CHAPTER 1

1.1 INTRODUCTION

Fixed point theorems give the conditions under which maps \( T x = x \) (single or multivalued) have solutions. The theory itself is a beautiful mixture of Analysis, Topology, and Geometry. Over the last 50 years or so the theory of fixed points has been revealed as a very powerful and important tool in the study of nonlinear phenomena. In particular fixed point techniques have been applied in diverse fields such as Engineering, Game theory and Physics.

The most important result in the fixed point theorem is the Brouwer’s fixed point theorem. It states that every continuous self mapping of the closed unit ball in \( \mathbb{R}^n \), the \( n \)-dimensional Euclidean space possess a fixed point. A more general form of Brouwer’s theorem is for continuous functions from a convex compact subset \( K \) of Euclidean space to itself. This result was published by Brouwer in 1910.

The concept of Banach space was introduced by Stefan Banach and obtained a fixed point theorem for contraction mappings in 1922. Banach fixed point theorem is also known as contraction mapping theorem or contraction mapping principle. It is an important tool in the theory of metric spaces. It guarantees the existence and uniqueness of fixed points of certain self-maps of metric spaces, and provides a constructive method to find those fixed points. Recently there have been numerous generalizations of Banach contraction principle by weakening its hypothesis while retaining the convergence property of the successive iterates to the unique fixed point of the mapping.

The Schauder fixed point theorem is an extension of the Brouwer fixed point theorem to topological vector spaces, which may be of infinite dimension. It asserts that if \( K \) is a convex subset of a topological vector space \( V \) and \( T \) is a continuous mapping of \( K \) into itself so that \( T(K) \) is contained in a compact subset of \( K \), then \( T \) has a fixed point. The theorem was
conjectured and proven for special cases, such as Banach spaces, by Juliusz Schauder in 1930. In 1934, Tychonoff proved the theorem for the case when $K$ is a compact convex subset of a locally convex space. This version is known as the Schauder – Tychonoff fixed point theorem.

In 1964, Krasnoselskii initiated the study of hybrid fixed point theorems in Banach spaces by combining metric fixed point theorem of Banach [5] with the topological fixed point theorem of schauder which is known as the Krasnoselskii fixed point theorem in nonlinear analysis. Schauder introduced the fixed point theorem, and since then several generalizations of this concept have been investigated by various authors, namely, Kirk [35], Baillon [4], Browder [8] and many others. Some more generalizations have been considered by S. A. Hussain [28], V. M. Sehgal, R. Kannan [34], S. B. Nadler [40], Hardy and Rodgers [27]. Recently, fuzzy version of various fixed point theorems were investigated by T. Bag and S. K. Samanta, V. Gregori [26], Romaguera, T. Som and R. N. Mukherjee, A. Chitra and P. V. Subrahmanyam [57] and many others.

Zadeh [72] introduced the concept of fuzzy sets in 1965. He showed successful applications in various fields and laid the foundation of fuzzy mathematics. He used it as a tool for dealing with uncertainty arising out of lack of information about certain complex system. Thus fuzzy set is a collection of objects with membership grade in continuum with each object being assign a value between 0 and 1. The only membership possibilities for an ordinary or crisp subset are non-membership or full membership. Such a set $A$ can thus be identified with the fuzzy set $\chi_A$ is given by the characteristic function $\chi_A : X \rightarrow [0,1]$, where $X$ is a nonempty set.

There are different ways of fuzzy metric spaces. Kramosil and Michalek [36] introduced the concept of fuzzy metric spaces in terms of $t$-norm. Using the concept of $t$-norm, Schweizer and Sklar [49] defined a probabilistic metric space. Kramosil and Michalek
[36] defined fuzzy metric spaces by generalizing the concept of probabilistic metric spaces to fuzzy situation. Kaleva and Seikkala introduced the concept of fuzzy metric spaces where the distance between two points is non-negative, upper semi-continuous, normalized and convex fuzzy number. George and Veeramani [24] modified the concept of fuzzy metric spaces introduced by Kramosil and Michalek [36] and defined the Hausdorff topology of fuzzy metric spaces. Fixed point theorems in fuzzy mathematics are emerging with vigorous hope and vital trust. It appears that Kramosil and Michalek’s study of fuzzy metric spaces paves a way for very soothing machinery to develop fixed point theorems for contractive type maps. Grabiec [25] extended the well known fixed point theorems of Banach [5] and Edelstein [18] to fuzzy metric spaces in the sense of Kramosil and Michalek [36]. Fang [20] proved some fixed point theorems in fuzzy metric spaces which improve and generalize the results of Grabiec [25], also unify and extend some main results of V. M. Sehgal and A. T. Bharucha-Reid [7]. Recently S. Sedghi and N. Shobe [50] introduced the concept of \( \mathcal{M} \)-fuzzy metric spaces which is a generalization of fuzzy metric spaces due to George and Veeramani [30] and proved a common fixed point theorem.
1.2 NOTATIONS, DEFINITIONS AND PRELIMINARIES

In this section, the notations, definitions and preliminaries are listed for the purpose of the future use in the thesis.

\( N \) Set of all natural numbers

\( R \) Set of real numbers

\( \geq \) Greater than or equal to

\( \leq \) Less than or equal to

\( \phi \) Null set or Empty set

\( \in \) Belongs to

\( = \) Equal to

\( \neq \) Not equal to

\( \mathcal{B}(X) \) Set of all non-empty, bounded subsets of \( X \).

\( \{x_n\} \) Sequence \( x_n \)

\( x_n \to x \) \( x_n \) converges to \( x \)

\( B_{r'}(x, r) \) An open ball in a \( D' \) metric space with centre at \( x \) and radius \( r \).

