Chapter 2

On Integral Equations of the First Kind with Some Applications

2.1 Introduction

Integral equations are results of transformation of points in a given vector space of integrable functions to points in the same space by the use of integral operators [20]. One of the broad categories of integral equations are integral equations of the first kind, in which the unknown to be determined is present within the integral operator alone. It so happens that many physical problems boil down to solving integral equations of the first kind. Given below is an example illustrating how a physical problem leads to an integral equation of the first kind.

Example:

The problem is of mechanics that involves finding the curve of descent, given the time of descent as a function of the vertical distance of fall [49]. The problem is to find the unknown path in the plane along which a particle will fall, under the influence of gravity alone, so that at each instant the time of fall is a known function of the distance fallen [76, 15]. Supposing that the particle falls from height $Y$ and that the path of descent is
parameterized by arc length \( s \), then the length of arc traversed at time \( t \) is denoted as \( s(t) (s(0) = 0) \). Assuming that the particle starts from rest, the gain in kinetic energy to the loss in potential energy is equated such that:

\[
\frac{1}{2} \left( \frac{ds}{dt} \right)^2 = g(Y - y), \tag{2.1.1}
\]

where \( g \) is the gravitational acceleration, \( v = \frac{ds}{dt} \) represents the speed of the particle. Making use of the well known formula, that connects the arc length to shape of the curve as given by

\[
\frac{ds}{dy} = [1 + \sigma'(y)^2]^{1/2}, \tag{2.1.2}
\]

where \( \sigma \equiv \sigma(y) \) is the equation of the smooth curve, \( \sigma' = \frac{d\sigma}{dy} \), then equation (2.1.1) can be expressed in the form:

\[
\frac{dy}{dt} = \frac{ds}{dy} \frac{ds}{dt} = -\frac{[2g(Y - y)]^{1/2}}{[1 + \sigma'(y)^2]^{1/2}}, \tag{2.1.3}
\]

The minus sign in the above relation (2.1.3) denotes the descent of the particle implying that \( s \) decreases with time \( t \).

Integrating equation (2.1.3), gives

\[
g(Y) = \int_0^Y \frac{\phi(y)dy}{\sqrt{(Y - y)}}, \tag{2.1.4}
\]

with

\[
\phi(y) = \left[ \frac{1 + (\sigma')^2}{2g} \right]^{1/2}. \tag{2.1.5}
\]

The problem of finding the shape of the curve, when the time of fall \( T(= g(Y)) \) is given, is the historic Abél’s problem, which is an integral equation of the first kind. In the present chapter such integral equations of the first kind, specifically of the Fredholm type are taken up for studies.

Recollecting from the first chapter, the general structure of the integral equation of the first kind, is given by

\[
K\phi = \int_a^b k(x, t)\phi(t)dt = g(x), \quad (a \leq x \leq b) \tag{2.1.6}
\]
where \( k(x, t) \) is the kernel, \( g(x) \) is the free term and \( \phi(x) \) is the unknown to be determined.

Now the problem is to solve the integral equation. The solution of equation (2.1.6) exists only when the equation is consistent. The consistency condition can be determined by making the following observation:

If there exists a nontrivial function \( h(x) \) such that

\[
\int_a^b k(t, x)h(x)dx = 0 \quad (\forall \; t \; \text{such that} \; a \leq t \leq b) \quad (2.1.7)
\]

then it is obvious to have that

\[
\int_a^b g(x)h(x)dx = 0 \quad (2.1.8)
\]

The condition (2.1.8) is the CONSISTENCY condition for the existence of solution of the given integral equation (2.1.6). It must be noted that, the above condition is valid only for SYMMETRIC kernels, that is when \( k(x, t) = k(t, x) \). It is observed that many physical problems that lead to solution of integral equations possess symmetric kernels and hence the above consistency condition (2.1.8) could be generalized to all such integral equations.

Discussed next, are the possible methods of solution to such consistent first kind equations.

**The possible methods of solutions:**

Two possible methods of solution, which includes both closed form solution and approximate solution are briefed here.

1. **Closed form solution:**

Let there exist a function \( w(x, t) \), \( a \leq x \leq b, a \leq t \leq b \), such that

\[
\int_a^b w(x, t)k(x, \xi)dx = C_0\delta(\xi - t), \quad (2.1.9)
\]

where \( C_0 \) is a known non-zero constant, and \( \delta(x) \) is Dirac’s delta function.

Multiplying (2.1.6) by \( w(x, t) \) and integrating with respect to ‘x’, between \( x = a \) and \( x = b \), gives

\[
\int_a^b \left( \int_a^b k(x, \xi)\phi(\xi)d\xi \right) w(x, t)dx = G(t), \quad (2.1.10)
\]
where

\[ G(t) = \int_a^b g(x)w(x,t)dx. \quad (2.1.11) \]

Assuming that interchanging orders of integration is permissible in equation (2.1.10), and making use of the relation (2.1.9), equation (2.1.10) implies

\[ \int_a^b C_0 \delta(\xi - t)\phi(\xi)d\xi = G(t). \quad (2.1.12) \]

From the definition of delta function the solution of the equation (2.1.6) is thus obtained to be

\[ \phi(t) = \frac{1}{C_0} G(t) \quad (a \leq t \leq b). \quad (2.1.13) \]

**Example:**

Let

\[ \int_0^\infty \phi(t) \cos(\pi t)dt = g(x), \quad (2.1.14) \]

be the integral equation of the first kind whose solution is to be obtained by the above method.

It can be observed that by taking inverse Fourier transform the solution of the above equation (2.1.14) is obtained to be:

\[ \phi(t) = \frac{2}{\pi} \int_0^\infty g(x) \cos(\pi x)dx. \quad (2.1.15) \]

Now the problem is to find a \( w(x, t) \) by new method (see relations (2.1.13) and (2.1.11)) that yields the solution of the equation (2.1.14) same as (2.1.15).

Comparing relation (2.1.15) with relation (2.1.13), it is observed that

\[ w(x, t) = \frac{2}{\pi} \cos(\pi t). \quad (2.1.16) \]

The above result necessitates that the following relation must hold good, that is

\[ \frac{2}{\pi} \int_0^\infty \cos(\pi x)\cos(\pi \xi)dx = \delta(\xi - t). \quad (2.1.17) \]
The LHS of the above equation, can be written as

\[
\frac{2}{\pi} \int_0^\infty \cos(xt) \cos(x\xi) \, dx = \frac{1}{\pi} \int_0^\infty [\cos x(\xi - t) + \cos x(\xi + t)] \, dx
\]

(2.1.18)

and this can be equated to \(\delta(\xi - t)\) for the following reason:

From the definition of Fourier transform for \(\delta\) function, it follows that

\[
\int_{-\infty}^{\infty} \delta(x)e^{i\alpha x} \, dx = 1.
\]

(2.1.19)

The corresponding inverse Fourier transform, gives that

\[
\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\alpha x} \, d\alpha
\]

\[
= \frac{1}{2\pi} \left[ \int_{0}^{\infty} e^{i\alpha x} + \int_{0}^{\infty} e^{-i\alpha x} \right] \, d\alpha
\]

(2.1.20)

\[
\delta(x) = \frac{1}{\pi} \int_0^\infty \cos(\alpha x) \, d\alpha.
\]

Thus relation (2.1.16) is rather valid and thus for a given \(g(x)\) the solution of the integral equation can be obtained using relations (2.1.13) and (2.1.11). It must be noted that the example considered here is rather simple and the solution can be obtained by means of Fourier transform also. But generally the challenge of solving such first kind equations lies in determining the function \(w(x, t)\) that satisfies relation (2.1.9).

