)^* M \left( z, Tz, \frac{t}{2r} \right) \]
\[ \geq M \left( z, Tz, \frac{t}{2r} \right) \]
\[ > M(z, Tz, t), \]

which is again a contradiction.

Case 2: When \( 0.5 \leq r < 1 \)

Now, either

\[ M(x, Tx, t) \geq M(x, y, \theta(r)t) \text{ or} \]
\[ M(Tx, T^2 x, t) \geq M(Tx, y, \theta(r)t) \text{ holds.} \]

If not, then

\[ M(x, Tx, t) < M(x, y, \theta(r)t) \]
\[ M(Tx, T^2 x, t) < M(Tx, y, \theta(r)t). \]

Now,

\[ M(x, Tx, t / 2) \geq M(x, Tx, \theta(r)t) \]
\[ \geq M \left( x, y, \frac{\theta(r)t}{2} \right)^* M \left( Tx, y, \frac{\theta(r)t}{2} \right) \]
\[ > M \left( x, Tx, \frac{t}{2} \right)^* M \left( Tx, T^2 x, \frac{t}{2} \right) \]
\[ \geq M\left(x, Tx, \frac{t}{2}\right) \ast M\left(x, Tx, \frac{t}{2r}\right) \]
\[ \geq M\left(x, Tx, \frac{t}{2}\right) \ast M\left(x, Tx, \frac{t}{2}\right) \]
\[ \geq M\left(x, Tx, \frac{t}{2}\right) , \]

which is a contradiction.

So, either

\[ M(x, Tx, t) \geq M(x, y, \theta(r)t) \] or

\[ M(Tx, T^2x, t) \geq M(Tx, y, \theta(r)t) . \]

Which implies, either

\[ M(x_n, x_{n+1}, t) \geq M(x_n, z, \theta(r)t) \] or

\[ M(x_{n+1}, x_{n+2}, t) \geq M(x_{n+1}, z, \theta(r)t) . \]

This further implies, either

\[ M(x_{n+1}, Tz, rt) \geq M(x_n, z, t) \] or

\[ M(x_{n+2}, Tz, rt) \geq M(x_{n+1}, z, t) . \]

As we know \( \{x_n\} \rightarrow z \).

So, we have \( z = Tz \).

The uniqueness of fixed point is obvious from (3.4).

We can easily see that a \( B \)-contraction is a special case of an \( S \)-contraction. However \( S \)-contraction and \( K \)-contraction are independent, see the examples below.

**Example. 3.3.1.** Let us consider a complete FMS \( (X, M, *) \); where, \( X = \{(0, 0), (0, 5), (5, 0), (0, 6), (6, 0), (5, 6), (6, 5)\} \) and fuzzy metric is defined as

\[ M((x_1, x_2), (y_1, y_2), t) = \frac{1}{e^{K(x_1-y_1)+(x_2-y_2)|t|}} . \]
The mapping $T$ on $X$ is defined as

$$T(x_1, x_2) = \begin{cases} (x_1, 0) & \text{if } x_1 \leq x_2 \\ (0, x_2) & \text{if } x_1 > x_2 \end{cases}$$

Here,

$$M\left(x, y, \frac{5t}{6}\right) \leq M(x, Tx, t)$$

$$\forall x, y \notin \{(5, 6), (6, 5)\}$$

For $x = (5, 6)$ and $y = (6, 5)$, we have

$$M(x, y, \theta(r)t) = \frac{1}{e^{2\theta(r)t}} > \frac{1}{e^{2\theta(r)t}} > \frac{1}{e^{2\theta(r)t}} = M(x, Tx, t).$$

Similarly, for $x = (6, 5)$ and $y = (5, 6)$, we have

$$M(x, y, \theta(r)t) > M(x, Tx, t).$$

So, $T$ is an $S$-contraction.

Now, for $x = (5, 6)$ and $y = (6, 5)$, we have

$$M\left(T(5, 6), T(6, 5), \frac{kt}{2}\right) = \frac{1}{e^{2\theta(r)t}} \text{ and }$$

$$M((5, 6), T(5, 6)) = M((6, 5), T(6, 5)) = \frac{1}{e^{2\theta(r)t}}.$$ 

So, it is not a $K$-contraction.

**Example 3.3.2.** Let us consider a complete FMS $(X, M, *)$; where, $X = \{-1, 0, 1, 2\}$ and fuzzy metric is given by

$$M(x, y, t) = \frac{1}{e^{4|x-y|t}}.$$ 

The mapping $T: X \rightarrow X$ is defined as
\[ T(x) = \begin{cases} 
0 & \text{if } x \neq 2 \\
-1 & \text{if } x = 2.
\end{cases} \]

Here, for \( x = 1 \) and \( y = 2 \), we have

\[ M(1, T1, t) = \frac{1}{e^{3t}} \text{ and } M(2, T2, t) = \frac{1}{e^{3t}}. \]

So,

\[ M(1, T1, t) \land M(2, T2, t) = \frac{1}{e^{3t}} \text{ and } \]

\[ M(T1, T2, \frac{kt}{2}) = \frac{1}{e^{2kt}}. \]

Thus, for \( 0 < k < 2/3 \), \( T \) is a \( K \)-contraction.

\[ M(1, 2, \theta(r)t) = \frac{1}{e^{\theta(r)t}} \leq \frac{1}{e^{3t}}. \]

But

\[ M(T1, T2, rt) = \frac{1}{e^{rt}} \text{ and } M(1, 2, t) = \frac{1}{e^{t}}; \]

Which implies, there exists no \( r \in (0, 1] \) such that

\[ M(T1, T2, rt) \geq M(1, 2, t). \]

So, \( T \) does not satisfy (3.4).

