CHAPTER 5
SOME COMMON FIXED POINT THEOREMS FOR
SEQUENCE OF MAPPINGS

5.1 INTRODUCTION
In this chapter we establish some common fixed point theorems for a sequence of
contraction and generalized contraction mappings in two metric spaces, fuzzy metric spaces
and \( H \)-fuzzy metric spaces. The content of this chapter is published in the International
Journal of Mathematics trends and Technology.

5.2. SOME COMMON FIXED POINT THEOREMS FOR SEQUENCE OF
MAPPINGS IN TWO METRIC SPACES

Theorem 5.2.1: Let \((X, d)\) and \((Y, e)\) be complete metric spaces. If \(\{T_n\} (n \in \mathbb{N})\) be a
sequence of mappings from \(X\) into \(Y\) and \(\{S_n\} (n \in \mathbb{N})\) be a sequence of mappings from \(Y\)
into \(X\) satisfying the following conditions:
\[
e^2(T_i x, T_j y) \leq c_1 \max\{d(x, S_j y) e(y, T_i x), d(x, S_j y) e(y, T_j y), e(y, T_i x) e(y, T_j y)\} \quad \text{--- (5.2.1)}
\]
\[
d^2(S_j y, S_j T_i x) \leq c_2 \max\{e(y, T_i x) d(x, S_j y), e(y, T_i x) d(x, S_j T_i x), d(x, S_j y) d(x, S_j T_i x)\} \quad \text{--- (5.2.2)}
\]
for all \(i \neq j, x \text{ in } X\) and \(y \text{ in } Y\) where \(0 \leq c_1 < 1\) and \(0 \leq c_2 < 1\), then \(\{S_n T_n\}\) have a unique
common fixed point \(z\) in \(X\) and \(\{T_n S_n\}\) have a unique common fixed point \(w\) in \(Y\). Further,
\(\{T_n\} z = w\) and \(\{S_n\} w = z\).

Proof. Let \(x_0\) be an arbitrary point in \(X\). Define a sequence \(\{x_n\}\) in \(X\) and a sequence \(\{y_n\}\) in
\(Y\), as follows:
\[x_n = (S_n T_n)^n x_0, \quad y_n = T_n(x_{n-1}) \text{ for } n = 1, 2, \ldots .\]
We have using inequality (5.2.1) we have
\[ d^2(x_n, x_{n+1}) = d((S_n T_n)^n x_0, (S_n T_n)^{n+1} x_0) \]
\[ = d(S_n (S_n T_n)^{n-1} x_0, S_n T_n x_0) \]
\[ = d(S_n T_n(x_{n-1}), S_n T_n x_n) \]
\[ = d(S_n y_n, S_n T_n x_n) \]
\[ \leq c_2 \max \{ e(y_n, T_n x_n) d(x_n, S_n T_n x_n), e(y_n, T_n x_{n-1}) d(x_n, S_n T_n x_n), e(y_n, S_n y_n) d(x_n, S_n T_n x_n), e(y_n, x_0) d(x_n, x_{n-1}) \} \]
\[ = c_2 \max \{ 0, e(y_n, y_{n+1}) d(x_n, x_{n+1}), 0 \} \]
\[ \leq c_2 e(y_n, y_{n+1}) d(x_n, x_{n+1}) \]
Which implies \( d(x_n, x_{n+1}) \leq c_2 e(y_n, y_{n+1}) \) \[ \text{----- (5.2.3)} \]

Now using inequality (5.2.2) we have
\[ e^2(y_n, y_{n+1}) = e^2(T_n x_n, T_n x_n) \]
\[ = e^2(T_n x_n, T_n S_n y_n) \]
\[ \leq c_1 \max \{ d(x_n, S_n y_n) e(y_n, T_n x_n), d(x_{n-1}, S_n y_n) e(y_n, T_n x_{n-1}), e(y_n, T_n x_{n-1}) e(y_n, T_n x_n) \} \]
\[ = c_1 \max \{ d(x_n, x_n) e(y_n, y_n), d(x_{n-1}, x_n) e(y_n, y_{n+1}), e(y_n, y_{n+1}) e(y_n, y_{n+1}) \} \]
\[ \leq c_1 e(y_n, y_{n+1}) \]
Which implies \( e(y_n, y_{n+1}) \leq c_1 d(x_n, x_n) \) \[ \text{-------- (5.2.4)} \]

Hence using inequalities (5.2.3) and (5.2.4) we have
\[ d(x_n, x_{n+1}) \leq c_1 c_2 d(x_{n-1}, x_n) \]
\[ \vdots \]
\[ \leq (c_1 c_2)^n d(x_0, x_1) \rightarrow 0 \text{ as } n \rightarrow \infty \] (since \( 0 \leq c_1 c_2 < 1 \))

Thus \( \{x_n\} \) is a Cauchy sequence in \((X, d)\). Since \((X, d)\) is complete, it converges to a point \(z\) in \(X\). Similarly, we can prove that the sequence \( \{y_n\} \) is also a Cauchy sequence in \((Y, e)\). Since \((Y, e)\) is complete, it converges to a point \(w\) in \(Y\).

Now we prove \(T_n z = w\).
Suppose \( T_n z \neq w \)

We have

\[
e^2(T_n z, w) = \lim_{n \to \infty} e^2(T_n z, y_{n+1})
\]

\[
= \lim_{n \to \infty} e^2(T_n z, T_n S_n y_n)
\]

\[
\leq \lim_{n \to \infty} c_1 \max \{d(z, S_n y_n) e(y_n, T_n z), d(z, T_n y_n) e(y_n, T_n z) e(y_n, T_n S_n y_n)\}
\]

\[
= \lim_{n \to \infty} c_1 \max \{d(z, x_n) e(y_n, T_n z), d(z, T_n x_n) e(y_n, y_{n+1}), e(y_n, T_n z) e(y_n, y_{n+1})\}
\]

\[
\leq 0
\]

So that \( e(T_n z, w) = 0 \)

Thus \( T_n z = w \).

To prove that \( S_n w = z \).

Suppose that \( S_n w \neq z \).

\[
d^2(S_n w, z) = \lim_{n \to \infty} d^2(S_n w, x_{n+1})
\]

\[
= \lim_{n \to \infty} d^2(S_n w, S_n T_n x_n)
\]

\[
\leq \lim_{n \to \infty} c_2 \max \{e(w, T_n x_n) d(x_n, S_n w), e(w, S_n T_n x_n) d(x_n, S_n w) d(x_n, S_n T_n x_n)\}
\]

\[
= \lim_{n \to \infty} c_2 \max \{e(w, y_{n+1}) d(x_n, S_n w), e(w, y_{n+1}) d(x_n, S_n w) d(x_n, x_{n+1})\}
\]

\[
\leq 0
\]

Thus \( S_n w = z \).

We have \( S_n T_n z = S_n w = z \) and \( T_n S_n w = T_n z = w \).

Thus the point \( z \) is a fixed point of \( \{S_n T_n\} \) in \( X \) and the point \( w \) is a fixed point of \( \{T_n S_n\} \) in \( Y \).

**Uniqueness:** Let \( z' \neq z \) be another fixed point of \( S_n T_n \) in \( X \).

We have

\[
d^2(z, z') = d(S_n w, S_n T_n z')
\]

\[
\leq c_2 \max \{e(w, T_n z') d(z', S_n w), e(w, T_n z') d(z', S_n T_n z'), d(z', S_n w) d(z', S_n T_n z')\}
\]
\[ = c_2 \max \{ e(w, Tnz')d(z', z), e(w, Tnz')d(z', z'), d(z', z)d(z', z') \} \]
\[ \leq c_2 e(w, Tnz')d(z', z) \]

Which implies \( d(z, z') \leq c_2 e(w, Tnz') \)

Now
\[ e^2(Tnz', w) = e(Tnz', TnS_n w) \]
\[ \leq c_1 \max \{ d(z', S_n w)e(w, Tnz'), d(z', S_n w)e(z', TnS_n z'), e(w, Tnz')e(z', TnS_n z') \} \]
\[ \leq c_1 d(z', z)e(w, Tnz') \]

Which implies \( e(Tnz', w) \leq c_1 d(z', z) \)

Hence
\[ d(z, z') \leq c_1 c_2 d(z, z') < d(z, z') \] (since \( c_1 c_2 < 1 \)), which is a contradiction.

Thus \( z = z' \).

So the point \( z \) is a unique common fixed point of \( \{ S_n T_n \} \). Similarly, we prove the point \( w \) is also a unique common point of \( \{ T_n S_n \} \). This completes the proof.

**Corollary 5.2.2:** Let \((X, d)\) and \((Y, e)\) be complete metric spaces. If \( T \) is a mapping from \( X \) into \( Y \) and \( S \) is a mapping from \( Y \) into \( X \) satisfying the following conditions
\[ e^2(Tx, TSy) \leq c_1 \max \{ d(x, Sy)e(y, Tx), d(x, Sy)d(y, TSy), e(y, Tx)e(y, TSy) \} \]
\[ d^2(Sy, STx) \leq c_2 \max \{ e(y, Tx)d(x, Sy), e(y, Tx)d(x, STx), d(x, Sy)d(x, STx) \} \]
for all \( x \) in \( X \) and \( y \) in \( Y \) where \( 0 \leq c_1 < 1 \) and \( 0 \leq c_2 < 1 \), then \( ST \) have a unique fixed point \( z \) in \( X \) and \( TS \) have a unique fixed point \( w \) in \( Y \). Further, \( Tz = w \) and \( Sw = z \).

**Remark 5.2.3.** If \((X, d)\) and \((Y, e)\) are the same metric spaces, then by the above theorem 5.2.1, we get the following theorem as corollary.

**Corollary 5.2.4.** Let \((X, d)\) be a complete metric space. If \( S \) and \( T \) are mappings from \( X \) into itself satisfying the following conditions:
\[ d^2(Tx, TSy) \leq c_1 \max \{ d(x, Sy)d(y, Tx), d(x, Sy)d(y, TSy), d(y, Tx)d(y, TSy) \} \]
\[ d^2(Sy, STx) \leq c_2 \max \{ d(y, Tx)d(x, Sy), d(y, Tx)d(x, STx), d(x, Sy)d(x, STx) \} \]
for all \( x, y \) in \( X \) where \( 0 \leq c_1, c_2 < 1 \), then \( ST \) has a unique fixed point \( z \) in \( X \) and \( TS \) has a unique fixed point \( w \) in \( X \). Further, \( Tz = w \) and \( Sw = z \) and if \( z = w \), then \( z \) is the unique common fixed point of \( S \) and \( T \).

**Theorem 5.2.5:** Let \((X, d)\) and \((Y, e)\) be complete metric spaces. Let \( \{A_n\}, \{B_n\} (n \in \mathbb{N}) \) be sequence of mappings of \( X \) into \( Y \) and \( \{S_n\}, \{T_n\} (n \in \mathbb{N}) \) be sequence of mappings of \( Y \) into \( X \) satisfying the inequalities.

\[
d(S_n A_n x, T_n B_n x') \leq c_1 \max\{d(x, x'), d(x, S_n A_n x), d(x', T_n B_n x'), e(A_n x, B_n x'), d(x, T_n B_n x'), d(S_n A_n x, x')\}
\]

--- (5.2.5)

\[
e(B_n S_n y, A_n T_n y') \leq c_2 \max\{e(y, y'), e(y, B_n S_n y), e(y', A_n T_n y'), d(S_n y, T_n y'), e(y, A_n T_n y'), e(B_n S_n y, y')\}
\]

--- (5.2.6)

for all \( i \neq j \neq p \neq q, x, x' \) in \( X \) and \( y, y' \) in \( Y \) where \( 0 \leq c_1 < 1 \) and \( 0 \leq c_2 < 1 \). If one of the mappings \( \{A_n\}, \{B_n\}, \{S_n\} \) and \( \{T_n\} \) is continuous, then \( \{S_n A_n\} \) and \( \{T_n B_n\} \) have a common fixed point \( z \) in \( X \) and \( \{B_n S_n\} \) and \( \{A_n T_n\} \) have a common fixed point \( w \) in \( Y \). Further, \( \{A_n\} z = \{B_n\} z = w \) and \( \{S_n\} w = \{T_n\} w = z \).