\( n \to \infty \) \( n \) tends to infinity
DEFINITIONS AND PRELIMINARIES

**Definition 1.2.1:** A sequence \( \{x_n\} \) in a metric space \((X, d)\) is said to be convergent to a point \( x \in X \) if given \( \varepsilon > 0 \) there exists a positive integer \( n_0 \) such that \( d(x_n, x) < \varepsilon \) for all \( n \geq n_0 \).

**Definition 1.2.2:** A sequence \( \{x_n\} \) in a metric space \((X, d)\) is said to be a cauchy sequence in \( X \) if given \( \varepsilon > 0 \) there exists a positive integer \( n_0 \) such that \( d(x_m, x_n) < \varepsilon \) for all \( m, n \geq n_0 \).

**Definition 1.2.3:** A metric space \((X, d)\) is said to be complete if every cauchy sequence in \( X \) converges to a point in \( X \).

**Definition 1.2.4:** Let \( X \) be a non-empty set and \( f: X \to X \) be a map. An element \( x \) in \( X \) is called a fixed point of \( X \) if \( f(x) = x \).

**Definition 1.2.5:** Let \( X \) be a non-empty set and \( f, g: X \to X \) be two maps. An element \( x \) in \( X \) is called a common fixed point of \( f \) and \( g \) if \( f(x) = g(x) = x \).

**Definition 1.2.6** [72]: A fuzzy set \( A \) in \( X \) is a function with domain \( X \) and values in \([0,1]\).

**Definition 1.2.7** [24]: A binary operation \( * : [0,1] \times [0,1] \to [0,1] \) is a continuous t-norm if it satisfies the following conditions:

(i) \( * \) is commutative and associative,

(ii) \( * \) is continuous,

(iii) \( a \ast 1 = a \) for all \( a \in [0,1] \),

(iv) \( a \ast b \leq c \ast d \) whenever \( a \leq c \) and \( b \leq d \), for each \( a, b, c, d \in [0,1] \).

Examples of t-norm are \( a \ast b = ab \) and \( a \ast b = \min(a,b) \).
**Definition 1.2.8.** [36] (Kramosil and Michalek): A 3-tuple $(X, M, \ast)$ is said to be a fuzzy metric space if $X$ is an arbitrary set, $\ast$ is a continuous $t$-norm and $M$ is a fuzzy set on $X^2 \times [0, \infty)$ satisfying the following conditions, $\forall x, y, z \in X$ and $t, s > 0$

1. $M(x, y, 0) = 0$,
2. $M(x, y, t) = 1$ for all $t > 0$ and if and only if $x = y$,
3. $M(x, y, t) = M(y, x, t)$,
4. $M(x, y, t) \ast M(y, z, s) \leq M(x, z, t+s)$,
5. $M(x, y, t) : [0, \infty) \to [0,1]$ is left continuous and
6. $\lim_{t \to \infty} M(x, y, t) = 1$.

Then $M$ is called a fuzzy metric on $X$ and $M(x, y, t)$ denotes the degree of nearness between $x$ and $y$ with respect to $t$.

**Example 1.2.9[9]:** Let $X = R$ with the usual metric defined by $d(x, y) = |x - y|$ for all $x, y \in X$. Let $\ast$ be the usual multiplication. Define

$$M(x, y, t) = \begin{cases} \frac{t}{t + |x-y|} & \text{if } x \neq y \text{ and } t \geq 0 \\ 0 & \text{if } x = y \text{ and } t = 0 \end{cases}$$

Then $(X, M, \ast)$ is a fuzzy metric space.

Kaleva and Seikkala (1984) introduced the concept of fuzzy metric spaces where the distance between two points is non-negative, upper semi continuous, normalized and convex fuzzy number.

George and Veeramani (1994) modified the definition of fuzzy metric space given by Kramosil and Michalek (1975) as follows.
**Definition 1.2.10**[24] George and Veeramani: A 3-tuple \((X, M, \ast)\) is said to be a fuzzy metric space if \(X\) is an arbitrary set, \(\ast\) is a continuous \(t\)-norm and \(M\) is a fuzzy set on \(X^2 \times (0, \infty)\) satisfying the following conditions, for all \(x, y, z \in X\) and \(s, t > 0\),

\[
\begin{align*}
(f-1) & \quad M(x, y, t) > 0, \\
(f-2) & \quad M(x, y, t) = 1 \text{ if and only if } x = y, \\
(f-3) & \quad M(x, y, t) = M(y, x, t), \\
(f-4) & \quad M(x, y, t) \ast M(y, z, s) \leq M(x, z, t+s), \\
(f-5) & \quad M(x, y, \cdot) : (0, \infty) \to [0,1] \text{ is continuous } \\
(f-6) & \quad \lim_{t \to \infty} M(x, y, t) = 1.
\end{align*}
\]

Then \(M\) is called a fuzzy metric on \(X\) and \(M(x, y, t)\) denotes the degree of nearness between \(x\) and \(y\) with respect to \(t\).

**Example 1.2.11**[24]: (Induced fuzzy metric): Let \((X, d)\) be a metric space, define \(a \ast b = \min\{a, b\}\) for all \(a, b \in [0,1]\) and \(M_d\) be fuzzy set on \(X^2 \times [0, \infty)\) defined as,

\[
M_d(x, y, t) = \frac{t}{t + d(x, y)} \quad \text{for all } x, y \in X \text{ and } t > 0.
\]

Then \((X, M_d, \ast)\) is a fuzzy metric space. We call this fuzzy metric \(M_d\) induced by the metric \(d\) as the standard intuitionistic fuzzy metric.