Thus, the independence of \( K \)-contraction and \( S \)-contraction are justified.

If we relax the condition imposed on \( \theta(r) \) in Theorem 3.3.1, we get the following result of Gregori and Sapena (Th 4.8, [109]).

**Corollary 3.3.1** [109]. Let \((X, M, *)\) be a complete fuzzy metric space. Let \( T: X \to X \) be a \( B \)-contraction such that
for all \( x, y \) in \( X \), \( 0 < k < 1 \). Then \( T \) has a unique fixed point.

Next, we define iterated function system in fuzzy metric space.

**Definition 3.3.2.** Let \((X, M, \ast)\) be a complete FMS and \( T_i : X \to X \), \( i = 1, 2, 3, \ldots, n \), satisfy (3.4). Then \( S \)-fuzzy IFS consists of a finite set of \( T_i \)'s in a complete FMS and it is denoted by \((X, M, T_1, T_2, \ldots, T_n)\).

The following proposition is fundamental for the collage theorem in fuzzy metric space.

**Proposition 3.3.2.** Let \((X, M, \ast)\) be a fuzzy metric space and \( T \) an \( S \)-contraction on \( X \). If \( z \) is a fixed point of \( T \) in \( X \), then

\[
M(x, z, t) \geq M\left(x, Tx, \frac{rt}{2r-1}\right).
\]

**Proof.** For \( x \in X \), we have \( \lim_{n \to \infty} x_n = z \).

Now, consider

\[
M(x, z, t) \geq \lim_{n \to \infty} M(x, x_n, t)
\]

\[
\geq \lim_{n \to \infty} \{M(x, x_{i_1}, t/2)*M(x, x_{i_2}, t/2^2)*\ldots*M(x, x_n, t/2^{n-1})\}
\]

\[
\geq \lim_{n \to \infty} \{M(x, x_{i_1}, t/2)*M(x, x_{i_2}, t/2^2 r)*\ldots*M(x, x_n, t/2^{n-1} r^{n-2})\}
\]

\[
= M\left(x, x_{i_1}, \frac{rt}{2r-1}\right).
\]

This completes the proof.

**Lemma. 3.3.3.** Suppose \( T_i : H(X) \to H(X) \) are \( S \)-contractions on \((H(X), H_M, \ast)\) for \( i = 1, 2, \ldots, n \), where \( n \) is some natural number. Then \( \bigcup_{i=1}^n T_i \) is also an \( S \)-contraction on \((H(X), H_M, \ast)\).

**Proof.** For \( n = 1 \), it is very obvious. For \( n = 2 \), we have

\[
H_M(B, C, \theta(\tau_1) t) \leq H_M(B, TB, t) \Rightarrow H_M(T_i B, T_i C, \tau_i t) \geq H_M(B, C, t) \quad \text{and}
\]
Similarly, it can be proved for any natural number $n$.

**Theorem 3.3.4.** Let $\{X, M, T_1, T_2, \ldots, T_n\}$ be an $S$-fuzzy IFS. Then $(H(X), H_M, T_1, T_2, \ldots, T_n)$ will also be an $S$-fuzzy IFS. Moreover, $\bigcup_{i=1}^{n} T_i$, will have a unique fixed point $A \in H(X)$, which is also called an attractor of the IFS.

**Proof.** According to Theorem 3.2, if $(X, M, *)$ is a complete FMS, then its corresponding H-FMS $(H(X), H_M, *)$ is also complete. Now, if $T_1, T_2, \ldots, T_n$ are $S$-contractions, then according to Theorem 3.2.2, the mapping $\bigcup_{i=1}^{n} T_i$ will also be an $S$-contraction.

Here, we have an $S$-contraction in a complete FMS $(H(X), H_M, *)$. So, using Lemma 3.3.3, we find that $\bigcup_{i=1}^{n} T_i$ has a unique fixed point or an attractor $A \in H(X)$.

Now, if we relax the condition on $\theta(r)$ in Theorem 3.3.4, we get the following result of Uthayakumar and Easwarmoorthy [54].

**Corollary 3.3.2 [54].** Let $\{X, M, T_1, T_2, \ldots, T_n\}$ be a fuzzy IFS where $T_1, T_2, \ldots, T_n$ are $B$-contraction on $X$. Let $H(X)$ be the collection of all non empty compact subsets of $X$. Then, there exists unique attractor $A \in H(X)$ of $T$.

From Proposition 3.3.2 and Theorem 3.3.4, the following collage theorem for $S$-contraction in fuzzy metric space follows immediately.

**Theorem 3.3.5.** Let $(X, M, *)$ be a fuzzy metric space and $(K(X), H_M, *)$ be the corresponding Hausdorff fuzzy metric space. Suppose $(K(X), T_1, T_2, \ldots, T_n)$ be a Fuzzy IFS, where $T_1, T_2, \ldots, T_n$ are $S$-contractions in $K(X)$ and if $A$ is the attractor of the IFS. Then we have
\[ M(L, A, t) \geq M\left( L, TL, \frac{rt}{2r-1} \right) \forall L \in K(X) \text{and } 0 < r < 1. \]