**Proof:** Let \( x_0 \) be an arbitrary point in \( X \) and we define the sequences \( \{x_n\} \) in \( X \) and \( \{y_n\} \) in \( Y \) by

\[
A_n x_{2n-1} = y_{2n-1}, S_n y_{2n-1} = x_{2n-1}, B_n x_{2n-1} = y_{2n}, T_n y_{2n} = x_{2n} \quad \text{for } n = 1, 2, 3 \ldots
\]

Now using inequality (5.2.5) we have

\[
d(x_{2n+1}, x_{2n}) = d(S_n A_n x_{2n}, T_n B_n x_{2n-1})
\]

\[
\leq c_1 \max\{d(x_{2n}, x_{2n-1}), d(x_{2n}, S_n A_n x_{2n}), d(x_{2n-1}, T_n B_n x_{2n}), e(A_n x_{2n}, B_n x_{2n-1})\,
\]

\[
\quad d(x_{2n}, T_n B_n x_{2n-1}), d(S_n A_n x_{2n}, x_{2n-1})\}\}
\]

\[
= c_1 \max\{d(x_{2n}, x_{2n-1}), d(x_{2n}, x_{2n+1}), d(x_{2n-1}, x_{2n}), e(y_{2n+1}, y_{2n}), d(x_{2n}, x_{2n}), d(x_{2n-1}, x_{2n+1})\}
\]

\[
= c_1 \max\{d(x_{2n}, x_{2n-1}), d(x_{2n}, x_{2n+1}), d(x_{2n-1}, x_{2n}), e(y_{2n+1}, y_{2n}), 0\}
\]

\[
\leq c_1 \max\{d(x_{2n-1}, x_{2n}), e(y_{2n+1}, y_{2n})\}\}
\]

--- (5.2.7)
Now using inequality (5.2.6) we have
\[ e(y_{2n}, y_{2n+1}) = e(B_nS_ny_{2n-1}, A_nT_ny_{2n}) \]
\[ \leq c_2 \max\{ e(y_{2n-1}, y_{2n}), e(y_{2n-1}, B_nS_ny_{2n-1}), e(y_{2n}, A_nT_ny_{2n}), d(S_ny_{2n-1}, T_ny_{2n}), e(y_{2n-1}, A_nT_ny_{2n}), e(B_nS_ny_{2n-1}, y_{2n}) \} \]
\[ = c_2 \max\{ e(y_{2n-1}, y_{2n}), e(y_{2n-1}, y_{2n}), e(y_{2n}, y_{2n+1}), d(x_{2n-1}, x_{2n}), e(y_{2n-1}, y_{2n+1}), e(y_{2n}, y_{2n}) \} \]
\[ \leq c_2 \max\{ e(y_{2n-1}, y_{2n}), d(x_{2n-1}, x_{2n}) \} \quad (5.2.8) \]

Similarly using inequalities (5.2.5) and (5.2.6), we have
\[ d(x_{2n}, x_{2n-1}) \leq c_1 \max\{ d(x_{2n-2}, x_{2n-1}), e(y_{2n-1}, y_{2n}) \} \quad (5.2.9) \]
\[ e(y_{2n}, y_{2n-1}) \leq c_2 \max\{ e(y_{2n-1}, y_{2n-2}), d(x_{2n-1}, x_{2n-2}) \} \quad (5.2.10) \]

From inequalities (5.2.7) and (5.2.9), we have
\[ d(x_{2n+1}, x_n) \leq (c_1c_2)^{n-1} \max\{ d(x_1, x_2), e(y_1, y_2) \} \to 0 \quad \text{as } n \to \infty \]

Thus \{x_n\} is a Cauchy sequence in \((X, d)\). Similarly using inequalities (5.2.8) and (5.2.10), we have \{y_n\} is a Cauchy sequences in \((Y, e)\). Since \((X, d)\) and \((Y, e)\) are complete, \{x_n\} converges to a point \(z\) in \(X\) and \{y_n\} converges to a point \(w\) in \(Y\).

Suppose \{\(A_n\)\} is continuous, then
\[ \lim_{n \to \infty} A_nx_{2n} = A_nz = \lim_{n \to \infty} y_{2n+1} = w. \]

Now we prove \(S_nA_nz = z\).

Suppose \(S_nA_nz \neq z\).

We have
\[ d(S_nA_nz, z) = \lim_{n \to \infty} d(S_nA_nz, T_nB_nx_{2n-1}) \]
\[ \leq \lim_{n \to \infty} c_1 \max\{ d(z, x_{2n-1}), d(z, S_nA_nz), d(x_{2n-1}, T_nB_nx_{2n-1}), e(A_nz, B_nx_{2n-1}), d(z, T_nB_nx_{2n-1}), d(S_nA_nz, x_{2n-1}) \} \]
\[ \leq \lim_{n \to \infty} c_1 \max\{ d(z, x_{2n-1}), d(z, S_nA_nz), d(x_{2n-1}, T_nB_nx_{2n-1}), e(A_nz, B_nx_{2n-1}), d(z, x_{2n}), d(S_nA_nz, x_{2n-1}) \} \]
\[ < d(z, S_nA_nz) \quad (\text{Since } 0 \leq c_1 < 1), \text{ which is a contradiction.} \]
Thus $S_nA_nz = z$.

Hence $S_nw = z$. (Since $A_nz = w$)

Now we prove $B_nS_nw = w$.

Suppose $B_nS_nw \neq w$.

We have

\[
ed(B_nS_nw, w) = \lim_{n \to \infty} e(B_nS_nw, A_nT_ny_{2n}) \leq \lim_{n \to \infty} c_2 \max\{ e(w, y_{2n}), e(w, B_nS_nw), e(y_{2m}, A_nT_ny_{2n}), d(S_nw, T_ny_{2n}), e(w, A_nT_ny_{2n}) \} \cdot e(B_nS_nw, y_{2n})
\]

\[
\leq \max\{ e(w, w), e(w, B_nS_nw), e(w, w), d(z, z), e(w, w) \} < e(w, B_nS_nw) \quad (\text{Since } 0 \leq c_2 < 1),
\]

which is a contradiction.

Thus $B_nS_nw = w$.

Hence $B_nz = w$. (Since $S_nw = z$)

Similarly we prove $T_nB_nz = z$ and $A_nT_nw = w$.

Hence $T_nw = z$ (Since $B_nz = w$)

The same results hold if one of the mappings $\{B_n\}, \{S_n\}$ and $\{T_n\}$ is continuous. So the point $z$ is the common fixed point of $\{S_nA_n\}$ and $\{T_nB_n\}$. Similarly we prove $w$ is a common fixed point of $\{B_nS_n\}$ and $\{A_nT_n\}$.

**Theorem 5.2.6:** Let $(X, d)$ and $(Y, e)$ be complete metric spaces. Let $\{A_n\}, \{B_n\}$ $(n \in \mathbb{N})$ be sequence of mappings of $X$ into $Y$ and $\{S_n\}, \{T_n\}$ $(n \in \mathbb{N})$ be sequence of mappings of $Y$ into $X$ satisfying the inequalities.

\[
d(S_nA_nx, T_nB_nx') \leq c_1 \max\{d(x, x'), d(x, S_nA_nx), e(A_nx, B_nx'), d(x, T_nB_nx')/2, d(S_nA_nx, x')/2, d(x, S_nA_nx).d(x', T_nB_nx')/ d(x, x') \} \quad (5.2.11)
\]

\[
e(B_nS_ny, A_nT_ny') \leq c_2 \max\{ e(y, y'), e(y, B_nS_ny), d(S_ny, T_ny'), e(y, A_nT_ny')/2, e(B_nS_ny, y')/2 \}
\]

\[
e(y, B_nS_ny).e(y', A_nT_ny') / e(y, y') \quad (5.2.12)
\]
for all \( i \neq j \neq p \neq q, \) \( x, x' \) in \( X \) and \( y, y' \) in \( Y \) where \( 0 \leq c_1 < 1 \) and \( 0 \leq c_2 < 1 \). If one of the mappings \( \{ A_n \}, \{ B_n \}, \{ S_n \} \) and \( \{ T_n \} \) is continuous, then \( \{ S_nA_n \} \) and \( \{ T_nB_n \} \) have a common fixed point \( z \) in \( X \) and \( \{ B_nS_n \} \) and \( \{ A_nT_n \} \) have a common fixed point \( w \) in \( Y \). Further, \( \{ A_n \} z = \{ B_n \} z = w \) and \( \{ S_n \} w = \{ T_n \} w = z \).

Proof: Let \( x_0 \) be an arbitrary point in \( X \) and we define the sequences \( \{ x_n \} \) in \( X \) and \( \{ y_n \} \) in \( Y \) by

\[
A_n x_{2n} = y_{2n-1}, \quad S_n y_{2n-1} = x_{2n-1}, \quad B_n x_{2n-1} = y_{2n}, \quad T_n y_{2n} = x_{2n}
\]

for \( n = 1, 2, 3 \ldots \).

Now using inequality (5.2.11), we have

\[
d(x_{2n+1}, x_{2n}) = d(S_nA_n x_{2n}, T_nB_n x_{2n})
\]

\[
\leq c_1 \max \{ d(x_{2n}, x_{2n-1}), d(x_{2n}, S_nA_n x_{2n}), e(A_n x_{2n}, B_n x_{2n-1}), d(x_{2n}, T_nB_n x_{2n-1}), \}
\]

\[
d(x_{2n}, S_n x_{2n}), d(x_{2n}, S_n x_{2n}), \} / 2 \}
\]

\[
= c_1 \max \{ d(x_{2n}, x_{2n-1}), d(x_{2n}, x_{2n-1}), e(y_{2n+1}, y_{2n}), d(x_{2n}, x_{2n}) / 2, \}
\]

\[
d(x_{2n-1}, x_{2n+1}) / 2, d(x_{2n-1}, x_{2n}), d(x_{2n}, x_{2n-1}) / d(x_{2n}, x_{2n-1}) \}
\]

\[
\leq c_1 \max \{ d(x_{2n}, x_{2n}), e(y_{2n+1}, y_{2n}) \} \quad \text{--- (5.2.13)}
\]

Now

\[
e(y_{2n}, y_{2n+1}) = e(B_nS_n y_{2n-1}, A_nT_n y_{2n})
\]

\[
\leq c_2 \max \{ e(y_{2n-1}, y_{2n}), e(y_{2n-1}, B_nS_n y_{2n-1}), d(S_n y_{2n-1}, y_{2n}), e(y_{2n-1}, A_nT_n y_{2n}), e(B_n, y_{2n-1}), e(B_nS_n y_{2n-1}), e(y_{2n}, A_nT_n y_{2n}) / e(y_{2n-1}), \}
\]

\[
= c_2 \max \{ e(y_{2n-1}, y_{2n}), e(y_{2n-1}, y_{2n}), d(x_{2n-1}, x_{2n}), e(y_{2n-1}, y_{2n}) / 2, \}
\]

\[
e(y_{2n}, y_{2n}) / 2, e(y_{2n-1}, y_{2n}), e(y_{2n}, y_{2n+1}) / e(y_{2n-1}, y_{2n}) \}
\]

\[
\leq c_2 \max \{ e(y_{2n-1}, y_{2n}), d(x_{2n-1}, x_{2n}) \} \quad \text{--- (5.2.14)}
\]

Similarly,

\[
d(x_{2n}, x_{2n-1}) \leq c_1 \max \{ d(x_{2n-2}, x_{2n-1}), e(y_{2n-1}, y_{2n}) \} \quad \text{--- (5.2.15)}
\]

\[
e(y_{2n}, y_{2n-1}) \leq c_2 \max \{ e(y_{2n-1}, y_{2n}), d(x_{2n-1}, x_{2n}) \} \quad \text{--- (5.2.16)}
\]
from inequalities (5.2.13) and (5.2.15), we have
\[ d(x_{n+1}, x_n) \leq (c_1 c_2)^{n-1} \max\{ d(x_1, x_0), e(y_1, y_2)\} \to 0 \text{ as } n \to \infty \]

Thus \( \{x_n\} \) is a Cauchy sequence in \((X, d)\). Since \((X, d)\) is complete, \( \{x_n\} \) converges to a point \( z \) in \( X \). Similarly using inequalities (5.2.14) and (5.2.16), we prove \( \{y_n\} \) is a Cauchy sequence in \((Y, e)\) with the limit \( w \) in \( Y \).