2. Approximate solution:

Here the approximate solution of equation (2.1.6) in expressed in the form of a polynomial, as given by

\[
\phi(x) = \sum_{i=1}^{N+1} a_{i-1} x^{i-1}
\]

(2.1.21)

where \(a_{i-1}(i = 1, 2, ..., N + 1)\) are unknown constants to be determined. Substituting (2.1.21) in equation (2.1.6) it is observed that there exists many solutions for equation (2.1.6), for different values of \(N\).

Given below are a few examples, having at least two solutions.

1. \(\int_0^1 (x + t)\phi(t) \, dt = x \quad (0 \leq x \leq 1)\)

The two different solutions are:
\[ \phi_1(t) = 4 - 6t, \]
\[ \phi_2(t) = 18t - 24t^2. \]

2. \[ \int_0^1 (x^2 + t^2) \phi(t) dt = x^2 \quad (0 \leq x \leq 1) \]

The two different solutions are:

\[ \phi_1(t) = 3 - 4t, \]
\[ \phi_2(t) = 12t - 15t^2. \]

3. \( \int_0^1 (\sqrt{x} + \sqrt{t}) \phi(t) dt = 1 + \sqrt{x} \quad (0 \leq x \leq 1) \)

The two different solutions are:

\[ \phi_1(t) = -3/2 + 5t, \]
\[ \phi_2(t) = -5t + (21/2)t^2. \]

A close observation on the examples of both forms of solutions reveal that there could exist many \( w(x, t) \)'s and polynomial approximations of different orders leading to non-uniqueness of solution of the integral equation (2.1.6). However, the solution of the integral equation (2.1.6) is unique if and only if the solution of the corresponding transposed homogeneous equation

\[ \int_a^b k(t, x) \phi(t) dt = 0 \quad (a \leq x \leq b) \quad (2.1.22) \]

is trivial, that is

\[ \phi_0(t) = 0. \quad (2.1.23) \]

Though it is fairly less complicated, at times, to establish the existence of solution of the first kind integral equations, it is not easy to prove their uniqueness, especially when the kernel \( k(x, t) \) involves some known parameter/s. It is not just integral equations of the first kind that pose problems with regard to existence and uniqueness of solution [20, 51, 57], in
the presence of a parameter in the kernel, integral equations of the second kind also exhibit similar challenges.

However, it is interesting to note that the common property enjoyed by all the three examples, considered above (under approximate solution), is that the kernels of these integral equations are all symmetric. Even though the solutions of the above integral equations are not unique, a special functional connecting the forcing term $g(x)$ of any of these integral equations and any solution $\phi(x)$ of the corresponding equation as given by the relation

\[ < g, \phi > = \int_a^b g(x)\phi(x)dx, \]  

(2.1.24)

possesses a "unique" value as presented below:

For example 1. $< g, \phi_1 > = < g, \phi_2 > = 0$,

For example 2. $< g, \phi_1 > = < g, \phi_2 > = 0$,

For example 3. $< g, \phi_1 > = < g, \phi_2 > = 2$.

In fact, the following important result holds good for any symmetric integral equation of the first kind (2.1.6), for which $k(x, t) = k(t, x)$, that is

\[ < g, \phi_1 > = < g, \phi_2 >, \]  

(2.1.25)

where $\phi_1$ and $\phi_2$ are any two solutions of the integral equation, and therefore it can be written that

\[ < g, \phi_1 > \equiv \int_a^b g(t)\phi_1(t)dt = \int_a^b \phi_1(t) \left( \int_a^b k(u, t)\phi_2(u)du \right)dt \]

\[ = \int_a^b \phi_2(u) \left( \int_a^b k(u, t)\phi_1(t)dt \right)du \]  

(2.1.26)

(\text{because } k \text{ is symmetric})

In the present work, studies on problems dealt by Ursell [92], Williams [97], Schenck [51], Jones [57] involving problems of water waves, electrostatics and acoustics respectively are taken up for studies. The problems dealt with in all the above areas gives rise to integral equations of the first kind or of the second kind.

The problems are viewed from three perspectives:
1. Studies on some existing methods of solution of the integral equations of both first and second kind and the difficulties involved therein.

2. Conversion of certain integral equations of the first kind to integral equations of the second kind based on the method proposed by Williams [97].

3. New developments in the analysis of the existence and uniqueness of solutions of the first and the second kind integral equations when the kernel involves parameter/s.

A brief study is made in the next section on some of the physical problems and their methods of solution. The problems of acoustics [51] and electromagnetic theory [97] are considered in Sub Sections-2.2.1 and 2.2.2, as examples to highlight the fact that existence and uniqueness of solution of integral equations either of the first kind or of the second kind in the presence of parameter/s are rather difficult to be established. Problems of scattering of surface water waves studied by Ursell [92] that leads to integral equations of the first kind are dealt with in detail in Sub Section-2.2.3. The integral equations of the first kind obtained by Ursell [92] is then taken up again in Section-2.3 and is converted to integral equations of the second kind using William’s [97] method. The method of conversion is presented in detail in the same section. The analysis of the existence and uniqueness of solutions of the integral equations is taken up in Section-2.4. Some Numerical results are tabulated in Section-2.5. The observations and inferences are briefed in Section-2.6.

2.2 A brief study

As mentioned earlier many physical problems relating to the field of electrostatics [97], acoustics [57] and water waves [92] reduce to problems of integral equations either of the first kind or of the second kind. The integral equations thus obtained depend much on the representation used. It is found that these integral equations invariably involve some parameter/s in their kernel. The parameter/s represent some physical quantity which are
vital in the study of physical problems. Such problems are of great interest mathematically.

One of the ways of solving a physical problem involving partial differential equation is, by employing appropriate integral representation for the solution of the partial differential equation [97]. Such a representation often reduces the problem to Fredholm integral equation of the first kind. As is well known, these integral equations of the First kind do not posses unique solution always. This difficulty was overcome by Williams [97], whereby he presented a general procedure for reformulating certain integral equations of the first kind to integral equations of the second kind.

The procedure proposed by Williams [97] is summarized as given below:
Consider integral equations of the form,

\[(K_0 + K_1)\phi = f,\]  \hspace{1cm} (2.2.1)

where the operators \(K_0\) and \(K_1\) are of the form

\[K_i\phi = \int_a^b k_i(x, t)\phi(t)dt \quad (i = 0, 1).\]

If \(K_0^{-1}\) exists, then, we can write equation (2.2.1) as:

\[\phi + K_0^{-1}(K_1\phi) = K_0^{-1}f.\]  \hspace{1cm} (2.2.2)

Setting

\[L\phi = \psi,\]  \hspace{1cm} (2.2.3)

with \(L\) being a known operator of special form, equation (2.2.2) results in the following equation

\[\psi + L(K_0^{-1}(K_1(L^{-1}\psi))) = L(K_0^{-1}f),\]  \hspace{1cm} (2.2.4)

which simplifies to the integral equation of the second kind as given by

\[\psi + \tilde{K}\psi = g,\]  \hspace{1cm} (2.2.5)

if \(\tilde{K} \equiv L(K_0^{-1}(K_1L^{-1}))\) happens to be an integral operator.