**Definition 1.2.12**[13]: Let \((X, M, \ast)\) be a fuzzy metric space. For \(t > 0\), the open ball \(B(x, r, t)\) with centre \(x \in X\) and radius \(0 < r < 1\) is defined by

\[
B(x, r, t) = \{ y \in X / M(x, y, t) > 1 - r \}.
\]

**Definition 1.2.13**[13]: Let \((X, M, \ast)\) be a fuzzy metric space. Then

(i) A sequence \(\{x_n\}\) in \(X\) is said to be convergent to a point \(x \in X\),

\[
\text{if } \lim_{n \to \infty} M(x_n, x, t) = 1 \text{ for all } t > 0.
\]
A sequence \( \{x_n\} \) in \( X \) is said to be a cauchy sequence if
\[
\lim_{n \to \infty} M(x_{n+p}, x_n, t) = 1 \quad \text{for all } t > 0 \text{ and } p > 0.
\]

A fuzzy metric space is said to be complete if every cauchy sequence in \( X \) is convergent to a point in it.

**Remark 1.2.14.** Since \( * \) is continuous, it follows from (f-4), that the limit of the sequence in fuzzy metric space is uniquely determined.

**Lemma 1.2.15[24].** For all \( x, y \in X \), \( M(x, y,.) \) is non-decreasing.

**Lemma 1.2.16[23].** Let \( \{x_n\} \) be a sequence in a fuzzy metric space \( (X, M, \ast) \) with the condition (f-6). If there exists a number \( q \in (0, 1) \) such that
\[
M(x_{n+2}, x_{n+1}, qt) \geq M(x_{n+1}, x_n, t) \quad \text{--- (f-7)}
\]
for all \( t > 0 \) and \( n = 1, 2, \ldots \), then \( \{x_n\} \) is a cauchy sequence in \( X \).

**Lemma 1.2.17.** If for all \( x, y \in X, t > 0 \) and for a number \( q \in (0, 1) \)
\[
M(x, y, qt) \geq M(x, y, t), \text{ then } x = y.
\]

**Definition 1.2.18.** Let \( X \) be a non-empty set and \( f, g: X \to X \) be two maps. An element \( x \) in \( X \) is called a common fixed point of \( f \) and \( g \) if \( f(x) = g(x) = x \).

**Definition 1.2.19.** Let \( (X, d) \) be a complete metric spaces and \( B(X) \) be family of all non-empty bounded subsets of \( X \). The function \( \delta (A, B) \) for \( A, B \) in \( B(X) \) is defined by
\[
\delta (A, B) = \sup \{ d(a,b): a \in A, b \in B \}
\]
and
\[
\delta (A) = \text{diameter}(A)
\]
If \( A \) consists of a single point \( a \) we write \( \delta (A, B) = \delta (a, B) \). If \( B \) also consists of a single point \( b \) we write \( \delta (A, B) = \delta (a, B) = \delta (a, b) \). It follows immediately that \( \delta (A, B) = \delta (B, A) \geq 0 \), and \( \delta (A, B) \leq \delta (A, C) + \delta (C, B) \) for all \( A, B \) in \( B(X) \).

Let \( T \) be a mapping of \( X \) into \( B(X) \). If \( A \) is in \( B(X) \), we define the set \( TA = \bigcup_{a \in A} Ta \).
**Definition 1.2.20** [22]. If \( \{ A_n : n = 1, 2, 3 \ldots \} \) is a sequence of sets in \( \mathcal{B}(X) \), we say that it converges to the closed set \( A \) in \( \mathcal{B}(X) \) if

(i) Each point \( a \) in \( A \) is the limit of some convergent sequence \( \{ a_n \in A_n : n = 1, 2, 3 \ldots \} \),

(ii) For arbitrary \( \epsilon > 0 \), there exists an integer \( N \) such that \( A_n \subseteq A \subseteq \) for \( n > N \) where \( A \subseteq \) is the union of all open spheres with centres in \( A \) and radius \( \epsilon \).

The set \( A \) is then said to be the limit of the sequence \( \{ A_n \} \).

**Definition 1.2.21.** Let \( f \) be a multivalued mapping of \( X \) into \( \mathcal{B}(X) \). \( f \) is continuous at \( x \) in \( X \) if whenever \( \{ x_n \} \) is a sequence of points in \( X \) converging to \( x \), the sequence \( \{ f(x_n) \} \) in \( \mathcal{B}(X) \) converges to \( fx \) in \( \mathcal{B}(X) \). If \( f \) is continuous at each point \( x \) in \( X \), then \( f \) is continuous mapping of \( X \) into \( \mathcal{B}(X) \).

**Definition 1.2.22.** Let \( T \) be a multifunction of \( X \) into \( \mathcal{B}(X) \). \( z \) is a fixed point of \( T \) if \( Tz = \{ z \} \).

**Lemma 1.2.23** [22]. If \( \{ A_n \} \) and \( \{ B_n \} \) are sequences of bounded subsets of a complete metric space \( (X, d) \) which converges to the bounded subsets \( A \) and \( B \), respectively, then the sequence \( \{ \delta(A_n, B_n) \} \) converges to \( \delta(A, B) \).

**Lemma 1.2.24**: Let \( \{ A_n \} \) be a sequence of nonempty subsets of \( X \) and let \( x \) be a point of \( X \) such that

\[
\lim_{n \to \infty} \delta(A_n, x) = 0.
\]

Then the sequence \( \{ A_n \} \) converges to the set \( \{ x \} \).

**Definition 1.2.25.** Let \( (X, M, *) \) be a fuzzy metric space. Let \( \mathcal{B}(X) \) be the set of all non-empty, bounded subsets of \( X \). For every \( t > 0 \), the function \( \delta (A, B, t) \) for \( A, B \) in \( \mathcal{B}(X) \) is defined by

\[
\delta (A, B, t) = \inf \{ M(a, b, t) : a \in A, b \in B \}
\]
If \( A \) consists of a single point \( a \) we write \( \delta(A, B, t) = \delta(a, B, t) \). If \( B \) also consists of a single point \( b \) we write \( \delta(A, B, t) = \delta(a, b, t) \).