Suppose \( \{A_n\} \) is continuous, then
\[ \lim_{n \to \infty} A_n x_{2n} = A_n z = \lim_{n \to \infty} y_{2n+1} = w. \]

Now we prove \( S_n A_n z = z \).

Suppose \( S_n A_n z \neq z \).

We have
\[
d(S_n A_n z, z) = \lim_{n \to \infty} d(S_n A_n z, T_n B_n x_{2n-1}) \]
\[
\leq \lim_{n \to \infty} c_1 \max\{ d(z, x_{2n-1}), d(z, S_n A_n z), e(A_n z, B_n x_{2n-1}), d(z, T_n B_n x_{2n-1}) / 2, \]
\[
d(S_n A_n z, x_{2n-1}) / 2, d(z, S_n A_n z).d(x_{2n-1}, T_n B_n x_{2n-1}) / d(z, x_{2n-1})\} \]
\[
\leq \lim_{n \to \infty} c_1 \max\{ d(z, x_{2n-1}), d(z, S_n A_n z), e(A_n z, y_{2n}), d(z, x_{2n}) / 2, \]
\[
d(S_n A_n z, x_{2n-1}) / 2, d(z, S_n A_n z).d(x_{2n-1}, x_{2n}) / d(z, x_{2n-1})\} \]
\[
=c_1. \max\{ d(z, z), d(z, S_n A_n z), e(w, w), d(z, z)/2, d(S_n A_n z, z) /2, d(z, z), d(S_n A_n z, z) / d(z, z)\} \]
\[
\leq c_1. d(z, S_n A_n z) \]
\[
< d(z, S_n A_n z) \quad (Since \ 0 \leq c_1 < 1), \text{ which is a contradiction.} \]

Thus \( S_n A_n z = z \).

Hence \( S_n w = z \). (Since \( A_n z = w \))

Now we prove \( B_n S_n w = w \).

Suppose \( B_n S_n w \neq w \).

We have
\[
e(B_n S_n w, w) = \lim_{n \to \infty} e(B_n S_n w, y_{2n+1}) \]
\[
= \lim_{n \to \infty} e(B_n S_n w, A_n T_n y_{2n}) \]
\[
\leq \lim_{n \to \infty} c_2 \max\{ e(w, y_{2n}), e(w, B_n S_n w), d(S_n w, T_n y_{2n}), e(w, A_n T_n y_{2n}) / 2, \]
\]

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\[ e(B_n S_n w, y_{2n}) / 2, e(w, B_n S_n w), e(y_{2n}, A_n T_n y_{2n}) / e(w, y_{2n}) \]

\[ < e(w, B_n S_n w) \quad (\text{Since } 0 \leq c_2 < 1) \]

Which is a contradiction.

Thus \( B_n S_n w = w \).

Hence \( B_n z = w \) (Since \( S_n w = z \))

Similarly we prove \( T_n B_n z = z \) and \( A_n T_n w = w \).

Hence \( T_n w = z \) (Since \( B_n z = w \))

The same results hold if one of the mappings \( \{B_n\}, \{S_n\} \) and \( \{T_n\} \) is continuous.

So the point \( z \) is the common fixed point of \( \{S_n A_n\} \) and \( \{T_n B_n\} \). Similarly we prove \( w \) is a common fixed point of \( \{B_n S_n\} \) and \( \{A_n T_n\} \).

**Uniqueness:** Let \( z' \) be another common fixed point of \( \{S_n A_n\} \) and \( \{T_n B_n\} \) in \( X \), \( w' \) be another common fixed point of \( \{B_n S_n\} \) and \( \{A_n T_n\} \) in \( Y \).

We have \( d(z, z') = d(S_n A_n z, T_n B_n z') \)

\[ \leq c_1 \cdot \max \{ d(z, z'), d(z, S_n A_n z), e(A_n z, B_n z'), d(z, T_n B_n z') / 2, d(S_n A_n z, z') / 2, \]

\[ d(z, S_n A_n z), d(z', T_n B_n z') / d(z, z') \} \]

\[ = c_1 \cdot \max \{ d(z, z'), d(z, z), e(w, w'), d(z, z') / 2, d(z, z') / 2, d(z, z).d(z', z') / d(z, z') \} \]

\[ < e(w, w') \]

\[ e(w, w') = e(B_n S_n w, A_n T_n w') \]

\[ \leq c_2 \cdot \max \{ e(w, w'), e(w, B_n S_n w), e(S_n w, T_n w'), e(w, A_n T_n w') / 2, \]

\[ e(B_n S_n w, w') / 2, e(w, B_n S_n w).e(w', A_n T_n w') / e(w, w') \} \]

\[ < d(z, z') \]

Hence \( d(z, z') < e(w, w') < d(z, z') \), which is a contradiction.

Thus \( z = z' \).

Thus the point \( z \) is the unique common fixed point of \( \{S_n A_n\} \) and \( \{T_n B_n\} \). Similarly we prove \( w \) is a unique common fixed point of \( \{B_n S_n\} \) and \( \{A_n T_n\} \).
5.3 SOME COMMON FIXED POINT THEOREMS FOR SEQUENCE OF

MAPPINGS IN TWO FUZZY METRIC SPACES

Theorem 5.3.1: Let \((X, M_i, *)\) and \((Y, M_j, *)\) be two complete fuzzy metric spaces with continuous \(t\)-norm* defined by \(a * b = \min\{a, b\}\) for all \(a, b \in [0,1]\). Let \(A_i, B_j\) be mappings of \(X\) into \(Y\) and \(S_p, T_q\) be mappings of \(Y\) into \(X\) satisfying the inequalities.

\[
M_1(S_p A_i x, T_q B_j x', qt) \geq \min\{M_1(x, x'), M_1(x, S_p A_i x, t), M_1(x', T_q B_j x', t), M_1(x, T_q B_j x', 2t), M_1(x, T_q B_j x', 2t)\} \quad (5.3.1)
\]

\[
M_2(B_j S_p y, A_i T_q y', qt) \geq \min\{M_2(y, y'), M_2(y, B_j S_p y, t), M_2(y', A_i T_q y', t), M_2(y, A_i T_q y', 2t), M_2(y, A_i T_q y', 2t)\} \quad (5.3.2)
\]

for all \(x, x'\) in \(X\) and \(y, y'\) in \(Y\). If one of the mappings \(\{A_n\}, \{B_n\}, \{S_n\}\) and \(\{T_n\}\) is continuous, then \(\{S_n A_n\}\) and \(\{T_n B_n\}\) have a unique common fixed point \(z\) in \(X\) and \(\{B_n S_n\}\) and \(\{A_n T_n\}\) have a unique common fixed point \(w\) in \(Y\). Further, \(\{A_n\} z = \{B_n\} w = w\) and \(\{S_n\} w = \{T_n\} w = z\).

Proof: Let \(x_0\) be an arbitrary point in \(X\) and we define the sequences \(\{x_n\}\) in \(X\) and \(\{y_n\}\) in \(Y\) by

\[
A_n x_{2n-2} = y_{2n-1}, S_n y_{2n-1} = x_{2n-1}, B_n x_{2n-1} = y_{2n}, T_n y_{2n} = x_{2n} \quad \text{for} \ n = 1, 2, 3 \ldots
\]

Now using inequality (5.3.1), we have

\[
M_1(x_{2n+1}, x_{2n}, qt) = M_1(S_n A_n x_{2n}, T_n B_n x_{2n-1}, qt)
\]

\[
\geq \min\{M_1(x_{2n}, x_{2n+1}, t), M_1(x_{2n}, S_n A_n x_{2n}, t), M_1(x_{2n+1}, T_n B_n x_{2n-1}, t), M_1(x_{2n}, T_n B_n x_{2n-1}, 2t), M_1(x_{2n}, T_n B_n x_{2n-1}, 2t) * M_1(x_{2n}, S_n A_n x_{2n}, 2t), M_2(B_n S_n y_{2n}, T_n y_{2n-1}, t)\} \quad (5.3.3)
\]

\[
\geq \min\{M_1(x_{2n}, x_{2n+1}, t), M_1(x_{2n}, x_{2n+1}, t), M_1(x_{2n+1}, x_{2n}, t), M_1(x_{2n}, x_{2n}, 2t), M_1(x_{2n}, x_{2n}, 2t) * M_1(x_{2n+1}, x_{2n+1}, 2t), M_2(y_{2n}, y_{2n} t)\} \quad (5.3.4)
\]

\[
= \min\{M_1(x_{2n}, x_{2n+1}, t), M_1(x_{2n}, x_{2n+1}, t), M_1(x_{2n}, x_{2n}, t), M_1(x_{2n}, x_{2n}, t), M_1(x_{2n}, x_{2n+1}, 2t), M_2(y_{2n}, y_{2n} t)\} \quad (5.3.5)
\]

\[
= \min\{M_1(x_{2n}, x_{2n+1}, t), M_1(x_{2n}, x_{2n+1}, t), M_1(x_{2n}, x_{2n}, t), M_1(x_{2n}, x_{2n}, t), l, M_1(x_{2n}, x_{2n+1}, 2t), M_2(y_{2n}, y_{2n} t)\} \quad (5.3.6)
\]
\[ M_2(y_{2n+1}, y_{2n}t) \geq M_2(B_n S_{2n} y_{2n-1}, t) \]

\[ M_2(y_{2n+1}, y_{2n}t) \geq M_2(y_{2n}, y_{2n}t, t), M_2(y_{2n}, y_{2n+1}, t), M_2(y_{2n+1}, y_{2n}t) \] --- (5.3.3)

Now

\[ M_2(y_{2n+1}, y_{2n}t) = M_2(y_{2n}, y_{2n}t, t) \]

\[ M_2(y_{2n+1}, y_{2n}t) \geq M_2(y_{2n}, y_{2n}t, t), M_2(y_{2n}, y_{2n+1}, t), M_2(y_{2n+1}, y_{2n}t) \]

\[ M_2(y_{2n+1}, y_{2n}t) \geq M_2(y_{2n}, y_{2n}t, t), M_2(y_{2n}, y_{2n+1}, t), M_2(y_{2n+1}, y_{2n}t) \] --- (5.3.4)

Similarly we have

\[ M_1(x_{2n}, x_{2n-1}, t) \geq \min\{ M_1(x_{2n}, x_{2n-1}, t), M_2(y_{2n}, y_{2n}t, t) \} \] --- (5.3.5)

\[ M_2(y_{2n}, y_{2n-1}, t) \geq \min\{ M_2(y_{2n}, y_{2n-1}, t), M_2(x_{2n-1}, x_{2n-1}, t) \} \] --- (5.3.6)

from inequalities (5.3.3) and (5.3.5), we have
\[ M_1(x_{n+1}, x_n, q) \geq \min \{ M_1(x_n, x_{n-1}, t), M_2(y_{n+1}, y_n, t/q) \} \]
\[
\vdots \geq \min \{ M_1(x_1, x_0, t/q^{n-1}), M_2(y_2, y_1, t/q^{n-1}) \} \rightarrow 1 \quad \text{as } n \rightarrow \infty
\]

Thus \( \{x_n\} \) is a Cauchy sequence in \( X \). Since \( (X, M_1, *) \) is complete, it converges to a point \( z \) in \( X \). Similarly using inequalities (5.3.4) and (5.3.6), we can prove that the sequence \( \{y_n\} \) is a Cauchy sequence in \( Y \) and it converges to a point \( w \) in \( Y \).