This method was employed to many physical problems by Williams [97]. One such problem is discussed next.
2.2.1 Problem of Electrostatics:

The problem is that of determining the potential at any point on a circular disc of radius \( a \) \((z = 0, 0 \leq \rho \leq a)\), placed between two parallel plates, represented by the equations \( z = b \) and \( z = c \), in cylindrical polar co-ordinates \((\rho, \phi, z)\). This was reduced to a problem of integral equation of the first kind by Williams [97], for its solution. Mathematically the above problem is represented as

\[
\int_0^a \hat{k}(\rho, t)g(t)dt = f(\rho); \quad (0 < \rho < a) \tag{2.2.6}
\]

where

\[
\hat{k}(t, \rho) = \int_0^{2\pi} \cos(n\psi)G_2(t, \rho, \psi, 0, 0)d\psi, \tag{2.2.7}
\]

\( n \) being an integer, and

\[
G_2(t, \rho, \psi, z, z_0) = \sum_{r=0}^{\infty} (2 - \delta_{0r})G^{(r)}_2(\rho, r, z, z_0) \cos r\psi + R^{-1}, \tag{2.2.8}
\]

with

\[
R^2 = (z - z_0)^2 + \rho^2 + t^2 - 2\rho t \cos(\phi - \phi'), \tag{2.2.9}
\]

and

\[
G^{(r)}_2 = \int_0^\infty \left[ e^{-(c+z_0)p} \sinh p(z-b) - e^{-(b+z_0)p} \sinh p(z+c) \right] \frac{J_r(\rho \rho')J_r(pt)dp}{\sinh p(b+c)}, \tag{2.2.10}
\]

\( \delta_{mn} \) denoting Kroneckar delta and \( J_n \) denoting the Bessel function of first kind of order \( n \).

It is observed that the kernel \( \hat{k} \) in the relation (2.2.7) possesses two parameters \( b \) and \( c \) (see (2.2.10)).

Williams [97] then reduced the integral equation (2.2.6) to an integral equation of the second kind, as described below:

The equation (2.2.6) can be re-written as (see [97] for details):

\[
\rho^{-n} \int_0^\rho \frac{t^{2n}S(t)dt}{(\rho^2 - t^2)^{1/2}} = \frac{1}{4}f(\rho) - \int_0^a G(t, \rho)g(t)dt, \tag{2.2.11}
\]
where
\[ S(t) = \int_t^a \frac{\rho^{-n} g(\rho)}{(\rho^2 - t^2)^{1/2}} \, d\rho, \]  
(2.2.12)
and
\[ G(t, \rho) = \frac{\pi}{2} \int_0^\infty \left[ \frac{2 \sinh p b \sinh p c}{\sinh p(a + b)} - 1 \right] J_n(p\rho)J_n(pt) \, dp, \]  
(2.2.13)
with
\[ J_n(p\rho)J_n(pt) = \frac{2p}{\pi} \rho^n \left[ \int_0^p \frac{J_{n-1/2}(pu)u^{n+1/2} \, du}{(p^2 - u^2)^{1/2}} \right] \left[ \int_0^t \frac{J_{n-1/2}(pv)v^{n+1/2} \, dv}{(t^2 - v^2)^{1/2}} \right]. \]  
(2.2.14)

Further simplification of equation (2.2.11) leads to
\[ W(\rho) = \frac{1}{2\pi \rho^n} \frac{d}{d\rho} \int_0^\rho \frac{t^{n+1} f(t) dt}{\sqrt{\rho^2 - t^2}} - \int_0^\rho W(t)L(t, \rho) \, dt, \]  
(2.2.15)
where
\[ L(t, \rho) = (tp)^{1/2} \int_0^\infty \rho J_{n-1/2}(pp)J_{n-1/2}(pt) \left[ \frac{2 \sinh p b \sinh p c}{\sinh p(b + c)} - 1 \right] \, dp, \]  
(2.2.16)
and
\[ \rho^n S(\rho) = W(\rho). \]  
(2.2.17)

Our observation:
The observation is that equation (2.2.15) is an integral equation of the second kind involving the parameters \(b\) and \(c\) in the kernel \(L(t, \rho)\), and that the values of these parameters may affect the existence and uniqueness of solution of the integral equation.

2.2.2 Problem of Acoustics:

Schenck [51], studied the problem of acoustic radiation which basically deals with the study of the pressure field perturbed by real objects in an acoustic medium. The real objects could be of various geometries like infinite rigid planes or cylinders, rigid geometrically perfect spheres or spheroids, and so on. He made a study on the analytical methods that were studied then and highlighted the difficulties innate in the previous integral equation.
methods of solution, thereby proposing an improved method of solution. The challenges involved in integral equation methods of solution are of interest here, as much of the theory developed in the present chapter revolves around analyzing and finding a feasible solution to these difficulties.

The problem that Schenck [51] took up for study was a boundary value problem involving Helmholtz equation,

$$\nabla^2 p(x) + k^2 p(x) = 0, \quad x \in R_0,$$  \hspace{1cm} (2.2.18)

where $p(x)$ is the pressure, $k = \omega/c$, and $\omega$ is the angular frequency. The pressure is such that it must be finite in the region of interest and that it must satisfy the boundary condition,

$$\frac{\partial p}{\partial n_\zeta} \equiv \hat{n}_\zeta \cdot [\nabla p(x)]_{x=\zeta} = -j\omega p v(\zeta), \quad \zeta \in S,$$  \hspace{1cm} (2.2.19)

where $S$ is the total surface area of an arbitrarily shaped finite object immersed in an infinite ideal homogeneous fluid whose density is $\rho$ and speed of sound is $c$ with $j^2 = -1$. The region exterior to $S$ is $R_0$ while the region interior to $S$ is called $R_i$. $\zeta$ is any point on the surface and $x$ is any arbitrary point in the exterior region. $\hat{n}_\zeta$ is the positive unit normal directed from the point $\zeta$ on $S$ into $R_0$ and $v(\zeta)$ is the velocity component normal at any surface point $\zeta$.

The pressure is also required to satisfy the radiation condition,

$$\lim_{R \to \infty} \int_{S_R} \left| \frac{\partial p(x)}{\partial r} + jkp(x) \right|^2 dS = 0,$$  \hspace{1cm} (2.2.20)

where $r$ is the radial distance from the origin of co-ordinates and $S_R$ is a sphere of radius $R$ centered at the origin, surrounding $x$ and the object $S$. The problem is to find $p(x)$.

Schenck [51] studied three integral equation methods of solution which had some inherent difficulties. A brief of all the three methods of solution involving three different representations are given below:
1. Simple-Source Formulation:

The integral representation used to solve the Helmholtz equation is

\[ p(x) = j\omega \rho \int_S \sigma(\xi) \frac{e^{-jkd(x, \xi)}}{d(x, \xi)} dS(\xi), \quad x \in R_0, \]  

(2.2.21)

where \( \sigma(\xi) \) is the source-density function (independent of \( x \), and initially unknown), and \( d(x, \xi) \) is the distance between a point \( x \in R_0 \) and a point \( \xi \in S \). Equation (2.2.21) satisfies equation (2.2.18) and equation (2.2.20) for arbitrary \( \sigma(\xi) \). The source density function \( \sigma(\xi) \) is required to satisfy equation (2.2.19) also.