It follows immediately that every \( t > 0 \)

\[
\delta(A, B, t) = 1 \quad \text{iff} \quad A = B = \{a\} \quad \text{for all} \quad A, B \in \mathcal{B}(X).
\]

\[
\delta(A, B, t) = \delta(B, A, t) \geq 0
\]

\[
\delta(A, B, t + s) \geq \delta(A, C, t) * \delta(C, B, s)
\]

In particular for \( A = B \neq \emptyset \) in \( \mathcal{B}(X) \), we have \( \delta(A, A, t) = \inf \{ M(a, b, t) : a, b \in A \} \)

**Lemma 1.2.26.** Let \( \{A_n\} \) be a sequence in \( \mathcal{B}(X) \) and \( y \) a point in \( X \) such that \( \delta(A_n, y, t) \to 1 \).

Then the sequence \( \{A_n\} \) converges to the set \( \{y\} \) in \( \mathcal{B}(X) \).

**Definition 1.2.27.** Let \( A, B : X \to \mathcal{B}(X) \) be two multi maps. An element \( x \) in \( X \) is called a common fixed point of \( A \) and \( B \) if \( A(x) = B(x) = \{x\} \).

**Definition: 1.2.28([68])** Let \( X \) be a nonempty set. A \( D' \)-metric (or generalized metric) on \( X \) is a function: \( D' : X^3 \to [0, \infty) \), that satisfies the following conditions for each \( x, y, z, a \in X \)

(i) \( D'(x, y, z) \geq 0 \),

(ii) \( D'(x, y, z) = 0 \quad \text{iff} \quad x = y = z \),

(iii) \( D'(x, y, z) = D'(p(x, y, z)) \), (symmetry) where \( p \) is a permutation function,

(iv) \( D'(x, y, z) \leq D'(x, y, a) + D'(a, z, z) \).

The pair \( (X, D') \) is called a generalized metric (or \( D' \)-metric) space.

Examples of \( D' \)-metric are

(a) \( D'(x, y, z) = \max \{d(x, y), d(y, z), d(z, x)\} \),

(b) \( D'(x, y, z) = d(x, y) + d(y, z) + d(z, x) \).

Here, \( d \) is the ordinary metric on \( X \).
Using generalized metric concepts, Sedghi and Shobe[50] defined $\mathcal{M}$-fuzzy metric space which is a generalization of fuzzy metric space as follows

**Definition 1.2.29 [50]:** A 3-tuple $\langle X, \mathcal{M}, \ast \rangle$ is called a $\mathcal{M}$-fuzzy metric space. If $X$ is an arbitrary non-empty set, $\ast$ is a continuous $t$-norm, and $\mathcal{M}$ is a fuzzy set on $X^3 \times (0,\infty)$, satisfying the following conditions for each $x, y, z, a \in X$ and $t, s > 0$

(FM – 1) $\mathcal{M}(x, y, z, t) > 0$

(FM – 2) $\mathcal{M}(x, y, z, t) = 1$ iff $x = y = z$

(FM – 3) $\mathcal{M}(x, y, z, t) = \mathcal{M}(p \{x, y, z\}, t)$, where $p$ is a permutation function

(FM – 4) $\mathcal{M}(x, y, a, t) \ast \mathcal{M}(a, z, z, s) \leq \mathcal{M}(x, y, z, t+s)$

(FM – 5) $\mathcal{M}(x, y, z, \cdot) : (0,\infty) \to [0,1]$ is continuous

(FM – 6) $\lim_{t \to \infty} \mathcal{M}(x, y, z, t) = 1$.

**Example 1.2.30:** Let $X$ be a nonempty set and $D'$ is the $D'$ - metric on $X$. Denote $a \ast b = a.b$ for all $a, b \in [0,1]$. For each $t \in (0,\infty)$, define

$$\mathcal{M}(x, y, z, t) = \frac{t}{t+D'(x,y,z)}$$

for all $x, y, z \in X$, then $\langle X, \mathcal{M}, \ast \rangle$ is a $\mathcal{M}$- fuzzy metric space.

**Lemma 1.2.31:** Let $\langle X, \mathcal{M}, \ast \rangle$ be a $\mathcal{M}$- fuzzy metric space. Then for every $t > 0$ and for every $x, y \in X$ we have $\mathcal{M}(x, x, y, t) = \mathcal{M}(x, y, y, t)$.

**Proof:**

For each $\epsilon > 0$ by triangular inequality we have

(i) $\mathcal{M}(x, x, \epsilon + t) \geq \mathcal{M}(x, x, \epsilon) \ast \mathcal{M}(x, y, y, t)$

$$= \mathcal{M}(x, y, y, t)$$
(ii) $\mathcal{M}(y, y, x, \in + t) \geq \mathcal{M}(y, y, y, \in) \ast \mathcal{M}(y, x, x, t) = \mathcal{M}(y, x, x, t)$.

By taking limits of (i) and (ii) when $\in \to 0$,

We obtain $\mathcal{M}(x, x, y, t) = \mathcal{M}(x, y, y, t)$.

**Lemma 1.2.32:** Let $(X, \mathcal{M}, \ast)$ be a $\mathcal{M}$-fuzzy metric space. Then $\mathcal{M}(x, y, z, t)$ is non-decreasing with respect to $t$, for all $x, y, z$ in $X$.

**Definition 1.2.33[50]:** Let $(X, \mathcal{M}, \ast)$ be a $\mathcal{M}$-fuzzy metric space. For $t > 0$, the open ball $B_{\mathcal{M}}(x, r, t)$ with center $x \in X$ and radius $0 < r < 1$ is defined by

$B_{\mathcal{M}}(x, r, t) = \{y \in X : \mathcal{M}(x, y, y, t) > 1 - r\}$.