Suppose \( A \) is continuous, then \( \lim_{n \to \infty} A_n x_{2n} = A_n z = y_{2n+1} = w \).

Now we prove \( S_n A_n z = z \).

Suppose \( S_n A_n z \neq z \).

We have
\[
M_1(S_n A_n z, z, q) = \lim_{n \to \infty} M_1(S_n A_n z, T_n B_n x_{2n-1}, q t)
\]
\[
\geq \lim_{n \to \infty} \min \{ M_1(z, x_{2n-1}, t), M_1(z, S_n A_n z, t), M_1(x_{2n-1}, T_n B_n x_{2n-1}, t), M_1(z, T_n B_n x_{2n-1}, 2t),
\]
\[\quad M_1(z, T_n B_n x_{2n-1}, 2t) \cdot M_1(x_{2n-1}, S_n A_n z, 2t), \ M_2(A_n z, B_n x_{2n-1}, t) \}\n\[
\leq \lim_{n \to \infty} \min \{ M_1(z, z, t), M_1(z, S_n A_n z, t), M_1(x_{2n-1}, x_{2n}, t), M_1(z, x_{2n}, 2t),
\]
\[\quad M_1(z, x_{2n}, 2t) \cdot M_1(x_{2n-1}, S_n A_n z, 2t), \ M_2(z, y_{2n}, t) \}\n\[
\geq \min \{ 1, \ M_1(z, S_n A_n z, t), 1, 1, M_1(z, S_n A_n z, 2t), 1 \} \]
\[\geq M_1(z, S_n A_n z, t) \quad (\text{since } q < 1) \quad \text{which is a contradiction.}
\]

Thus \( S_n A_n z = z \).

Hence \( S_n w = z \). (Since \( A_n z = w \))

Now we prove \( B_n S_n w = w \).

Suppose \( B_n S_n w \neq w \).

We have
\[
M_2(B_n S_n w, w, q t) = \lim_{n \to \infty} M_2(B_n S_n w, y_{2n+1}, q t)
\]
\[
\geq \lim_{n \to \infty} \min \{ M_2(w, y_{2n}, t), M_2(w, B_n S_n w, t), M_2(y_{2n}, A_n T_n y_{2n}, t), M_2(w, A_n T_n y_{2n}, 2t),
\]
\[
\geq \lim_{n \to \infty} \min \{ M_2(w, y_{2n}, t), M_2(w, B_n S_n w, t), M_2(y_{2n}, A_n T_n y_{2n}, t), M_2(w, A_n T_n y_{2n}, 2t),
\]
\[
\geq \lim_{n \to \infty} \min \{ M_2(w, y_{2n}, t), M_2(w, B_n S_n w, t), M_2(y_{2n}, A_n T_n y_{2n}, t), M_2(w, A_n T_n y_{2n}, 2t),
\]
\[
\geq \lim_{n \to \infty} \min \{ M_2(w, y_{2n}, t), M_2(w, B_n S_n w, t), M_2(y_{2n}, A_n T_n y_{2n}, t), M_2(w, A_n T_n y_{2n}, 2t),
\]
\[
\geq \lim_{n \to \infty} \min \{ M_2(w, y_{2n}, t), M_2(w, B_n S_n w, t), M_2(y_{2n}, A_n T_n y_{2n}, t), M_2(w, A_n T_n y_{2n}, 2t),
\]
\[
\geq \lim_{n \to \infty} \min \{ M_2(w, y_{2n}, t), M_2(w, B_n S_n w, t), M_2(y_{2n}, A_n T_n y_{2n}, t), M_2(w, A_n T_n y_{2n}, 2t),
\]
\[
\geq \lim_{n \to \infty} \min \{ M_2(w, y_{2n}, t), M_2(w, B_n S_n w, t), M_2(y_{2n}, A_n T_n y_{2n}, t), M_2(w, A_n T_n y_{2n}, 2t),
\]
\[
\geq \lim_{n \to \infty} \min \{ M_2(w, y_{2n}, t), M_2(w, B_n S_n w, t), M_2(y_{2n}, A_n T_n y_{2n}, t), M_2(w, A_n T_n y_{2n}, 2t),
\]
\[
\geq \lim_{n \to \infty} \min \{ M_2(w, y_{2n}, t), M_2(w, B_n S_n w, t), M_2(y_{2n}, A_n T_n y_{2n}, t), M_2(w, A_n T_n y_{2n}, 2t),
\]
\[
\geq \lim_{n \to \infty} \min \{ M_2(w, y_{2n}, t), M_2(w, B_n S_n w, t), M_2(y_{2n}, A_n T_n y_{2n}, t), M_2(w, A_n T_n y_{2n}, 2t),
\]
\[
\geq \lim_{n \to \infty} \min \{ M_2(w, y_{2n}, t), M_2(w, B_n S_n w, t), M_2(y_{2n}, A_n T_n y_{2n}, t), M_2(w, A_n T_n y_{2n}, 2t),
\]
\[
\geq \lim_{n \to \infty} \min \{ M_2(w, y_{2n}, t), M_2(w, B_n S_n w, t), M_2(y_{2n}, A_n T_n y_{2n}, t), M_2(w, A_n T_n y_{2n}, 2t),
\]
\[
\geq \lim_{n \to \infty} \min \{ M_2(w, y_{2n}, t), M_2(w, B_n S_n w, t), M_2(y_{2n}, A_n T_n y_{2n}, t), M_2(w, A_n T_n y_{2n}, 2t),
\]
\[
\geq \lim_{n \to \infty} \min \{ M_2(w, y_{2n}, t), M_2(w, B_n S_n w, t), M_2(y_{2n}, A_n T_n y_{2n}, t), M_2(w, A_n T_n y_{2n}, 2t),
\]
\[
\geq \lim_{n \to \infty} \min \{ M_2(w, y_{2n}, t), M_2(w, B_n S_n w, t), M_2(y_{2n}, A_n T_n y_{2n}, t), M_2(w, A_n T_n y_{2n}, 2t),
\]
\[
\geq \lim_{n \to \infty} \min \{ M_2(w, y_{2n}, t), M_2(w, B_n S_n w, t), M_2(y_{2n}, A_n T_n y_{2n}, t), M_2(w, A_n T_n y_{2n}, 2t),
\]
\[ M_2(w, A_nT_ny_{2n}, 2t) \ast M_2(y_{2n}, B_nS_nw, 2t), \quad M_1(S_nw, T_ny_{2n}, t) \]
\[ = \min\{1, M_2(w, B_nS_nw, t), 1, 1, M_2(w, B_nS_nw, 2t), 1\} > M_2(w, B_nS_nw, t) \quad (\text{Since } q < 1), \text{ which is a contradiction.} \]

Thus \( B_nS_nw = w \).

Hence \( B_nz = w \). (Since \( S_nw = z \))

Similarly we prove \( T_nB_nz = z \) and \( A_nT_nw = w \).

Hence \( T_nw = z \). (Since \( B_nz = w \))

The same results hold if one of the mappings \( B_n, S_n \) and \( T_n \) is continuous.

**Uniqueness:** Let \( z' \) be another common fixed point of \( S_nA_n \) and \( T_nB_n \) in \( X \), \( w' \) be another common fixed point of \( B_nS_n \) and \( A_nT_n \) in \( Y \).

We have
\[ M_1(z, z', qt) = M_1(S_nA_nz, T_nB_nz', qt) \]
\[ \geq \min\{M_1(z, z', t), M_1(z, S_nA_nz, t), M_1(z', T_nB_nz', t), M_1(z, T_nB_nz', 2t), M_1(z, T_nB_nz', 2t) \ast M_1(z', S_nA_nz, 2t), M_2(A_nz, B_nz', t)\} \]
\[ = \min\{M_1(z, z', t), 1, 1, M_1(z, z', 2t), M_1(z, z', 2t) \ast M_1(z', z, 2t), M_2(w, w', t)\} \]
\[ > M_2(w, w', t) \quad (\text{Since } q < 1) \]

\[ M_2(w, w', qt) = M_2(B_nS_nw, A_nT_nw', qt) \]
\[ \geq \min\{M_2(w, w', t), M_2(w, B_nS_nw, t), M_2(w, A_nT_nw', t), M_2(w, A_nT_nw', 2t), M_2(w, A_nT_nw', 2t) \ast M_2(w', B_nS_nw, 2t), M_1(S_nw, T_nw', t)\} \]
\[ = \min\{M_2(w, w', t), 1, 1, M_2(w, w', 2t), M_2(w, w', 2t) \ast M_2(w', w, 2t), M_1(z, z', t)\} \]
\[ > M_1(z, z', t) \quad (\text{Since } q < 1) \]

Hence \( M_1(z, z', qt) > M_1(w, w', t) > M_1(z, z', t/q) \)

Which is a contradiction.

Thus \( z = z' \).

So the point \( z \) is the unique common fixed point of \( \{S_nA_n\} \) and \( \{T_nB_n\} \). Similarly we prove \( w \) is a unique common fixed point of \( \{B_nS_n\} \) and \( \{A_nT_n\} \).
5.4 SOME COMMON FIXED POINT THEOREMS FOR SEQUENCE OF

MAPPINGS IN TWO $\mathcal{M}$-FUZZY METRIC SPACES

Theorem 5.4.1: Let $(X, \mathcal{M}_1, *)$ and $(Y, \mathcal{M}_2, *)$ be two complete $\mathcal{M}$-fuzzy metric spaces. If $T_i$ is a mapping from $X$ into $Y$ and $S_j$ is a mapping from $Y$ into $X$ satisfying

\[
2 \mathcal{M}_1 (S_j y, S_j y, S_j T_i x, q t) \geq \mathcal{M}_1 (x, x, S_j T_i x, t) + \mathcal{M}_2 (y, y, T_i x, t) \quad (5.4.1)
\]

\[
2 \mathcal{M}_2 (T_i x, T_i x, T_i S_j y, q t) \geq \mathcal{M}_2 (y, y, T_i S_j y, t) + \mathcal{M}_1 (x, x, S_j y, t) \quad (5.4.2)
\]

for all $i \neq j$ in $N$, $x$ in $X$ and $y$ in $Y$ where $q < 1$, then $\{S_n T_n \}$ has a unique common fixed point $z$ in $X$ and $\{T_n S_n \}$ has a unique common fixed point $w$ in $Y$. Further $\{T_n z = w$ and $\{S_n w = z$.