The integrand becomes singular as \( x \to S \), but then the integral exists as a limiting value and is also continuous. The normal derivative of the integral (2.2.21) is discontinuous at the surface \( S \), resulting in two terms when equation (2.2.21) is differentiated. The boundary condition (2.2.19) when employed on equation (2.2.21) gives rise to the following integral equation

\[ v(\zeta) = 2\pi \sigma(\zeta) - \int_S \sigma(\xi) \frac{\partial}{\partial n_\zeta} \left[ \frac{e^{-jkd(\zeta, \xi)}}{d(\zeta, \xi)} \right] dS(\xi), \quad \zeta \in S. \]  

(2.2.22)

2. Surface Helmholtz Integral Equation Formulation:

In this, the pressure in the region exterior to \( S \) is given by the Helmholtz integral formula as

\[ p(x) = \frac{1}{4\pi} \int_S \left[ p(\xi) \frac{\partial}{\partial n_\zeta} \left[ \frac{e^{-jkd(x, \xi)}}{d(x, \xi)} \right] + j\omega \rho v(\xi) \frac{e^{-jkd(x, \xi)}}{d(x, \xi)} \right] dS(\xi), \quad x \in R_0. \]  

(2.2.23)

When the point \( x \) in equation (2.2.23) approaches a point \( \zeta \) on \( S \), the following integral equation is obtained in terms of the known velocity:

\[ 2\pi p(\zeta) - \int_S \left[ p(\xi) \frac{\partial}{\partial n_\zeta} \left[ \frac{e^{-jkd(\zeta, \xi)}}{d(\zeta, \xi)} \right] \right] dS(\xi) = j\omega \rho \int_S v(\xi) \frac{e^{-jkd(\zeta, \xi)}}{d(\zeta, \xi)} dS(\xi), \quad \zeta \in S. \]  

(2.2.24)

3. Interior Helmholtz Integral Equation Formulation:

This representation is used when the field point is restricted to lie in the region \( R_i \), interior to the surface \( S \) and the integral formula is as follows:

\[ 0 = \int_S \left[ p(\xi) \frac{\partial}{\partial n_\zeta} \left[ \frac{e^{-jkd(X, \xi)}}{d(X, \xi)} \right] + j\omega \rho v(\xi) \frac{e^{-jkd(X, \xi)}}{d(X, \xi)} \right] dS(\xi), \quad x \in R_i, \]  

(2.2.25)
As in the previous case, as $X \rightarrow S$ we obtain equation (2.2.24) for the interior Helmholtz integral formulation.

All the above Fredholm integral equations of the second kind (2.2.22), (2.2.24) and (2.2.25) are solved by converting the equations to a system of algebraic equations. It was shown by Schenck [51] that, there exists certain numbers called the characteristic wave numbers, represented by $k'$, for which the solution seize to exist, when these characteristic wavenumbers include the characteristic number, of the integral equation, that is when $k = k'$. It was also shown by Schenck [51] that unique solution to all the Fredholm integral equations (2.2.22), (2.2.24) and (2.2.25) does not exist for all values of the wavenumber $k$. He further proposed an improved method for solving such integral equations involving parameter/s in the kernel, known as Combined Helmholtz Integral Equation Formulation (CHIEF) [51] in which all the integral equations are solved numerically by cleverly choosing the nodes that would avoid the case $k = k'$.

**Jones's approach for the problem of acoustics:**

Jones [57], stated that such difficulties occur not because the original problem fails to have unique solution, but on account of the representation used. He proposed another modification for the solution of the same physical problem. Jones's [57] modification suggests a systematic way of choosing the interior points for CHIEF which is independent of the knowledge of the nodes of the eigen functions as opposed to CHIEF where the nodes are chosen arbitrarily.

**Our observation:**

Thus it is evident from from the above examples that integral equations either of the first kind or of the second kind, that involve parameter/s in their kernels pose difficulties in terms of existence and uniqueness of solution.

Another problem which gives rise to similar challenges is considered next. A comprehensive study of the problem and that of the existence and uniqueness of the solution is
presented by substantiating it with relevant proofs wherever necessary.

2.2.3 Problems of Water Waves

The mathematical problems of scattering of surface water waves by rigid vertical barriers, studied by Ursell (see [92],[65]), is as described below:

The motion is considered to be two dimensional in which the fluid is assumed to be incompressible and inviscid, and the motion to be irrotational and simple harmonic. The problems are to determine the complex velocity potentials \( \phi_j(x,y), (j = 1, 2) \), with \( i^2 = -1 \) in the two-dimensional cartesian \( xy \) co-ordinates, in the presence of barriers placed at two positions, in the half plane \( y > 0 \), satisfying

\[
\frac{\partial^2 \phi_j}{\partial x^2} + \frac{\partial^2 \phi_j}{\partial y^2} = 0, \quad -\infty < x < \infty , \, y > 0
\]

(2.2.26)

along with the free surface boundary condition

\[
\frac{\partial \phi_j}{\partial y} + K \phi_j = 0, \quad \text{on } y = 0
\]

(2.2.27)

The condition on the barrier is given by

\[
\frac{\partial \phi_j}{\partial x} = 0, \quad \text{on } x = 0\pm, \, y \in L_j = (a_j, b_j)
\]

(2.2.28)

\[
\phi_j(0-, y) = \phi_j(0-, y), \quad \text{for } y \in G_j : (0, \infty) - L_j
\]

(2.2.29)

denotes the homogeneity of the fluid.

Near the submerged edge of the barrier, \( \phi \) behaves as

\[
\frac{\partial \phi_j}{\partial x} \approx o(|y - t_j|^{-1/2}), \quad \text{as } x \to 0, \, y \to t_j
\]

(2.2.30)

with \( (L_j, j = 1, 2, \text{ representing the two barriers}) a_1 = a, b_2 = \infty; a_2 = 0, b_2 = b; \) and \( t_1 = a-, t_2 = b+ \).

The radiation condition is given by

\[
\phi_j \sim (1 - R_j) e^{iKx-Ky}, \quad \text{as } x \to \infty
\]

(2.2.31)

\[
\phi_j \sim e^{iKx-Ky} + R_j e^{-iKx-Ky}, \quad \text{as } x \to -\infty
\]
in which $R_j$'s are unknown constants (called the reflection co-efficients) to be determined which are related to $\phi_j$'s, and

$$\phi_j, |\nabla \phi_j| \to 0, \quad \text{as } y \to \infty$$  \hfill (2.2.32)

denotes the bottom boundary condition.

**Ursell’s Method of Solution**

Ursell [65, 92] expressed the unknown potential $\phi_j(x, y)$ in the following integral forms, which satisfy equations and conditions (2.2.26), (2.2.27), (2.2.31) and (2.2.32) as given in Section 2.2.3.

$$\phi_j = (1 - R_j)e^{iKx - Ky} + \int_0^\infty A_j(\xi)L(\xi, y)e^{-\xi y} \, d\xi, \quad x > 0,$$

$$\phi_j = e^{iKx - Ky} + R_j e^{-iKx - Ky} - \int_0^\infty A_j(\xi)L(\xi, y)e^{+\xi y} \, d\xi, \quad x < 0,$$  \hfill (2.2.33)

with $j = 1, 2$ and $L(\xi, y) = \xi \cos \xi y - K \sin \xi y$, where $A_j$'s, the unknown functions and $R_j$'s, the unknown constants, are determined from the following sets of dual integral equations:

$$\int_0^\infty A_j(\xi)L(\xi, y) \, d\xi = R_j e^{-Ky}, \quad \text{for } y \in G_j,$$

$$\int_0^\infty \xi A_j(\xi)L(\xi, y) \, d\xi = iK(1 - R_j)e^{-Ky}, \quad \text{for } y \in L_j(j = 1, 2).$$  \hfill (2.2.34)