A subset $A$ of $X$ is called open set if for each $x \in A$ there exist $t > 0$ and $0 < r < 1$ such that $B_{\mathcal{M}}(x, r, t) \subseteq A$.

**Definition 1.2.34[50]:** Let $(X, \mathcal{M}, \ast)$ be a $\mathcal{M}$-fuzzy metric space and $\{x_n\}$ be a sequence in $X$

(a) $\{x_n\}$ is said to converge to a point $x \in X$ if $\lim_{n \to \infty} \mathcal{M}(x, x, x_n, t) = 1$ for all $t > 0$

(b) $\{x_n\}$ is said to be a Cauchy sequence if $\lim_{n \to \infty} \mathcal{M}(x_n+p, x_{n+p}, x_n, t) = 1$ for all $t > 0$ and $p > 0$.

**Definition 1.2.35[51]:** A $\mathcal{M}$-fuzzy metric space $(X, \mathcal{M}, \ast)$ is said to be complete if every every cauchy sequence is convergent.

**Definition 1.2.36:** A point $x$ in $X$ is said to be a common fixed point of sequence of maps $T_n : X \to X$ if $T_n(x) = x$ for all $n$.

**Remark 1.2.37:** since $\ast$ is continuous, it follows from (FM-4) that the limit of the sequence is uniquely determined.
Lemma 1.2.38 [51]: Let \( \{x_n\} \) be a sequence in a \( \mathcal{M} \)-fuzzy metric space \((X, \mathcal{M}, \ast)\) with the condition (FM-6). If there exists a number \( q \in (0,1) \) such that
\[
\mathcal{M}(x_n, x_{n+1}, t) \geq \mathcal{M}(x_{n-1}, x_{n-1}, x_n, t/q)
\]
for all \( t > 0 \) and \( n = 1, 2, 3, \ldots \), then \( \{x_n\} \) is a cauchy sequence.

Lemma 1.2.39 [51]: Let \((X, \mathcal{M}, \ast)\) be a \( \mathcal{M} \)-fuzzy metric space with condition (FM-6). If for all \( x, y, z \in X, t > 0 \) with positive number \( q \in (0,1) \) and
\[
\mathcal{M}(x, y, z, qt) \geq \mathcal{M}(x, y, z, t),
\]
then \( x = y = z \).
1.3 SOME RESULTS OF THE THESIS

This thesis consists of five chapters. A brief resume of these chapters are given below.

In chapter 1, we give an introduction to the theory of fixed point theorems, basic definitions, notations and preliminaries needed to complete the thesis.

In chapter 2, we prove some fixed point theorems in two metric spaces, fuzzy metric spaces and $\mathcal{M}$-fuzzy metric spaces. Here we establish some fixed point theorems for contractive type mappings and non-expansive mappings. Some of the results proved in this chapter are listed below.

**Theorem 1.3.1:** Let $(X, d)$ and $(Y, e)$ be complete metric spaces. If $T$ is a mapping from $X$ into $Y$ and $S$ is a mapping from $Y$ into $X$ satisfying the following conditions:

\[
e(Tx, TSy) \leq c_1 \max\{d(x, Sy), e(y, Tx) + e(y, TSy)\}
\]
\[
d(Sy, STx) \leq c_2 \max\{d(x, Sy) + d(x, STx), e(y, Tx)\}
\]

for all $x$ in $X$ and $y$ in $Y$ where $0 \leq c_1 < 1$ and $0 \leq c_2 < 1$, then $ST$ has a unique fixed point $z$ in $X$ and $TS$ has a unique fixed point $w$ in $Y$. Further $Tz = w$ and $Sw = z$.

**Theorem 1.3.2:** Let $(X, d)$ and $(Y, e)$ be complete metric spaces. If $T$ is a mapping from $X$ into $Y$ and $S$ is a mapping from $Y$ into $X$ satisfying the following conditions

\[
e^3(Tx, TSy) \leq c_1 \max\{e(y, Tx)d(x, Sy)e(y, Tx), e(y, Tx)e(y, Tx)e(y, TSy), e(y, TSy)d(x, Sy)e(y, TSy)\}
\]
\[
d^3(Sy, STx) \leq c_2 \max\{d(x, Sy)e(y, Tx)d(x, Sy), d(x, Sy)d(x, Sy)d(x, STx), e(y, Tx)d(x, STx)d(x, STx)\}
\]

for all $x$ in $X$ and $y$ in $Y$ where $0 \leq c_1 < 1$ and $0 \leq c_2 < 1$, then $ST$ has a unique fixed point $z$ in $X$ and $TS$ has a unique fixed point $w$ in $Y$. Further $Tz = w$ and $Sw = z$.

**Theorem 1.3.3:** Let $(X, M_1, *)$ and $(Y, M_2, *)$ be two complete fuzzy metric spaces. If $T$ is a mapping from $X$ into $Y$ and $S$ is a mapping from $Y$ into $X$, satisfying following conditions:
for all \(x \in X \) and \(y \in Y\) where \(q < 1\), then \(ST\) has a fixed point \(z \) in \(X\) and \(TS\) has a fixed point \(w \) in \(Y\). Further \(Tz = w\) and \(Sw = z\).

**Theorem 1.3.4:** Let \((X, M_1, \ast)\) and \((Y, M_2, \ast)\) be two \(M\)-fuzzy metric spaces. If \(T\) is a mapping from \(X\) into \(Y\) and \(S\) is a mapping from \(Y\) into \(X\), such that for all \(x \in X, y \in Y\)

\[
M_2(Tx, TSy, qt) \geq \min\{M_1(x, Sy, t), M_2(y, Tx, t), M_2(y, TSy, t)\}
\]

\[
M_1(Sy, STx, qt) \geq \min\{M_2(y, Tx, t), M_1(x, Sy, t), M_1(x, STx, t)\}
\]

and for any point \(x \in X, y \in Y\), the sequences \(\{(ST)^n(x)\}\) and \(\{(TS)^n(y)\}\) have subsequences converges to \(z\) and \(w\) respectively, then \(ST\) has a unique fixed point \(z\) in \(X\) and \(TS\) has a unique fixed point \(w\) in \(Y\). Further \(Tz = w\) and \(Sw = z\).