Proof. Let $x_0$ be an arbitrary point in $X$. Define two sequences $\{x_n\}$ and $\{y_n\}$ in $X$ and $Y$, respectively, as follows:

\[
x_n = (S_n T_n)^n x_0, \quad y_n = T_n(x_{n-1}) \quad \text{for} \quad n = 1, 2, \ldots
\]

By (5.4.1) we have

\[
2 \mathcal{M}_1 (x_n, x_n, x_{n+1}, q t) \geq 2 \mathcal{M}_1 ((S_n T_n)^n x_0, (S_n T_n)^n x_0, (S_n T_n)^{n+1} x_0, q t)
\]

\[
= 2 \mathcal{M}_1 (S_n T_n(S_n T_n)^{n-1} x_0, S_n T_n(S_n T_n)^{n-1} x_0, S_n T_n(S_n T_n)^n x_0, q t)
\]

\[
= 2 \mathcal{M}_1 (S_n T_n(x_{n-1}), S_n T_n(x_{n-1}), S_n T_n x_{n-1}, q t)
\]

\[
= 2 \mathcal{M}_1 (S_n y_n, S_n y_n, S_n T_n x_n, q t)
\]

\[
\geq \mathcal{M}_1 (x_n, x_n, S_n T_n x_n, t) + \mathcal{M}_2 (y_n, y_n, T_n x_n, t)
\]

\[
= \mathcal{M}_1 (x_n, x_n, x_{n+1}, t) + \mathcal{M}_2 (y_n, y_n, y_{n+1}, t)
\]

\[
\geq \mathcal{M}_1 (x_n, x_n, x_{n+1}, q t) + \mathcal{M}_2 (y_n, y_n, y_{n+1}, t)
\]

Thus we have

\[
\mathcal{M}_1 (x_n, x_n, x_{n+1}, q t) \geq \mathcal{M}_2 (y_n, y_n, y_{n+1}, t) \quad (5.4.3)
\]
Similarly, by (5.4.2)

\[ 2 \mathcal{M}_2 (y_n, y_n, y_{n+1}, qt) = 2 \mathcal{M}_2 (T_n x_{n-1}, T_n x_n, T_n x_m, qt) \]

\[ = 2 \mathcal{M}_2 (T_n x_{n-1}, T_n x_{n-1}, T_n S_0 y_n, qt) \]

\[ \geq \mathcal{M}_2 (y_n, y_n, T_n S_0 y_n, t) + \mathcal{M}_1 (x_{n-1}, x_{n-1}, S_0 y_n, t) \]

\[ = \mathcal{M}_2 (y_n, y_n, y_{n+1}, qt) + \mathcal{M}_1 (x_{n-1}, x_{n-1}, x_n, t) \]

\[ \geq \mathcal{M}_2 (y_n, y_n, y_{n+1}, qt) + \mathcal{M}_1 (x_{n-1}, x_{n-1}, x_n, t) \]

Thus we have

\[ \mathcal{M}_2 (y_n, y_n, y_{n+1}, qt) \geq \mathcal{M}_1 (x_{n-1}, x_{n-1}, x_n, t) \] --- (5.4.4)

Therefore, by (5.4.3) and (5.4.4)

\[ \mathcal{M}_1 (x_n, x_{n-1}, x_{n+1}, qt) \geq \mathcal{M}_2 (y_n, y_n, y_{n+1}, qt) \]

\[ \geq \mathcal{M}_1 (x_{n-1}, x_{n-1}, x_n, t) \]

\[ \vdots \]

\[ \geq \mathcal{M}_1 (x_{n-1}, x_{n-1}, t/q^n) \rightarrow 1 \text{ as } n \rightarrow \infty \]

Thus \( \{x_n\} \) is a Cauchy sequence in \( X \). Since \( (X, \mathcal{M}_1, \ast) \) is complete, \( \{x_n\} \) converges to a point \( z \) in \( X \). Similarly we prove \( \{y_n\} \) converges to a point \( w \) in \( Y \).

Again by (5.4.2) we have

\[ 2 \mathcal{M}_2 (T_n z, T_n z, y_{n+1}, qt) = \mathcal{M}_2 (T_n z, T_n z, T_n S_0 y_n, qt) \]

\[ \geq \mathcal{M}_2 (y_n, y_n, T_n S_0 y_n, t) + \mathcal{M}_1 (z, z, S_0 y_n, t) \]

\[ = \mathcal{M}_2 (y_n, y_n, y_{n+1}, t) + \mathcal{M}_1 (z, z, x_n, t) \] --- (5.4.5)

Letting \( n \rightarrow \infty \) in (5.4.5) we have

\[ 2 \mathcal{M}_2 (T_n z, T_n z, w, qt) \geq 2 \]

That is \( \mathcal{M}_2 (T_n z, T_n z, w, qt) \geq 1 \)
which implies that $M_2(T_nz, T_nz, w, qt) = 1$ so that $T_nz = w$.

On the other hand, by (5.4.1) we have
\[
2 M_1(S_nw, S_nw, S_nx_{n+1}, qt) = 2 M_1(S_nw, S_nw, S_nx_n, t) \\
\geq M_1(x_n, x_n, S_nT_nx_n, t) + M_2(w, w, T_nx_n, t) \\
= M_1(x_n, x_n, x_{n+1}, t) + M_2(w, w, y_{n+1}, t) \quad (5.4.6)
\]

Letting $n \to \infty$ in (5.4.6), it follows that $S_nw = z$.

Therefore we have $S_nT_nz = S_nw = z$ and $T_nS_nw = T_nz = w$, which means that the point $z$ is a common fixed point of $S_n T_n$ and the point $w$ is a common fixed point of $T_n S_n$.

To prove the uniqueness of the common fixed point $z$, let $z'$ be the second common fixed point of $S_n T_n$.

By (5.4.1) we have
\[
2 M_1(z, z, z', qt) = 2 M_1(S_nT_nz, S_nT_nz, S_nT_nz', qt) \\
= 2 M_1(S_n(T_nz), S_n(T_nz), S_nT_nz', qt) \\
\geq M_1(z', z', S_nT_nz', t) + M_2(T_nz, T_nz, T_nz', t) \\
\geq M_1(z', z', z', qt) + M_2(T_nz, T_nz, T_nz', t)
\]

Which implies that
\[
M_1(z, z, z', qt) \geq M_2(T_nz, T_nz, T_nz', t) \quad (5.4.7)
\]

Similarly by (5.4.2), we have
\[
2 M_2(T_nz, T_nz, T_nz', qt) = 2 M_2(T_nz, T_nz, T_nS_nT_nz', qt) \\
\geq M_2(T_nz', T_nz', T_nS_nT_nz', t) + M_1(z, z, S_nT_nz', t) \\
\geq M_2(T_nz', T_nz', T_nz', qt) + M_1(z, z, z', t)
\]

Which implies that
\[
M_2(T_nz, T_nz, T_nz', qt) \geq M_1(z, z, z', t) \quad (5.4.8)
\]
Therefore by (5.4.7) and (5.4.8)

\[ M_i(z, z, z', t) \geq M_i(T_n z, T_n z, T_n z', t) \geq M_i(z, z, z', t/q) \] (since \( q < 1 \)),

which is a contradiction.

Thus \( z = z' \).

So the point \( z \) is the unique common fixed point of \( ST \). Similarly, we prove the point \( w \) is also a unique common fixed point of \( TS \).

**Theorem 5.4.2:** Let \( (X, M_i, \ast) \) and \( (Y, M_j, \ast) \) be two complete \( M \)-fuzzy metric spaces. Let \( A_i \) and \( B_j \) be mappings of \( X \) into \( Y \) and \( S_n, T_q \) be mappings of \( Y \) into \( X \) satisfying the inequalities.

\[ 3 M_i(S_p A_i x, T_q B_j x', T_q B_j x, t) \geq M_i(x, x, S_p A_i x, t) + M_i(x', x', T_q B_j x', t) \] ---(5.4.9)
\[ 3 M_i(B_j S_p y, A_i T_q y, A_i T_q y, t) \geq M_i(y, y, B_j S_p y, t) + M_i(y', y', A_i T_q y, t) \] ---(5.4.10)

for all \( i \neq j \neq q \) in \( N \), \( x, x' \) in \( X \) and \( y, y' \) in \( Y \) where \( q < 1 \). If one of the mappings \( \{A_n\}, \{B_n\}, \{S_n\} \) and \( \{T_n\} \) is continuous, then \( \{S_n A_n\} \) and \( \{T_n B_n\} \) have a unique common fixed point \( z \) in \( X \) and \( \{B_n S_n\} \) and \( \{A_n T_n\} \) have a unique common fixed point \( w \) in \( Y \). Further, \( \{A_n\} z = \{B_n\} z = w \) and \( \{S_n\} w = \{T_n\} w = z \).

**Proof:** Let \( x_0 \) be an arbitrary point in \( X \) and we define the sequences \( \{x_n\} \) in \( X \) and \( \{y_n\} \) in \( Y \) by

\[ A_n x_{2n-2} = y_{2n-1}, S_n y_{2n-1} = x_{2n-1}, B_n x_{2n-1} = y_{2n}, T_n y_{2n} = x_{2n} \] for \( n = 1, 2, 3 \ldots \).

Now we have

\[ 3 M_i(x_{2n+1}, x_{2n}, x_{2n}, t) = 3 M_i(S_p A_i x_{2n}, T_q B_j x_{2n-1}, T_q B_j x_{2n-1}, t) \]
\[ \geq M_i(x_{2n}, x_{2n}, x_{2n}, t) + M_i(x_{2n}, x_{2n}, S_p A_i x_{2n}, t) + M_i(x_{2n}, x_{2n}, T_q B_j x_{2n}, t) \]
\[ = M_i(x_{2n}, x_{2n}, x_{2n}, t) + M_i(x_{2n}, x_{2n}, x_{2n}, t) + M_i(x_{2n}, x_{2n}, x_{2n}, t) \]
\[ = 2 M_i(x_{2n}, x_{2n}, x_{2n}, t) + M_i(x_{2n}, x_{2n}, x_{2n}, t) \]
\[ \geq 2 M_i(x_{2n}, x_{2n}, x_{2n}, t) + M_i(x_{2n+1}, x_{2n}, x_{2n}, qt) \]
Which implies that

\[ 2 \mathcal{M}_1(x_{2n+1}, x_{2n}, x_{2n}, qt) \geq 2 \mathcal{M}_1(x_{2n}, x_{2n-1}, x_{2n-1}, t) \]

That is \( \mathcal{M}_1(x_{2n+1}, x_{2n}, x_{2n}, qt) \geq \mathcal{M}_1(x_{2n}, x_{2n-1}, x_{2n-1}, t) \) --- (5.4.11)