Ursell's [65, 92] method of solution begins by setting

$$\int_0^\infty \xi A_j(\xi)L(\xi, y) \, d\xi = f_j(y) + iK(1 - R_j)e^{-Ky} \quad \forall y \in (0, \infty)(j = 1, 2)$$  \hfill (2.2.35)

where $f(y)$ denotes the unknown velocity function along the vertical line below the origin, such that

$$f(y) = \frac{\partial \phi}{\partial x}(0, y), \quad 0 < y < \infty$$  \hfill (2.2.36)

The integral equation for $f(y)$ is obtained by differentiating both sides of second of relations (2.2.34) and then substituting it in relation (2.2.36), to get

$$f(y) = iK(1 - R)e^{-Ky} - \int_0^\infty \xi A_j(\xi)L(\xi, y) \, d\xi.$$  \hfill (2.2.37)
Using Havelock’s expansion [21] the unknown functions \( A_j(\xi) \) and the unknown constants \( R_j \)'s are determined in terms of the velocity \( f(y) \). Thus the relations are as given below:

\[
A_j(\xi) = -\frac{2}{\pi} \frac{1}{\xi(\xi^2 + K^2)} \int_{G_j} f_j(t) L(\xi, t) \, dt, \quad (2.2.38)
\]

and

\[
R_j = 1 + 2i \int_{G_j} f_j(y)e^{-Ky} \, dy \quad (j = 1, 2). \quad (2.2.39)
\]

Substituting equation (2.2.38) into the first of the dual relations (2.2.34), in the integral equation of the first kind in \( f_j(y) \) is obtained to be

\[
\int_{G_j} f_j(t) M(y, t) \, dt = -R_j e^{-Ky} \text{ for } y \in G_j, \quad (2.2.40)
\]

where

\[
M(y, t) = \frac{2}{\pi} \int_0^\infty \frac{(\xi \cos \xi y - K \sin \xi y)(\xi \cos \xi t - K \sin \xi t) \, d\xi}{\xi(\xi^2 + K^2)}. \quad (2.2.41)
\]

The function \( M(y, t) \) is further simplified as follows:

Let

\[
I = \int_0^\infty \frac{(\xi \cos \xi y - K \sin \xi y)(\xi \cos \xi t - K \sin \xi t) \, d\xi}{\xi(\xi^2 + K^2)}. \quad (2.2.42)
\]

Writing

\[
I = \int_0^\infty \frac{N(\xi, y, t)}{D(\xi)} \, d\xi, \quad (2.2.43)
\]

with

\[
D(\xi) = \xi(\xi^2 + K^2), \quad (2.2.44)
\]
where

\[ N(\xi, y, t) = (\xi \cos \xi y - K \sin \xi y)(\xi \cos \xi t - K \sin \xi t), \]
\[ = \frac{1}{2}(\xi^2 + K^2) \cos \xi (y + t) + \frac{1}{2}(\xi^2 + K^2) \cos \xi |y - t|
- K^2 \cos \xi (y + t) - K \xi \sin (y + t). \] (2.2.45)

Therefore

\[ I = \int_0^\infty \frac{N(\xi, y, t)}{D(\xi)} \, d\xi = \lim_{\epsilon \to 0} \int_0^\infty \frac{e^{-\epsilon \xi} \left[ \cos \xi |y - t| + \cos \xi (y + t) \right]}{\xi} \, d\xi
- \lim_{\epsilon \to 0} \int_0^\infty \frac{K^2 \cos \xi (y + t) + K \xi \sin \xi (y + t)}{\xi(\xi^2 + K^2)} \, d\xi, \] (2.2.46)

\[ = \frac{1}{2} \lim_{\epsilon \to 0} \int_0^\infty \frac{e^{-\epsilon \xi} \left[ \cos \xi |y - t| - \cos \xi (y + t) \right]}{\xi} \, d\xi
+ \lim_{\epsilon \to 0} \int_0^\infty \frac{e^{-\epsilon \xi} \left[ \cos \xi (y + t) - K \xi \sin \xi (y + t) \right]}{\xi(\xi^2 + K^2)} \, d\xi, \] (2.2.47)

\[ = \frac{1}{2} \lim_{\epsilon \to 0} \int_0^\infty e^{-\epsilon \xi} d\xi \int_{|y-t|}^{y+t} \sin(\xi u) \, du
+ \lim_{\epsilon \to 0} \int_0^\infty e^{-\epsilon \xi} \left[ \frac{\xi \cos \xi (y + t) - K \xi \sin \xi (y + t)}{\xi(\xi^2 + K^2)} \right] \, d\xi. \] (2.2.48)

Interchanging orders of integration of the first integral of the above equation (2.2.48) we get

\[ I \equiv \frac{1}{2} \lim_{\epsilon \to 0} \int_{|y-t|}^{y+t} du \int_0^\infty e^{-\epsilon \xi} \sin(\xi u) \, d\xi + J \text{(say)}, \]
\[ = \frac{1}{2} \int_{|y-t|}^{y+t} \frac{du}{u} + J, \] (2.2.49)

\[ = \frac{1}{2} \ln \left| \frac{y + t}{y - t} \right| + J, \] (2.2.50)

where

\[ J = \lim_{\epsilon \to 0} \int_0^\infty e^{-\epsilon \xi} \left[ \frac{\xi \cos \xi (y + t) - K \xi \sin \xi (y + t)}{\xi(\xi^2 + K^2)} \right] \, d\xi. \] (2.2.51)

Noticing that

\[ \frac{dJ}{dy} + KJ = -\lim_{\epsilon \to 0} \int_0^\infty e^{-\epsilon \xi} \sin \xi (y + t) \, d\xi
= -\frac{1}{y + t}. \] (2.2.52)
Solving the ordinary differential equation (2.2.52), gives

\[ J e^{Ky} = - \int_{y}^{y+t} \frac{e^{Ky}}{y+t} dy + C \]

\[ = -e^{Kt} \int_{K(y+t)}^{y} \frac{e^{K(y+t)}}{y+t} dy + C. \quad (2.2.53) \]

As \( y \to -\infty \), \( C' \to 0 \).

Therefore

\[ J = -e^{-K(y+t)} \int_{-\infty}^{K(y+t)} \frac{e^{v}}{v} dv. \quad (2.2.54) \]

Substituting (2.2.54) in (2.2.49), gives

\[ I = \frac{1}{2} \ln \left| \frac{y+t}{y-t} \right| - e^{-K(y+t)} \int_{-\infty}^{K(y+t)} \frac{e^{v}}{v} dv. \quad (2.2.55) \]

Substituting (2.2.55) in equation (2.2.40) and eliminating \( R \) from equation (2.2.40) using equation (2.2.39), gives

\[ \frac{1}{\pi} \int_{G_j} \left[ \ln \left| \frac{y+t}{y-t} \right| - 2e^{-K(y+t)} \int_{-\infty}^{K(y+t)} \frac{e^{v}}{v} dv + 2ie^{-K(y+t)} \right] f_j(t) dt = -e^{-Ky} \]

for \( y \in G_j(j = 1, 2) \).