**Theorem 1.3.5:** Let \((X, M_1, \ast)\) and \((Y, M_2, \ast)\) be two \(M\)-fuzzy metric spaces and \(T: X \to Y\), \(S: Y \to X\) be two mappings satisfying

\[
M_1(Sy, Sy, STx, t) > \min\{M_1(x, x, Sy, t), M_1(x, x, STx, t), M_2(y, y, Tx, t), M_2(y, y, TSy, t), M_2(Tx, Tx, TSy, t)\}
\]

\[
M_2(Tx, Tx, TSy, t) > \min\{M_2(y, y, Tx, t), M_2(x, x, Sy, t), M_1(x, x, STx, t), M_1(Sy, Sy, STx, t)\}
\]

\(\forall x \in X, y \in Y\) with \(x \neq Sy, y \neq Tx\) and \(\forall t > 0\).

Suppose one of the following is true:

a) \((X, M_1, \ast)\) is sequentially compact, and \(ST\) is continuous on \(X\).

b) \((Y, M_2, \ast)\) is sequentially compact and \(TS\) is continuous on \(Y\).
Then $ST$ has a unique fixed point $z \in X$ and $TS$ has a unique fixed point $w \in Y$. Further $Tz = w$ and $Sw = z$.

In chapter 3, we establish some common fixed point theorems for some generalized contraction mappings in two metric spaces, fuzzy metric spaces and $\mathcal{M}$-fuzzy metric spaces.

Some of the results proved in this chapter are listed below.

**Theorem 1.3.6:** Let $(X, d)$ and $(Y, e)$ be complete metric spaces. Let $A, B$ be mappings of $X$ into $Y$ and $S, T$ be mappings of $Y$ into $X$ satisfying the inequalities.

\[
d(SAx, TBx') \leq c_1 \max \{ d(x,SAx), d(x',TBx'), e(Ax,Bx'), \frac{d(x,TBx')}{2}, \frac{d(SAx,x')}{2} \}
\]

\[
e(BSy, ATy') \leq c_2 \max \{ e(y,BSy), e(y',ATy'), d(Sy,Ty'), \frac{e(y,ATy')}{2}, \frac{e(BSy,y')}{2} \}
\]

for all $x, x'$ in $X$ and $y, y'$ in $Y$ where $0 \leq c_1 < 1$ and $0 \leq c_2 < 1$. If one of the mappings $A, B, S$ and $T$ is continuous, then $SA$ and $TB$ have a unique common fixed point $z$ in $X$ and $BS$ and $AT$ have a unique common fixed point $w$ in $Y$. Further, $Az = Bz = w$ and $Sw = Tw = z$.

**Theorem 1.3.7:** Let $(X, M_1, *)$ and $(Y, M_2, *)$ be two complete fuzzy metric spaces with continuous $t$-norm* defined by $a*b = \min\{a,b\}$ for all $a, b \in [0,1]$. Let $A, B$ be mappings of $X$ into $Y$ and $S, T$ be mappings of $Y$ into $X$ satisfying the inequalities.

\[
M_1(SAx, TBx',qt) \geq \min\{M_1(x,x',t), M_1(x,SAx,t), M_1(x',TBx',t), M_1(x,TBx',2t)*M_1(x',SAx,2t), M_2(Ax,Bx',t)\}
\]

\[
M_2(BSy, ATy',qt) \geq \min\{M_2(y,y',t), M_2(y,BSy,t), M_2(y',ATy',t), M_2(y,ATy',2t)*M_2(y',BSy,2t), M_1(Sy,Ty',t)\}
\]

for all $x, x'$ in $X$ and $y, y'$ in $Y$. If one of the mappings $A, B, S$ and $T$ is continuous, then $SA$ and $TB$ have a unique common fixed point $z$ in $X$ and $BS$ and $AT$ have a unique common fixed point $w$ in $Y$. Further, $Az = Bz = w$ and $Sw = Tw = z$. 
**Theorem 1.3.8:** Let \((X, \mathcal{M}_1, \ast)\) and \((Y, \mathcal{M}_2, \ast)\) be two complete \(\mathcal{M}\)-fuzzy metric spaces. Let \(A, B\) be mappings of \(X\) into \(Y\) and \(S, T\) be mappings of \(Y\) into \(X\) satisfying the inequalities.

\[
4 \mathcal{M}_1(SAx, SAx, TBx', qt) \geq \mathcal{M}_1(x, x, x, t) + \mathcal{M}_1(x, x, SAx, t) + \mathcal{M}_1(x', x', TBx', t) + \\
\left[\mathcal{M}_1(x, x, SAx, t). \mathcal{M}_1(x', x', TBx', t)\right] / \mathcal{M}_1(x, x, x, t)
\]

\[
4 \mathcal{M}_2(BSy, BSy, ATy', qt) \geq \mathcal{M}_2(y, y, y, t) + \mathcal{M}_2(y, y, BSy, t) + \mathcal{M}_2(y', y', ATy') + \\
\left[\mathcal{M}_2(y, y, BSy, t). \mathcal{M}_2(y', y', ATy')\right] / \mathcal{M}_2(y, y, y, t)
\]

for all \(x, x'\) in \(X\) and \(y, y'\) in \(Y\) where \(0 < q < 1\). If one of the mappings \(A, B, S\) and \(T\) is continuous, then \(SA\) and \(TB\) have a unique common fixed point \(z\) in \(X\) and \(BS\) and \(AT\) have a unique common fixed point \(w\) in \(Y\). Further, \(Az = Bz = w\) and \(Sw = Tw = z\).