Again using (5.4.9)

\[ 3 \mathcal{M}_1(x_{2n}, x_{2n-1}, x_{2n-1}, qt) = 3 \mathcal{M}_1(x_{2n-1}, x_{2n-1}, x_{2n-1}, qt) \]

\[ = 3 \mathcal{M}_1(S_nA_nx_{2n-2}, T_nB_nx_{2n-1}, T_nB_nx_{2n-1}, qt) \]

\[ \geq \mathcal{M}_1(x_{2n-2}, x_{2n-1}, x_{2n-1}, t) + \mathcal{M}_1(x_{2n-2}, x_{2n-2}, S_nA_nx_{2n-2}, t) + \]

\[ \mathcal{M}_1(x_{2n-1}, x_{2n-1}, T_nB_nx_{2n-1}, t) \]

\[ = \mathcal{M}_1(x_{2n-2}, x_{2n-1}, x_{2n-1}, t) + \mathcal{M}_1(x_{2n-2}, x_{2n-2}, x_{2n-1}, t) + \mathcal{M}_1(x_{2n-1}, x_{2n-1}, x_{2n-1}, t) \]

\[ \geq 2 \mathcal{M}_1(x_{2n-2}, x_{2n-1}, x_{2n-1}, t) + \mathcal{M}_1(x_{2n-1}, x_{2n-1}, x_{2n}, qt) \]

Which implies that

\[ 2 \mathcal{M}_1(x_{2n}, x_{2n-1}, x_{2n-1}, qt) \geq 2 \mathcal{M}_1(x_{2n-2}, x_{2n-1}, x_{2n-1}, t) \]

That is \( \mathcal{M}_1(x_{2n}, x_{2n-1}, x_{2n-1}, qt) \geq \mathcal{M}_1(x_{2n-2}, x_{2n-2}, x_{2n-2}, t) \) --- (5.4.12)

Thus from inequalities (5.4.11) and (5.4.12), we have

\[ \mathcal{M}_1(x_{n+1}, x_{2n}, x_{2n}, qt) \geq \mathcal{M}_1(x_n, x_{n-1}, x_{n-1}, t) \]

\[ \geq \mathcal{M}_1(x_{n-1}, x_{n-2}, x_{n-2}, t/q) \]

\[ \vdots \]

\[ \geq \mathcal{M}_1(x_1, x_0, x_0, t/q^{n-1}) \rightarrow 1 \text{ as } n \rightarrow \infty \]

Thus \( \{x_n\} \) is a Cauchy sequence in \( X \). Since \( (X, \mathcal{M}_1, *) \) is complete, \( \{x_n\} \) converges to a point \( z \) in \( X \). Similarly applying inequality (5.4.10) and proceeding as above, we prove \( \{y_n\} \) converges to a point \( w \) in \( Y \).

Suppose \( \{A_n\} \) is continuous, then
\[ \lim_{n \to \infty} A_n x_{2n} = A_n z = \lim_{n \to \infty} y_{2n+1} = w. \]

Now we prove \( S_n A_n z = z \) for all \( n \).

We have

\[ 3 \mathcal{M}(S_n A_n z, z, z, \omega) = \lim_{n \to \infty} 3 \mathcal{M}(S_n A_n z, T_n B_n x_{2n+1}, T_n B_n x_{2n+1}, \omega) \]

\[ \geq \lim_{n \to \infty} \left\{ \mathcal{M}(z, x_{2n+1}, x_{2n+1}, t) + \mathcal{M}(z, z, S_n A_n z, t) + \mathcal{M}(x_{2n+1}, x_{2n+1}, T_n B_n x_{2n+1}, t) \right\} \]

\[ = \mathcal{M}(z, z, t) + \mathcal{M}(z, z, S_n A_n z, t) + \mathcal{M}(z, z, t) \]

\[ = 1 + \mathcal{M}(z, S_n A_n z, t) + 1 \]

\[ \geq 2 + \mathcal{M}(z, S_n A_n z, \omega) \]

Which implies

\[ \mathcal{M}(S_n A_n z, z, z, \omega) \geq 1. \]

That is \( \mathcal{M}(S_n A_n z, z, z, \omega) = 1 \)

Thus \( S_n A_n z = z \) for all \( n \).

Hence \( S_n w = z \) for all \( n \). (Since \( A_n z = w \) for all \( n \).)

Now we prove \( B_n S_n w = w \) for all \( n \).

We have

\[ 3 \mathcal{M}(B_n S_n w, w, w, \omega) = \lim_{n \to \infty} 3 \mathcal{M}(B_n S_n w, y_{2n+1}, y_{2n+1}, \omega) \]

\[ = \lim_{n \to \infty} 3 \mathcal{M}(B_n S_n w, A_n T_n y_{2n}, A_n T_n y_{2n}, \omega) \]

\[ \geq \lim_{n \to \infty} \left\{ \mathcal{M}(w, y_{2n}, y_{2n}, t) + \mathcal{M}(w, w, B_n S_n w, t) + \mathcal{M}(y_{2n}, y_{2n}, A_n T_n y_{2n}, t) \right\} \]

Which implies \( \mathcal{M}(B_n S_n w, w, w, \omega) = 1. \)

Thus \( B_n S_n w = w \) for all \( n \).

Hence \( B_n z = w \) for all \( n \). (Since \( S_n w = z \)
Now we prove $T_nB_nz = z$ for all $n$.

$$3\, M_1(z, T_nB_nz, T_nB_nz, qt) = \lim_{n \to \infty} 3\, M_1(x_{2n+1}, T_nB_nz, T_nB_nz, qt)$$

$$= \lim_{n \to \infty} 3\, M_4(SA_{2n}T_nB_nz, T_nB_nz, qt)$$

$$\geq \lim_{n \to \infty} \{ M_1(x_{2n}, z, z, t) + M_1(x_{2n}, x_{2n}, S_nA_nx_{2n}, t) + M_1(z, z, T_nB_nz, t) \}$$

Which implies $M_1(z, T_nB_nz, T_nB_nz, qt) = 1$.

Thus $T_nB_nz = z$ for all $n$.

Hence $T_nw = z$ for all $n$.  (Since $B_nz = w$)

Now we prove $A_nT_nw = w$ for all $n$.

$$3\, M_2(w, A_nT_nw, A_nT_nw, qt) = \lim_{n \to \infty} 3\, M_2(y_{2n}, A_nT_nw, A_nT_nw, qt)$$

$$= \lim_{n \to \infty} 3\, M_2(B_nS_ny_{2n-1}, A_nT_nw, A_nT_nw, qt)$$

$$\geq \lim_{n \to \infty} \{ M_2(y_{2n-1}, w, w, t) + M_2(y_{2n-1}, y_{2n-1}, B_nS_ny_{2n-1}, t) + M_2(w, w, A_nT_nw, t) \}$$

Thus $A_nT_nw = w$ for all $n$.

The same results hold if one of the mappings $\{B_n\}, \{S_n\}$ and $\{T_n\}$ is continuous.

**Uniqueness:** Let $z'$ be another common fixed point of $\{S_nA_n\}$ and $\{T_nB_n\}$ in $X$, $w'$ be another common fixed point of $\{B_nS_n\}$ and $\{A_nT_n\}$ in $Y$.

We have

$$3\, M_1(z, z', z', qt) = 3\, M_1(S_nA_nz, T_nB_nz', T_nB_nz', qt)$$

$$\geq M_1(z, z', z', t) + M_1(z, z, S_nA_nz, t) + M_1(z', z', T_nB_nz', t)$$

$$= M_1(z, z', z', t) + M_1(z, z, z, t) + M_1(z', z', z', t)$$

$$\geq M_1(z, z', z', qt) + 2$$

Which implies $M_1(z, z', z', qt) \geq 1$

Thus $z = z'$.  

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So the point $z$ is the unique common fixed point of $\{S_nA_n\}$ and $\{T_nB_n\}$ in $X$. Similarly we prove $w$ is a unique common fixed point of $\{B_nS_n\}$ and $\{A_nT_n\}$ in $Y$.

**Remark 5.4.3:** In the above theorem 5.4.2, we have

$$\mathcal{M}_i(S_pA_ix, T_qB_ix', T_qB_ix', qt) \geq 1/3 \{ \mathcal{M}_i(x,x',x',t) + \mathcal{M}_i(x, x, S_pA_ix, t) + \mathcal{M}_i(x', x', T_qB_ix', t) \} \geq \min \{ \mathcal{M}_i(x,x',x',t), \mathcal{M}_i(x, x, S_pA_ix, t), \mathcal{M}_i(x', x', T_qB_ix', t) \}$$

$$\mathcal{M}_2(B_pS_py, A'T_qy', A'T_qy', qt) \geq 1/3 \{ \mathcal{M}_2(y,y',y',t) + \mathcal{M}_2(y, y, B_pS_py, t) + \mathcal{M}_2(y', y', A'T_qy', t) \} \geq \min \{ \mathcal{M}_2(y,y',y',t), \mathcal{M}_2(y, y, B_pS_py, t), \mathcal{M}_2(y', y', A'T_qy', t) \}$$

for all $i \neq j \neq p \neq q$ in $N$, $x, x'$ in $X$ and $y, y'$ in $Y$ where $q < 1$. If one of the mappings $\{A_n\}, \{B_n\}, \{S_n\}$ and $\{T_n\}$ is continuous, then $\{S_nA_n\}$ and $\{T_nB_n\}$ have a unique common fixed point $z$ in $X$ and $\{B_nS_n\}$ and $\{A_nT_n\}$ have a unique common fixed point $w$ in $Y$. Further, $\{A_n\}z = \{B_n\}w = z$ and $\{S_n\}w = \{T_n\}w = z$.

**Corollary 5.4.4:** Let $(X, \mathcal{M}_i, *)$ and $(Y, \mathcal{M}_2, *)$ be two complete $\mathcal{M}$-fuzzy metric spaces. Let $A_i, B_j$ be mappings of $X$ into $Y$ and $S_p, T_q$ be mappings of $Y$ into $X$ satisfying the inequalities.

$$\mathcal{M}_i(S_pA_ix, T_qB_ix', T_qB_ix', qt) \geq \min \{ \mathcal{M}_i(x,x',x',t), \mathcal{M}_i(x, x, S_pA_ix, t), \mathcal{M}_i(x', x', T_qB_ix', t) \}$$

$$\mathcal{M}_2(B_pS_py, A'T_qy', A'T_qy', qt) \geq \min \{ \mathcal{M}_2(y,y',y',t), \mathcal{M}_2(y, y, B_pS_py, t), \mathcal{M}_2(y', y', A'T_qy', t) \}$$

for all $i \neq j \neq p \neq q$ in $N$, $x, x'$ in $X$ and $y, y'$ in $Y$ where $q < 1$. If one of the mappings $\{A_n\}, \{B_n\}$, $\{S_n\}$ and $\{T_n\}$ is continuous, then $\{S_nA_n\}$ and $\{T_nB_n\}$ have a unique common fixed point $z$ in $X$ and $\{B_nS_n\}$ and $\{A_nT_n\}$ have a unique common fixed point $w$ in $Y$. Further, $\{A_n\}z = \{B_n\}w = z$ and $\{S_n\}w = \{T_n\}w = z$.

**Corollary 5.4.5:** Let $(X, \mathcal{M}_i, *)$ and $(Y, \mathcal{M}_2, *)$ be two complete $\mathcal{M}$-fuzzy metric spaces. Let $A, B$ be mappings of $X$ into $Y$ and $S, T$ be mappings of $Y$ into $X$ satisfying the inequalities.

$$3 \mathcal{M}_i(SAx, TBx', TBx', qt) \geq \mathcal{M}_i(x,x',x',t) + \mathcal{M}_i(x, x, SAx, t) + \mathcal{M}_i(x', x', TBx', t)$$

$$3 \mathcal{M}_2(BSy, ATy', ATy', qt) \geq \mathcal{M}_2(y,y',y',t) + \mathcal{M}_2(y, y, BSy, t) + \mathcal{M}_2(y', y', ATy', t)$$

for all $x, x'$ in $X$ and $y, y'$ in $Y$ where $q < 1$. If one of the mappings $A, B, S$ and $T$ is continuous, then $SA$ and $TB$ have a unique common fixed point $z$ in $X$ and $BS$ and $AT$ have a unique common fixed point $w$ in $Y$. Further, $Az = Bz = w$ and $Sw = Tw = z$.
**Remark 5.4.6:** If \((X, \mathcal{M}_1, *)\) and \((Y, \mathcal{M}_2, \ast)\) are the same \(\mathcal{M}\)-fuzzy metric spaces in the above theorem 5.4.2, then we obtain the following theorem as corollary.