By operating on both sides of equation (2.2.56) by \( \left( \frac{d}{dy} + K \right) \) and after some assumptions and simplifications, Ursell obtained the solution as

\[ f(u) = \frac{d}{du} \left[ e^{-Ku} \int_{a}^{u} \frac{v}{\sqrt{v^2 - a^2}} dv \right]. \quad (2.2.57) \]

Further, he obtained \( A \) (for \( y \in [a, \infty) \)) as:

\[ A = \int_{a}^{\infty} e^{-Ky} \frac{y}{\sqrt{y^2 - a^2}} dy, \quad (2.2.58) \]

and \( R \) (for \( y \in [a, \infty) \)) as:

\[ R = \frac{\pi I_1(n^2a/g)}{\sqrt{\pi^2F_1(n^2a/g) + K_1^2(n^2a/g)}} \quad (2.2.59) \]
2.3 New Approach

A new approach for solving problems of scattering of surface water waves which reduce to integral equations of the first kind is presented here by converting these integral equations of the first kind to integral equations of the second kind. The advantage of such a conversion is that, integral equations of second kind are fairly easy to handle for solutions and so is the analysis for existence and uniqueness of solutions given the standard theorems of Fredholm [59] in place.

Recalling from section 2.2, the general procedure for conversion of integral equations of the first kind to integral equations of the second kind, proposed by Williams [97], solutions for two cases are worked out here, that is when the barrier is placed at two different positions as given below:

Case 1: when the barrier is given by: \( x = 0, y \in [a, \infty) \) and

Case 2: when the barrier is given by: \( x = 0, y \in [0, b] \).

Thus equation (2.2.26) along with the conditions (2.2.27) to (2.2.30) are handled for solutions for two cases, that is when \( j = 1 \) and \( j = 2 \).

2.3.1 The case \( j = 1 \), for the problem involving \( \phi_1 \), with \( a_1 = a \),

\[ b_1 \to \infty \]

Ursell obtained an integral equation of the first kind for the unknown function \( f_1(y) = \frac{\partial \phi_1}{\partial x} \bigg|_{x=0, y \in G_1=(a, \infty)} \) as

\[ \int_a^\infty f_1(u) \left[ \ln \left| \frac{y+u}{y-u} \right| - 2e^{-K(y+u)} \int_{-\infty}^{K(y+u)} \frac{e^v}{v} dv + 2te^{-K(y+u)} \right] du = -e^{-Ky}, \quad (K > 0) \]

(2.3.1)

along with the functional relation as given by:

\[ R_1 = 1 + 2i \int_a^\infty f_1(y)e^{-Ky} dy. \]

(2.3.2)
The general method of reformulating integral equations of the first kind to integral equations of the second kind, as described in Section (2.2)(see the relations (2.2.1-2.2.5)) is followed here, to reduce the equation (2.3.1) to an integral equation of the second kind. The details are explained below:

Making use of the result

\[
\ln \left| \frac{y + u}{y - u} \right| = 2uy \int_{\max(y,u)}^{\infty} \frac{dv}{v \sqrt{(v^2 - y^2)(v^2 - u^2)}},
\]

(2.3.3)
gives

\[
\int_{a}^{\infty} f_1(u) \left[ \ln \left| \frac{y + u}{y - u} \right| \right] du = \int_{a}^{u} \left[ 2uy \int_{y}^{\infty} \frac{dv}{v \sqrt{(v^2 - y^2)(v^2 - u^2)}} \right] f_1(u) du + \int_{u}^{\infty} \left[ 2uy \int_{a}^{\infty} \frac{dv}{v \sqrt{(v^2 - y^2)(v^2 - u^2)}} \right] f_1(u) du.
\]

(2.3.4)

Interchanging the orders of integration of the above equation, gives

\[
\int_{a}^{\infty} f_1(u) \left[ \ln \left| \frac{y + u}{y - u} \right| \right] du = 2y \int_{y}^{\infty} \frac{dv}{v \sqrt{(v^2 - y^2)}} \int_{a}^{v} \frac{uf_1(u) du}{\sqrt{(v^2 - u^2)}},
\]

(2.3.5)

\[
\int_{a}^{\infty} f_1(u) \left[ \ln \left| \frac{y + u}{y - u} \right| \right] du = 2y \int_{y}^{\infty} \frac{F_1(v) dv}{v \sqrt{(v^2 - y^2)}},
\]

(2.3.6)

where

\[
\int_{a}^{v} \frac{uf_1(u) du}{\sqrt{(v^2 - u^2)}} = F_1(v) \text{ (say)}.
\]

(2.3.7)

Using equation (2.3.6) in equation (2.3.1) and applying Abel inversion ([76],[15]), gives

\[
\frac{F_1(v)}{v} + \frac{1}{\pi} \frac{d}{dv} \left[ \int_{v}^{\infty} \frac{P(y, u)f_1(u) du}{\sqrt{(y^2 - u^2)}} dy \right] = g_1(v),
\]

(2.3.8)

where

\[
g_1(v) = \frac{1}{\pi} \frac{d}{dv} \int_{v}^{\infty} \frac{e^{-Kv} dy}{\sqrt{(y^2 - v^2)}},
\]

(2.3.9)

Equation (2.3.8) is further simplified as

\[
F_1(v) - \frac{2v}{\pi^2} \int_{a}^{\infty} F_1(s) \hat{k}_1(s, v; K) dv = v g_1(v),
\]

(2.3.10)
where

\[
\tilde{k}_1(s, v) = \frac{d}{dv} \left[ \int_v^\infty \frac{\left( \frac{\partial Q}{\partial s} \right) dy}{\sqrt{(y^2 - v^2)}} \right],
\]  
(2.3.11)

with

\[
\frac{\partial Q}{\partial s} = \frac{\partial}{\partial s} \int_s^\infty \frac{P(y, u) du}{\sqrt{(u^2 - s^2)}},
\]  
(2.3.12)

and

\[
P(y, u) = 2e^{-K(y+u)} \int_{-\infty}^{K(y+u)} \frac{e^t}{t} dt - 2ie^{-K(y+u)},
\]  
(2.3.13)

which involves the known parameter \(K\) in the kernel of the Fredholm integral equation (2.3.10), of the second kind.

2.3.2 The case \(j = 2\), for the problem involving \(\phi_2\), with \(a_2 = 0, b_2 = b\)

In this case, for which \(f_2(y) = \frac{\partial \phi}{\partial x} \big|_{x=0, y \in G_2 = (0, b)}\), the corresponding integral equation of the first kind, turns out to be

\[
\int_0^b f_2(u) \left[ \ln \left| \frac{y + u}{y - u} \right| - 2e^{-K(y+u)} \int_{-\infty}^{K(y+u)} \frac{e^v}{v} dv + 2ie^{-K(y+u)} \right] du = -e^{-Ky},
\]  
(2.3.14)

along with the functional relation

\[
R_2 = 1 + 2i \int_0^b f_2(y)e^{-Ky} dy.
\]  
(2.3.15)

Making use of the result

\[
\ln \left| \frac{y + u}{y - u} \right| = 2 \int_0^{\min(y, u)} \frac{\xi d\xi}{\sqrt{(y^2 - \xi^2)(u^2 - \xi^2)}},
\]  
(2.3.16)

in this case, gives

\[
\int_0^b f_2(u) \left[ \ln \left| \frac{y + u}{y - u} \right| \right] du = 2 \int_0^y \frac{\xi F_2(\xi) d\xi}{\sqrt{(y^2 - \xi^2)}},
\]  
(2.3.17)
where
\[
\int_{\xi}^{b} \frac{f_2(u)du}{\sqrt{u^2 - \xi^2}} = F_2(\xi) \text{ (say)},
\] (2.3.18)