**Theorem 1.3.9:** Let \((X, \mathcal{M}_1, \ast)\) and \((Y, \mathcal{M}_2, \ast)\) be two complete \(\mathcal{M}\)-fuzzy metric spaces. Let \(A\) and \(B\) be mappings from \(X\) to \(Y\) and \(S\) and \(T\) be mappings from \(Y\) to \(X\) satisfying the following inequalities.

\[
\mathcal{M}(x, x, x', t). \mathcal{M}_1(SAx, SAx, TBx', qt) > \min\{ \mathcal{M}_1(x, x, x', t). \mathcal{M}_1(x', x', TBx', t), \\
\mathcal{M}_1(x, x, SAx, t). \mathcal{M}_1(x, x', TBx', t), \mathcal{M}_2(Ax, Ax, Bx', t). \mathcal{M}_1(x, x', TBx', t), \\
\mathcal{M}_1(x, x', x', t). \mathcal{M}_1(x, x', TBx', t)\}
\]

\[
\mathcal{M}_2(BSy, BSy, ATy, QT) > \min\{ \mathcal{M}_2(y, y, y', t). \mathcal{M}_2(y', y', BSy', t), \\
\mathcal{M}_2(y, y', BSy', t). \mathcal{M}_2(y, y, ATy, t), \mathcal{M}_2(y', y', BSy', t). \mathcal{M}_2(Sy', Sy', Ty, t), \\
\mathcal{M}_2(y, y', t). \mathcal{M}_2(y, y, ATy, t)\}
\]

for all \(x, x'\) in \(X\) and \(y, y'\) in \(Y\) and \(0 < q < 1\). If one of the mappings \(A, B, S\) and \(T\) is continuous then \(SA\) and \(TB\) have a common fixed point \(z\) in \(X\) and \(BS\) and \(AT\) have a common fixed point \(w\) in \(Y\). Further \(Az = Bz = w\) and \(Sw = Tw = z\).
**Theorem 1.3.10:** Let \((X, \mathcal{M}_1, *)\) and \((Y, \mathcal{M}_2, *)\) be two complete \(\mathcal{M}\)-fuzzy metric spaces. If \(A, B, C\) be three mappings of \(X\) to \(Y\) and \(T, S, R\) be three mappings of \(Y\) to \(X\) such that satisfies the following conditions:

\[
\mathcal{M}_1(SA x, TB x', RC x'', qt) \geq \min \{ (\mathcal{M}_1(x, x', x'', t), \mathcal{M}_2(Ax, Bx', Cx'', t)) \}
\]

\[
\mathcal{M}_2(CT y, AR y', BS y'', qt) \geq \min \{ \mathcal{M}_2(y, y', y'', t), \mathcal{M}_1(Ty, Ry', Sy'', t) \}
\]

for every \(x, x', x'' \in X\) and \(y, y', y'' \in Y\) where \(q < 1\). If one of the mappings \(A, B, C, T, S\) and \(R\) is continuous, then there exist a unique common fixed point \(z\) in \(X\) and \(w\) in \(Y\), such that \(SA z = TB z = RC z = z\) and \(AR w = BS w = CT w = w\). Moreover, \(Sw = Tw = Rw = z\) and \(Az = Bz = Cz = w\).

In chapter 4, we prove some fixed point theorems and common fixed point theorems for multivalued mappings in two metric spaces, fuzzy metric spaces and \(\mathcal{M}\)-fuzzy metric spaces. Some of the results proved in this chapter are listed below.

**Theorem 1.3.11:** Let \((X,d)\) and \((Y,e)\) be compact metric spaces. If \(T\) is a continuous mapping of \(X\) into \(B(Y)\) and \(S\) is a continuous mapping of \(Y\) into \(B(X)\) satisfying the inequalities

\[
\delta_1(ST x, ST x') < \max \{ \delta_1(x, x'), \delta_1(x, ST x), \delta_1(x', ST x'), \delta_2(Tx, Tx') \}
\]

\[
\delta_2(TS y, TS y') < \max \{ \delta_2(y, y'), \delta_2(y, TS y), \delta_2(y', TS y'), \delta_1(Sy, Sy') \}
\]

for all distinct \(x, x' \in X\) and \(y, y' \in Y\), then \(ST\) has a fixed point \(z\) in \(X\) and \(TS\) has a fixed point \(w\) in \(Y\). Further \(Tz = \{w\}\) and \(Sw = \{z\}\).

**Theorem 1.3.12:** Let \((X, M_1, *)\) and \((Y, M_2, *)\) be two complete fuzzy metric spaces. Let \(A, B\) be mappings of \(X\) into \(B(Y)\) and \(S, T\) be mappings of \(Y\) into \(B(X)\) satisfying the inequalities.

\[
\delta_1(SA x, TB x', qt) \geq \min \{ \delta_1(x, x', t), \delta_2(Ax, Bx', t) \}
\]

\[
\delta_2(BS y, AT y', qt) \geq \min \{ \delta_2(y, y', t), \delta_2(Sy, Ty', t) \}
\]
for all \( x, x' \) in \( X \) and \( y, y' \) in \( Y \) where \( 0 < q < 1 \). If one of the mappings \( A, B, S \) and \( T \) is continuous, then \( SA \) and \( TB \) have a unique common fixed point \( z \) in \( X \) and \( BS \) and \( AT \) have a unique common fixed point \( w \) in \( Y \). Further, \( Az = Bz = \{w\} \) and \( Sw = T w = \{z\} \).