**Corollary 5.4.7:** [69]: Let \((X, \mathcal{M}, *)\) be a complete \(\mathcal{M}\)-fuzzy metric space and \(T_n : X \rightarrow X\) be a sequence of maps such that for all \(t > 0\) and \(0 < k < 1\) satisfying the condition

\[
3 \mathcal{M}(T_x, T_y, T_y, t) \geq \{ \mathcal{M}(x, y, y, t/k) + \mathcal{M}(x, x, T_x, t/k) + \mathcal{M}(y, y, T_y, t/k) \}
\]

for all \(i \neq j\) and for all \(x, y\) in \(X\). Then \(\{T_n\}\) have a unique common fixed point.

**Theorem 5.4.8:** Let \((X, \mathcal{M}_1, \ast)\) and \((Y, \mathcal{M}_2, \ast)\) be two complete \(\mathcal{M}\)-fuzzy metric spaces with continuous t-norm \(*\) defined by \(a \ast b = \min\{a, b\}\) for all \(a, b \in [0,1]\). If \(T_i\) is a mapping from \(X\) into \(Y\) and \(S_j\) is a mapping from \(Y\) into \(X\) satisfying

\[
\mathcal{M}_1(S_jy, S_jy, S_jT_x, qt) \geq \mathcal{M}_1(S_jy, S_jy, x) \ast \mathcal{M}_1(x, x, S_jT_x, t) \ast \mathcal{M}_1(y, y, T_x, t) \quad --- (5.4.13)
\]

\[
\mathcal{M}_2(T_x, T_x, T_y, S_jy, qt) \geq \mathcal{M}_2(T_x, T_x, y) \ast \mathcal{M}_2(y, y, S_jy, t) \ast \mathcal{M}_2(x, x, S_jy, t) \quad --- (5.4.14)
\]

for all \(i \neq j\) in \(N\), \(x\) in \(X\) and \(y\) in \(Y\) where \(q < 1\), then \(\{S_nT_n\}\) has a unique fixed point \(z\) in \(X\) and \(\{T_nS_n\}\) has a unique fixed point \(w\) in \(Y\). Further \(\{T_n\}z = w\) and \(\{S_n\}w = z\).

**Proof.** Let \(x_0\) be an arbitrary point in \(X\). Define two sequences \(\{x_n\}\) and \(\{y_n\}\) in \(X\) and \(Y\), respectively, as follows:

\[
x_n = (S_nT_n)^n x_0, \quad y_n = T_n(x_{n-1}) \text{ for } n = 1, 2, \ldots
\]

By (5.4.13) we have

\[
\mathcal{M}_1(x_n, x_{n+1}, qt) = \mathcal{M}_1((S_nT_n)^n x_0, (S_nT_n)^n x_0, (S_nT_n)^n x_0, qt)
\]

\[
= \mathcal{M}_1(S_nT_n(S_nT_n)^n x_0, S_nT_n(S_nT_n)^n x_0, S_nT_n(S_nT_n)^n x_0, qt)
\]

\[
= \mathcal{M}_1(S_nT_n(x_{n-1}), S_nT_n(x_{n-1}), S_nT_n(x_{n-1}), qt)
\]

\[
= \mathcal{M}_1(S_ny_n, S_ny_n, S_nT_nx_n, qt)
\]
\[
\begin{align*}
\geq M_1(\text{Sn}y_n, \text{Sn}y_n, \text{Sn}x_n, t) & \ast M_1(x_n, x_n, S_nT_n x_n, t) \ast M_2(y_n, y_n, T_n x_n, t) \\
= M_1(x_n, x_n, x_n, t) & \ast M_1(x_n, x_n, x_n+1, t) \ast M_2(y_n, y_n, y_n+1, t) \\
\geq 1 & \ast M_1(x_n, x_n, x_n+1, qt) \ast M_2(y_n, y_n, y_n+1, t) \\
\geq M_2(y_n, y_n, y_n+1, t)
\end{align*}
\]

Thus we have

\[M_1(x_n, x_n, x_n+1, qt) \geq M_2(y_n, y_n, y_n+1, t) \quad \text{--- (5.4.15)}\]

Similarly, by (5.4.14)

\[M_2(y_n, y_n, y_n+1, qt) = M_2(T_n x_n, T_n x_n, T_n x_n, qt) = M_2(T_n x_n, T_n x_n, T_n x_n, S_n y_n, qt) \geq M_2(T_n x_n, T_n x_n, y_n, t) \ast M_2(y_n, y_n, T_n x_n, t) \ast M_1(x_n, x_n, S_n y_n, t) \]

\[= M_2(y_n, y_n, y_n, t) \ast M_2(y_n, y_n, y_n+1, t) \ast M_1(x_n, x_n, x_n+1, t) \]

\[\geq M_2(y_n, y_n, y_n, qt) \ast M_1(x_n, x_n, x_n+1, t) \]

Thus we have

\[M_2(y_n, y_n, y_n+1, qt) \geq M_1(x_n, x_n, x_n+1, t) \quad \text{--- (5.4.16)}\]

Therefore, by (5.4.15) and (5.4.16)

\[M_1(x_n, x_n, x_n+1, qt) \geq M_2(y_n, y_n, y_n+1, t) \geq M_1(x_n, x_n, x_n, t) \geq \cdots \geq M_1(x_0, x_0, x_1, t) \rightarrow 1 \quad \text{as } n \rightarrow \infty\]
Thus \( \{x_n\} \) is a Cauchy sequence in \( X \). Since \((X, \mathcal{M}, \ast)\) is complete, \( \{x_n\} \) converges to a point \( z \) in \( X \). Similarly we prove \( \{y_n\} \) converges to a point \( w \) in \( Y \).

Again by (5.4.14) we have
\[
\mathcal{M}_2(T_nz, T_nz, y_{n+1}, qt) = \mathcal{M}_2(T_nz, T_nz, T_nS_ny_n, qt)
\]
\[
\geq \mathcal{M}_2(T_nz, T_nz, y_n, t) \ast \mathcal{M}_2(y_n, y_n, T_nS_ny_n, t) \ast \mathcal{M}_1(z, z, S_ny_n, t)
\]
\[
= \mathcal{M}_2(T_nz, T_nz, y_n, t) \ast \mathcal{M}_2(y_n, y_n, y_{n+1}, t) \ast \mathcal{M}_1(z, z, x_n, t) \quad --- (5.4.17)
\]
Letting \( n \to \infty \) in (5.4.17) we have
\[
\mathcal{M}_2(T_nz, T_nz, w, qt) \geq \mathcal{M}_2(T_nz, T_nz, y_n, t) \ast 1 \ast 1
\]
That is \( \mathcal{M}_2(T_nz, T_nz, w, qt) \geq \mathcal{M}_2(T_nz, T_nz, y_n, t) \)
which implies that \( \mathcal{M}_2(T_nz, T_nz, w, qt) = 1 \) so that \( T_nz = w \).

On the other hand, by (5.4.14) we have
\[
\mathcal{M}_1(S_nw, S_nw, x_{n+1}, qt) = \mathcal{M}_1(S_nw, S_nw, S_nT_nx_n, t)
\]
\[
\geq \mathcal{M}_1(S_nw, S_nw, x_n, t) \ast \mathcal{M}_1(x_n, x_n, S_nT_nx_n, t) \ast \mathcal{M}_2(w, w, T_nx_n, t)
\]
\[
= \mathcal{M}_1(S_nw, S_nw, x_n, t) \ast \mathcal{M}_1(x_n, x_n, x_{n+1}, t) \ast \mathcal{M}_2(w, w, y_{n+1}, t) \quad --- (5.4.18)
\]
Letting \( n \to \infty \) in (5.4.18), it follows that \( S_nw = z \).

Therefore we have \( S_nT_nz = S_nw = z \) and \( T_nS_nw = T_nz = w \), which means that the point \( z \) is a fixed point of \( S_nT_n \) and the point \( w \) is a fixed point of \( T_nS_n \).

To prove the uniqueness of the fixed point \( z \), let \( z' \) be the second fixed point of \( S_nT_n \).

By (5.4.13) we have
\[
\mathcal{M}_1(z, z, z', qt) = \mathcal{M}_1(S_nT_nz, S_nT_nz, S_nT_nz', qt)
\]
\[
= \mathcal{M}_1(S_n(T_nz), S_n(T_nz), S_nT_nz', qt)
\]
\[
\geq \mathcal{M}_1(S_nT_nz, S_nT_nz, z', t) \ast \mathcal{M}_1(z', z', S_nT_nz', t) \ast \mathcal{M}_2(T_nz, T_nz', T_nz', t)
\]
\[ M_1(z, z, z', t) \ast M_2(z', z', z', t) \ast M_2(T_n z, T_n z, T_n z', t) \]
\[ \geq M_2(T_n z, T_n z, T_n z', t) \]
\[ M_2(T_n z, T_n z, T_n z', t) \geq M_2(T_n z', T_n z', T_n z', t) \ast M_1(z, z, z', t) \]
\[ \geq M_2(T_n z, T_n z, T_n z', t) \ast I \ast M_1(z, z, z', t) \]
\[ \geq M_1(z, z, z', t) \]

Hence \[ M_1(z, z, z', q t) \geq M_2(T_n z, T_n z, T_n z', t) \geq M_1(z, z, z', t) \]

Thus \( z = z' \)

So the point \( z \) is the unique fixed point of \( S_n T_n \). Similarly, we prove the point \( w \) is also a unique fixed point of \( T_n S_n \). \( \square \)

**Theorem 5.4.9**: Let \((X, M_1, \ast)\) and \((Y, M_2, \ast)\) be two complete \( M \)-fuzzy metric spaces with continuous \( t \)-norm* defined by \( a \ast b = \min\{a, b\} \) for all \( a, b \in [0, 1] \). Let \( A_i, B_i \) be mappings of \( X \) into \( Y \) and \( S_n, T_q \) be mappings of \( Y \) into \( X \) satisfying the inequalities.

\[ M_1(S_n A_i x, T_q B_i x', T_q B_i x', q t) \geq M_1(x, x, x', t) \ast M_1(x, x, S_n A_i x, t) \ast M_1(x', x', T_q B_i x', t) \ast M_1(x, x, T_q B_i x', 2t) \quad \text{(5.4.19)} \]

\[ M_2(B_n S_p y, A_i T_q y', A_i T_q y', q t) \geq M_2(y, y, y', t) \ast M_2(y, y, B_n S_p y, t) \ast M_2(y', y', A_i T_q y', t) \ast M_2(y, y, A_i T_q y', 2t) \quad \text{(5.4.20)} \]

for all \( i \neq j \neq p \neq q, x, x', y, y' \) in \( X \) and \( Y \) where \( q < 1 \). If one of the mappings \( \{A_n\}, \{B_n\}, \{S_n\} \) and \( \{T_n\} \) is continuous, then \( \{S_n A_n\} \) and \( \{T_n B_n\} \) have a unique common fixed point \( z \) in \( X \) and \( \{B_n S_n\} \) and \( \{A_n T_n\} \) have a unique common fixed point \( w \) in \( Y \). Further, \( \{A_n\} z = \{B_n\} z = w \) and \( \{S_n\} w = \{T_n\} w = z \).
Proof: Let \( x_0 \) be an arbitrary point in \( X \) and we define the sequences \( \{x_n\} \) in \( X \) and \( \{y_n\} \) in \( Y \) by

\[
A_n x_{2n-2} = y_{2n-1}, \quad S_n y_{2n-1} = x_{2n-1}, \quad B_n x_{2n-1} = y_{2n}, \quad T_n y_{2n} = x_{2n} \quad \text{for } n = 1, 2, 3 \ldots .
\]