Equation (2.3.14) then reduces to,
\[
\int_{0}^{y} \frac{\xi F_2(\xi) d\xi}{\sqrt{(y^2 - \xi^2)}} = -\frac{e^{-Ky}}{2} + \frac{1}{2} \int_{0}^{b} P(y, u) f_2(u) du,
\] (2.3.19)

Applying Abel inversion (see [76],[15]) to the above equation gives
\[
F_2(\xi) = -\frac{1}{\pi \xi} \frac{d}{d\xi} \left[ \int_{0}^{\xi} \frac{ye^{-Ky}dy}{\sqrt{(\xi^2 - y^2)}} \right] + \frac{1}{\pi \xi} \frac{d}{d\xi} \left[ \int_{0}^{\xi} \frac{y[\int_{0}^{b} \frac{P(y, u) f_2(u) du}{\sqrt{(\xi^2 - y^2)}}]}{dy} \right],
\] (2.3.20)

which takes up the form as given by:
\[
F_2(\xi) = \frac{2}{\pi^2 \xi} \int_{0}^{b} \hat{k}_2(\xi, s; K) F_2(s) ds = g_2(\xi),
\] (2.3.21)

where
\[
g_2(\xi) = -\frac{1}{\pi \xi} \frac{d}{d\xi} \left[ \int_{0}^{\xi} \frac{ye^{-Ky}dy}{\sqrt{(\xi^2 - y^2)}} \right],
\] (2.3.22)

\[
\hat{k}_2(\xi, s) = \frac{d}{d\xi} \left[ \int_{0}^{\xi} \frac{y(\frac{\partial Q}{\partial s})dy}{\sqrt{(\xi^2 - y^2)}} \right],
\] (2.3.23)

with
\[
\frac{\partial Q}{\partial s} = \frac{\partial}{\partial s} \int_{0}^{s} \frac{uP(y, u) du}{\sqrt{(s^2 - u^2)}},
\] (2.3.24)

where
\[
P(y, u) = 2e^{-K(y+u)} \int_{-\infty}^{K(y+u)} \frac{e^v dv}{v} - 2ie^{-K(y+u)}.
\] (2.3.25)

Equation (2.3.21) is again a Fredholm integral equation of the second kind which involves the parameter $K$, as in the previous case of the equation (2.3.10).
2.4 Existence and Uniqueness of Solutions of the First Kind Equations

It is observed that there are many physical problems [92, 97, 98] in which the aspects of existence and uniqueness of solutions of the integral equations obtained have not been dealt with. While Schenck [51] and Jones [57] have stressed on this fact there are no explicit proofs available for the the problems dealt with by Williams [97] and Ursell [92]. Given the rather complicated form of kernels in equations (2.2.15), (2.3.10) and (2.3.21) it is difficult to establish proofs satisfying relations (2.1.8) and (2.1.23) for existence and uniqueness of solutions of the integral equations since the kernel involves parameters $b$, $c$ and $K$. However, sufficient proofs are presented here to establish the existence and uniqueness of the solutions for equations obtained by Ursell [92].

Before taking up the study on existence and uniqueness of solution of Ursell's method, a rather simple problem is presented here whose kernel involves a parameter and the existence and uniqueness of solution of this problem is discussed first for a better understanding.

Example:

An integral equation of the first kind as given by

$$
\int_0^a \phi(t) \left[ \ln \left| \frac{x + t}{x - t} \right| + Kxt \right] dt = f(x), \quad (0 < x < a)
$$

(2.4.1)

is considered here, where $K$ is some known real parameter.

The general method of conversion of integral equations of the first kind to integral equations of the second kind as discussed earlier is used here and the steps are as follows:

Making use of the result

$$
\ln \left| \frac{y + u}{y - u} \right| = 2 \int_0^{\min(y,u)} \frac{\xi d\xi}{\sqrt{(y^2 - \xi^2)(u^2 - \xi^2)}},
$$

(2.4.2)

gives
\[ \int_0^x \frac{\xi \varphi(\xi)}{\sqrt{x^2 - \xi^2}} d\xi = \frac{f(x)}{2} - \frac{1}{2} \int_0^a K x t \phi(t) dt, \quad (2.4.3) \]

where

\[ \varphi(\xi) = \int_\xi^a \frac{\phi(t) dt}{\sqrt{t^2 - \xi^2}}, \quad (2.4.4) \]

Applying Abel inversion (see [76],[15]) to the equation (2.4.3) and on further simplification we obtain an integral equation of the second kind involving the parameter \( K \) as given by

\[ \varphi(\xi) - \frac{K}{\xi} \int_0^a s \varphi(s) ds = g(\xi), \quad (2.4.5) \]

where

\[ g(\xi) = \frac{1}{\pi \xi} \frac{d}{d\xi} \left[ \int_0^\xi \frac{xf(x) dx}{\sqrt{\xi^2 - x^2}} \right], \quad (2.4.6) \]

It is easily shown that the integral equation (2.4.5) of the second kind possesses unique solution for all values of \( K \neq \frac{1}{a} \) and, hence, existence of non-unique solution of the original integral equation (2.4.1) is observed, if \( K = \frac{1}{a} \).

### 2.4.1 Uniqueness of Solution using Ursell’s method

The following observations for the integral equation (2.3.1) are made (Similar observations hold good, for the equation(2.3.14)):

Ursell obtained the solution of equation (2.3.1), for all \( K > 0 \), as given by:

\[ f_1(u) = \frac{d}{du} \left[ e^{-Ku} \int_a^u \frac{ve^{-Kv} dv}{\sqrt{v^2 - a^2}} \right] \quad (2.7) \]

Making use of the fact that the solution exists for equation (2.3.1), we have the following relation satisfied, that is

\[ \int_a^\infty e^{-Ky} f_0^T(y) dy = 0, \quad (2.4.8) \]
where $f_0^T$ is the solution of the transposed homogeneous equation

$$\int_{a}^{\infty} f(u) \left[ \ln \left| \frac{u + y}{u - y} \right| - 2e^{-K(u+y)} \int_{-\infty}^{K(u+y)} \frac{e^{-v}}{v} dv - 2ue^{-K(u+y)} \right] du = 0, \quad (K > 0)$$

(2.4.9)

The relation (2.4.8) can be re-written as

$$\int_{0}^{\infty} \hat{f}_0^T(y)e^{-Ky}dy = 0$$

(2.4.10)

where

$$\hat{f}_0(y) = \begin{cases} 0 & 0 \leq y < a, \\ f_0^T(y) & y \geq a, \end{cases}$$

(2.4.11)

Then by using Lerch’s theorem[87], we have that

$$f_0^T(y) = 0 = f_0(y) = 0 \text{ (since the kernel is symmetric)}$$

(2.4.12)

which satisfies the condition (2.1.23), thus establishing the uniqueness of the solution of Ursell’s integral equation (2.3.1) [92]. Similar conclusion holds good for the other integral equation (2.3.21) of Ursell [92].

### 2.4.2 Uniqueness of Solution using the New Approach

From the previous proof it is evident that the solution to the problem of scattering of surface water waves is unique, for all values of the parameter $K$. Therefore, we assume that the solution to this particular problem through the new approach must also be unique. Since the parameter $K$ is real, there exists no real eigen value [15, 59] for which the solution exists. Therefore, the only possibility is that all eigen values or the characteristic numbers must be complex in nature. It is only under this condition that the solution for the water wave problem exists and is unique for all values of the parameter $K$.