**Theorem 1.3.13:** Let \((X, M, *)\) and \((Y, M', *)\) be two complete \( M \)-fuzzy metric spaces. If \( T \) is a continuous mapping of \( X \) into \( B(Y) \) and \( S \) is a continuous mapping of \( Y \) into \( B(X) \) satisfying the inequalities.

\[
\delta_M(STx, STx, STx', qt) \geq \min \{ \delta_M(x, x, STx, t), \delta_M(x', x', STx', t), \delta_M(Tx, Tx, Tx', t) \}
\]

\[
\delta_{M'}(TSy, TSy, TSy', qt) \geq \min \{ \delta_{M'}(y, y, TSy, t), \delta_{M'}(y', y', TSy', t), \delta_{M'}(Sy, Sy, Sy', t) \}
\]

for all \( x, x' \) in \( X \) and \( y, y' \) in \( Y \), where \( 0 \leq q < 1 \), then \( ST \) has a fixed point \( z \) in \( X \) and \( TS \) has a fixed point \( w \) in \( Y \). Further \( Tz = \{w\} \) and \( Sw = \{z\} \).

**Theorem 1.3.14:** Let \((X, M, *)\) and \((Y, M', *)\) be two complete \( M \)-fuzzy metric spaces. Let \( A, B \) be mappings of \( X \) into \( B(Y) \) and \( S, T \) be mappings of \( Y \) into \( B(X) \) satisfying the inequalities.

\[
\delta_M(SAx, SAx, TBx', qt) \geq \min \{ \delta_M(x, x, x', t), \delta_M(x, x, SAx, t), \delta_M(x', x', Bx', t), \delta_M(Ax, Ax, Bx', t) \}
\]

\[
\delta_{M'}(BSy, BSy, ATy', qt) \geq \min \{ \delta_{M'}(y, y, y', t), \delta_{M'}(y, y, BSy, t), \delta_{M'}(y', y', ATy', t), \delta_{M'}(Sy, Sy, Ty', t) \}
\]

for all \( x, x' \) in \( X \) and \( y, y' \) in \( Y \) where \( 0 \leq q < 1 \). If one of the mappings \( A, B, S \) and \( T \) is continuous, then \( SA \) and \( TB \) have common fixed point \( z \) in \( X \) and \( BS \) and \( AT \) have common fixed point \( w \) in \( Y \). Further, \( Az = Bz = \{w\} \) and \( Sw = Tw = \{z\} \).

In chapter 5, we prove some common fixed point theorems for sequence of mappings in two metric spaces, fuzzy metric spaces and \( M \)-fuzzy metric spaces. Some of the results proved in this chapter are listed below.
Theorem 1.3.15: Let \((X, d)\) and \((Y, e)\) be complete metric spaces. If \(\{T_n\} \ (n \in \mathbb{N})\) be a sequence of mappings from \(X\) into \(Y\) and \(\{S_n\} \ (n \in \mathbb{N})\) be a sequence of mappings from \(Y\) into \(X\) satisfying the following conditions

\[
e^2(T_ix, T_is_jy) \leq \max\{d(x, S_jy)e(y, T_ix), d(x, S_jy)e(y, T_is_jy), d(x, S_jy)e(y, T_is_jy)\}
\]

\[
d^2(S_jy, S_jT_ix) \leq \max\{e(y, T_ix)d(x, S_jy), e(y, T_ix)d(x, S_jT_ix), d(x, S_jy)d(x, S_jT_ix)\}
\]

for all \(i \neq j\), \(x\) in \(X\) and \(y\) in \(Y\) where \(0 \leq c_1 < 1\) and \(0 \leq c_2 < 1\), then \(\{S_nT_n\}\) have a unique fixed point \(z\) in \(X\) and \(\{T_nS_n\}\) have a unique fixed point \(w\) in \(Y\). Further, \(\{T_n\}z = w\) and \(\{S_n\}w = z\).

Theorem 1.3.16: Let \((X, M, \ast)\) and \((Y, M, \ast)\) be two complete \(M\) - fuzzy metric spaces. If \(T_i\) is a mapping from \(X\) into \(Y\) and \(S_j\) is a mapping from \(Y\) into \(X\) satisfying

\[
2 M_i(S_jy, S_jy, S_jT_ix, qt) \geq M_i(x, x, S_jT_ix, t) + M_2(y, y, T_ix, t)
\]

\[
2 M_2(T_ix, T_ix, T_is_jy, qt) \geq M_2(y, y, T_is_jy, t) + M_1(x, x, S_jy, t)
\]

for all \(i \neq j\) in \(\mathbb{N}\), \(x\) in \(X\) and \(y\) in \(Y\) where \(q < 1\), then \(\{S_nT_n\}\) has a unique fixed point \(z\) in \(X\) and \(\{T_nS_n\}\) has a unique fixed point \(w\) in \(Y\). Further \(\{T_n\}z = w\) and \(\{S_n\}w = z\).

Theorem 1.3.17: Let \((X, M, \ast)\) and \((Y, M, \ast)\) be two complete \(M\) - fuzzy metric spaces with continuous t-norm \(\ast\) defined by \(a \ast b = \min\{a, b\}\) for all \(a, b \in [0,1]\). If \(T_i\) is a mapping from \(X\) into \(Y\) and \(S_j\) is a mapping from \(Y\) into \(X\) satisfying

\[
M_i(S_jy, S_jy, S_jT_ix, qt) \geq M_i(x, x, S_jT_ix, t) \ast M_i(y, y, T_ix, t)
\]

\[
M_2(T_ix, T_ix, T_is_jy, qt) \geq M_2(y, y, T_is_jy, t) \ast M_1(x, x, S_jy, t)
\]

for all \(i \neq j\) in \(\mathbb{N}\), \(x\) in \(X\) and \(y\) in \(Y\) where \(q < 1\), then \(\{S_nT_n\}\) has a unique fixed point \(z\) in \(X\) and \(\{T_nS_n\}\) has a unique fixed point \(w\) in \(Y\). Further \(\{T_n\}z = w\) and \(\{S_n\}w = z\).