However, it is observed that even if unique solutions fail to exist of the integral equations (2.3.10) and (2.3.21), due to the presence of the parameter $K$ in their kernels, the
values of the functionals $R_j(j = 1, 2)$, as given by the relations (2.3.2) and (2.3.15) are unique, which are certain important measures, called the reflection co-efficients of the corresponding scattering problems. This has been proved recently by Chakrbari et all [46], where in they have presented table of reflection co-efficients obtained for different values of the parameter $K$, for different approximate solutions. Further, for a particular value of $K$ irrespective of the method of solution, the reflection co-efficient remains the same.

Nevertheless, the solutions of the integral equations (2.3.10) and (2.3.21) can be obtained by the method of iteration, as described below:

The first two iterations are obtained as follows:

$$F_1^{(0)}(v) = vg_1(v)$$ (2.4.13)

where

$$g_1(v) = \frac{1}{\pi} \frac{d}{dv} \int_v^\infty e^{-K\sqrt{y^2-v^2}} dy$$

$$F_1^{(1)}(v) = \frac{2v}{\pi^2} \int_v^\infty \frac{d}{dv} \left[ \int_v^\infty \frac{\partial Q}{\partial s} dy \right] (sg_1(s)) ds + vg_1(v)$$ (2.4.14)

Similarly the solution of equation (2.3.21), by the method of iteration is obtained as

$$F_2^{(0)}(\xi) = g_2(\xi)$$ (2.4.15)

where

$$g_2(\xi) = \frac{1}{\pi} \frac{d}{d\xi} \int_0^\xi \frac{y e^{-K\sqrt{\xi^2-y^2}}}{\sqrt{\xi^2-y^2}} dy$$

and

$$F_2^{(1)}(\xi) = \frac{2v}{\pi^2} \int_v^\infty \frac{d}{d\xi} \left[ \int_v^\infty \frac{\partial Q}{\partial s} dy \right] (g_2(s)) ds + g_2(\xi)$$ (2.4.16)

It is observed that, in the equations (2.3.10) and (2.3.21) there could exist set of values of $K$ for which $\lambda = \frac{1}{\pi}$ becomes the characteristic value of the corresponding integral equation leading to break down of the solution.

However, table of numerical values of $F_1^0(v)$ and $F_2^0(\xi)$ for some specific values of $Ka$ and $Kb$ are presented in the next section (see table 2.1 and 2.2).
2.5 Table of Numerical Values

Table 2.1: Table of values for $F^{(0)}_1(v)$ and $Ka = 0.1$

<table>
<thead>
<tr>
<th>$v$</th>
<th>$p$</th>
<th>$n$</th>
<th>$\epsilon$</th>
<th>$g_1(v)$</th>
<th>$F^{(0)}_1(v)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>3</td>
<td>5</td>
<td>0.01</td>
<td>6.39462</td>
<td>38.3677</td>
</tr>
<tr>
<td>3</td>
<td>30</td>
<td>0.01</td>
<td>0.03190074</td>
<td>0.1914</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>15</td>
<td>0.01</td>
<td>0.2313828</td>
<td>1.3883</td>
<td></td>
</tr>
</tbody>
</table>

| 8   | 5   | 15  | 0.01      | 1.287236 | 10.2979        |
| 5   | 30  | 0.01| 0.03954739| 0.3164   |
| 5   | 50  | 0.01| 0.005454792| 0.0436 |
| 5   | 20  | 0.01| 0.2982839 | 2.3863   |

Table 2.2: Table of values for $F^{(0)}_2(\zeta)$ and $Kb = 0.1$

<table>
<thead>
<tr>
<th>$\zeta$</th>
<th>$p$</th>
<th>$n$</th>
<th>$\epsilon$</th>
<th>$g_2(\zeta)$</th>
<th>$F^{(0)}_2(\zeta)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3</td>
<td>15</td>
<td>0.01</td>
<td>0.2412964</td>
<td>0.2412964</td>
</tr>
<tr>
<td>0.5</td>
<td>4</td>
<td>20</td>
<td>0.01</td>
<td>0.2533695</td>
<td>0.2533695</td>
</tr>
<tr>
<td>0.2</td>
<td>5</td>
<td>25</td>
<td>0.01</td>
<td>0.2602368</td>
<td>0.2602368</td>
</tr>
<tr>
<td>0.6</td>
<td>6</td>
<td>30</td>
<td>0.01</td>
<td>0.2598554</td>
<td>0.2598554</td>
</tr>
<tr>
<td>0.8</td>
<td>6</td>
<td>35</td>
<td>0.01</td>
<td>0.2613966</td>
<td>0.2613966</td>
</tr>
</tbody>
</table>

2.6 Observations & Inferences

1. Relations (2.4.10) and (2.4.12) together show that the solution (2.4.7) of the integral equation (2.2.56) obtained by Ursell is unique for all values of the parameter $K$. However, the proof presented above is incidental, as it is shown from the fact that the solution already exists. It is observed that there exists many other physical problems which suffer from existence and uniqueness of solution due to the presence of some
parameter/s in the kernel of the corresponding integral equations. Sufficient examples are presented in this regard in the previous sections.

2. Jones (see [57]) stated that such complicacies of existence and uniqueness when faced could be due to the representation used to solve the physical problem. He also stated that various representations lead to non-unique integral equations. Given below is an example:

**Problem:**

To cast the Bessel equation

\[ x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + (n^2 x^2 - 1)y = 0 \]  

(2.6.1)

with the end conditions \( y(0) = 0 = y(1) \) into an integral equation.

**Solution:**

**Type 1:** Rewriting the Bessel equation in the form

\[ x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} - y = -n^2 x^2, \]

(2.6.2)

and determining the Green’s function \( g(x, t) \) satisfying

\[ x^2 \frac{d^2 g}{dx^2} + x \frac{dg}{dx} - g = \delta(x - t), \]

(2.6.3)

for a fixed \( t \in (0, 1) \), with the end conditions

\[ g(0, t) = g(1, t) = 0, \]

(2.6.4)

gives

\[ y(x) = -n^2 \int_0^t t^2 g(x, t)y(t) \, dt, \]

(2.6.5)

which is an integral equation for the determination of \( y(x) \).
Type 2: Suppose the Bessel equation is rewritten as

\[ x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} - p = -(n^2 x^2 + p - 1), \quad (2.6.6) \]

where \( p \) is any constant, then a corresponding Green’s function \( g(x, t) \) satisfying

\[ x^2 \frac{d^2 g}{dx^2} + x \frac{dg}{dx} - p = \delta(x - t), \quad (2.6.7) \]

for a fixed \( t \in (0, 1) \), for the same end conditions

\[ g(0, t) = g(1, t) = 0, \quad (2.6.8) \]

gives

\[ y(x) = p - 1 - n^2 \int_0^t t^2 g(x, t)y(t)dt, \quad (2.6.9) \]

which is a different integral equation from the one obtained in (2.6.5). Similarly a different integral equation is obtained when the problem is cast as:

\[ x^2 \frac{d^2 y}{dx^2} + n^2 x^2 - 1 = -x \frac{dy}{dx}, \quad (2.6.10) \]

Therefore, it is to be noted that several such integral equations can be obtained for different Green functions. Much depends on the way the problem is cast.

3. Though a given physical problem under consideration has a unique solution, the integral equation formulated for the mathematical solution of the problem may not possess a unique solution when the integral equation involves known parameter/s. Under such circumstances, as suggested by Jones [57], one has to look back to the representations used to solve the physical problem and reformulate the problem in terms of appropriate integral equations possessing unique solutions.