Applying equality (5.4.19), we have

\[
\mathcal{M}_1(x_{2n+1}, x_{2n}, x_{2n-1}, qt) = \mathcal{M}_1(S_n A_n x_{2n}, T_n B_n x_{2n-1}, T_n B_n x_{2n-1}, qt)
\]

\[
\geq \mathcal{M}_1(x_{2n}, x_{2n-1}, x_{2n-1}, t) \ast \mathcal{M}_1(x_{2n}, S_n A_n x_{2n-1}) \ast \mathcal{M}_1(x_{2n-1}, x_{2n-1}, T_n B_n x_{2n-1}, t) \ast \mathcal{M}_1(x_{2n-1}, x_{2n-1}, T_n B_n x_{2n-1}, 2t)
\]

\[
= \mathcal{M}_1(x_{2n}, x_{2n-1}, x_{2n-1}, t) \ast \mathcal{M}_1(x_{2n}, x_{2n+1}, x_{2n-1}, t) \ast \mathcal{M}_1(x_{2n-1}, x_{2n}, x_{2n-1}, t) \ast \mathcal{M}_1(x_{2n}, x_{2n}, x_{2n-1}, 2t)
\]

\[
= \mathcal{M}_1(x_{2n}, x_{2n-1}, x_{2n-1}, t) \ast \mathcal{M}_1(x_{2n}, x_{2n}, x_{2n-1}, t)
\]

Which implies

\[
\mathcal{M}_1(x_{2n+1}, x_{2n}, x_{2n-1}, qt) \geq \mathcal{M}_1(x_{2n}, x_{2n-1}, x_{2n-1}, t) \quad --- (5.4.21)
\]

Now

\[
\mathcal{M}_1(x_{2n}, x_{2n-1}, x_{2n-1}, qt) = \mathcal{M}_1(x_{2n-1}, x_{2n}, x_{2n}, qt)
\]

\[
= \mathcal{M}_1(S_n A_n x_{2n-2}, T_n B_n x_{2n-1}, T_n B_n x_{2n-1}, qt)
\]

\[
\geq \mathcal{M}_1(x_{2n-2}, x_{2n-1}, x_{2n-1}, t) \ast \mathcal{M}_1(x_{2n-2}, 2, S_n A_n x_{2n-2}, t) \ast \mathcal{M}_1(x_{2n-2}, x_{2n-2}, T_n B_n x_{2n-1}, t) \ast \mathcal{M}_1(x_{2n-2}, x_{2n-2}, T_n B_n x_{2n-1}, 2t)
\]

\[
= \mathcal{M}_1(x_{2n-2}, x_{2n-1}, x_{2n-1}, t) \ast \mathcal{M}_1(x_{2n-2}, x_{2n-1}, x_{2n-1}, t) \ast \mathcal{M}_1(x_{2n-2}, x_{2n-2}, x_{2n-1}, t) \ast \mathcal{M}_1(x_{2n-2}, x_{2n-2}, x_{2n-1}, t) \ast \mathcal{M}_1(x_{2n-2}, x_{2n-2}, x_{2n-1}, 2t)
\]

Which implies
\[ M(x_{2n}, x_{2n-1}, x_{2n-1}, qt) \geq M(x_{2n-1}, x_{2n-2}, x_{2n-2}, t) \quad \text{(5.4.22)} \]

Thus from inequalities (5.4.21) and (5.4.22) we have
\[ M(x_{n+1}, x_{n}, x_{n}, t) \geq M(x_{n}, x_{n-1}, x_{n-1}, t) \]
\[ \geq M(x_{2n-1}, x_{2n-2}, x_{2n-2}, t/q) \]
\[ \vdots \]
\[ \geq M(x_1, x_0, x_0, t/q^{n-1}) \to 1 \quad \text{as } n \to \infty \]

Thus \( \{x_n\} \) is a Cauchy sequence in \( X \). Since \( (X, M_1, *) \) is complete, \( \{x_n\} \) converges to a point \( z \) in \( X \). Similarly applying inequality (5.4.20) and proceeding as above, we prove \( \{y_n\} \) converges to a point \( w \) in \( Y \).

Suppose \( \{A_n\} \) is continuous, then \( \lim_{n \to \infty} A_n x_{2n} = A_n z = \lim_{n \to \infty} y_{2n+1} = w \).

Now we prove \( S_n A_n z = z \).

We have
\[ M_1(S_n A_n z, z, z, t) = \lim_{n \to \infty} M_1(S_n A_n z, T_n B_n x_{2n-1}, T_n B_n x_{2n-1}, qt) \]
\[ \geq \lim_{n \to \infty} M_1(z, x_{2n-1}, x_{2n-1}, t) * M_1(z, S_n A_n z, t) * M_1(x_{2n-1}, x_{2n-1}, T_n B_n x_{2n-1}, t) \]
\[ \quad * M_1(z, z, T_n B_n x_{2n-1}, 2t) \]
\[ = M_1(z, z, z, t) * M_1(z, S_n A_n z, t) * M_1(z, z, z, t) * M_1(z, z, z, 2t) \]
\[ = 1 * M_1(z, S_n A_n z, t) * 1 * 1 \]
\[ = M_1(z, z, S_n A_n z, t) \]
\[ \geq M_1(z, z, S_n A_n z, t/q^n) \to 1 \quad \text{as } n \to \infty \]

Thus \( S_n A_n z = z \).

Hence \( S_n w = z \). (Since \( A_n z = w \))

Now we prove \( B_n S_n w = w \).
We have
\[ M_2(B_n S_n w, w, t) = \lim_{n \to \infty} M_2(B_n S_n w, y_{2n+1}, y_{2n+1}, t) \]
\[ = \lim_{n \to \infty} M_2(B_n S_n w, A_n T_n y_{2n}, A_n T_n y_{2n}, t) \]
\[ \geq \lim_{n \to \infty} M_2(w, y_{2n}, y_{2n}, t) \] 
\[ \geq M_2(w, w, B_n S_n w, t) \] 
\[ \geq M_2(w, w, B_n S_n w, t) \] 
\[ M_2(w, w, w, 2t) \]
\[ = M_2(w, w, w, t) \] 
\[ M_2(w, w, w, t) \] 
\[ M_2(w, w, w, t) \]
\[ \rightarrow 1 \text{ as } n \to \infty \]

Thus \( B_n S_n w = w \).

Hence \( B_n z = w \). (Since \( S_n w = z \))

Similarly we prove \( T_n B_n z = z \) and \( A_n T_n w = w \).

Hence \( T_n w = z \). (Since \( B_n z = w \))

The same results hold if one of the mappings \( \{B_n\}, \{S_n\} \) and \( \{T_n\} \) is continuous.

**Uniqueness:** Let \( z' \) be another common fixed point of \( \{S_n A_n\} \) and \( \{T_n B_n\} \) in \( X \), \( w' \) be another common fixed point of \( \{B_n S_n\} \) and \( \{A_n T_n\} \) in \( Y \).

We have
\[ M_1(z, z', z', t) = M_1(S_n A_n z, T_n B_n z', T_n B_n z', t) \]
\[ \geq M_1(z, z', z', t) \] 
\[ \geq M_1(z, z, z, t) \] 
\[ M_1(z, z, z, t) \] 
\[ M_1(z, z, z, 2t) \]
\[ \geq M_1(z, z, z, t) \] 
\[ \geq M_1(z, z, z, t) \] 
\[ \rightarrow 1 \text{ as } n \to \infty \]

Thus \( z = z' \).

So the point \( z \) is the unique common fixed point of \( \{S_n A_n\} \) and \( \{T_n B_n\} \) in \( X \). Similarly we prove \( w \) is a unique common fixed point of \( \{B_n S_n\} \) and \( \{A_n T_n\} \) in \( Y \).
Remark 5.4.10: If \((X, \mathcal{M}, *)\) and \((Y, \mathcal{M}, *)\) are the same \(\mathcal{M}\)-fuzzy metric spaces in the above theorem 5.4.9, then we obtain the following theorem as corollary.

Corollary 5.4.11[68]: Let \((X, \mathcal{M}, *)\) be a complete \(\mathcal{M}\)-fuzzy metric space with continuous \(t\)-norm* defined by \(a * b = \min\{a, b\}\) and \(T_n : X \to X\) be a sequence of maps such that for all \(t > 0\) and \(0 < k < 1\) satisfying the condition
\[
\mathcal{M}(T_i x, T_j y, T_j y, t) \geq \{ \mathcal{M}(x, y, y, t/k) * \mathcal{M}(x, x, T_i x, t/k) * \mathcal{M}(y, y, T_j y, t/k) * \mathcal{M}(x, x, T_j y, 2t/k) \}
\]
for all \(i \neq j\) and for all \(x, y\) in \(X\). Then \(\{T_n\}\) have a unique common fixed point.

Theorem 5.4.12: Let \((X, \mathcal{M}, *)\) and \((Y, \mathcal{M}, *)\) be two complete \(\mathcal{M}\)-fuzzy metric spaces with continuous \(t\)-norm* defined by \(a * b = \min\{a, b\}\) for all \(a, b \in [0,1]\). Let \(A_i, B_j\) be mappings of \(X\) into \(Y\) and \(S_p, T_q\) be mappings of \(Y\) into \(X\) satisfying the inequalities.
\[
\mathcal{M}_1(S_p A_i x, T_q B_j x', T_q B_j x', qt) \geq \min\{ \mathcal{M}_1(x, x', x', t), \mathcal{M}_1(x, S_p A_i x, S_p A_i x, t), \mathcal{M}_1(x', T_q B_j x', T_q B_j x', t) \} \quad (5.4.23)
\]
\[
\mathcal{M}_2(B_j S_p y, A_i T_q y', A_i T_q y', qt) \geq \min\{ \mathcal{M}_2(y, y', y', t), \mathcal{M}_2(y, B_j S_p y, B_j S_p y', t), \mathcal{M}_2(y', A_i T_q y', A_i T_q y', t) \} \quad (5.4.24)
\]
for all \(i \neq j \neq p \neq q\), \(x, x'\) in \(X\) and \(y, y'\) in \(Y\) where \(q < 1\). If one of the mappings \(\{A_n\}, \{B_n\}, \{S_n\}\) and \(\{T_n\}\) is continuous, then \(\{S_n A_n\}\) and \(\{T_n B_n\}\) have a unique common fixed point \(z\) in \(X\) and \(\{B_n S_n\}\) and \(\{A_n T_n\}\) have a unique common fixed point \(w\) in \(Y\). Further, \(\{A_n\}z = \{B_n\}z = w\) and \(\{S_n\}w = \{T_n\}w = z\).

Proof: Let \(x_0\) be an arbitrary point in \(X\) and we define the sequences \(\{x_n\}\) in \(X\) and \(\{y_n\}\) in \(Y\) by
\[
A_n x_{2n-2} = y_{2n-1}, S_n y_{2n-1} = x_{2n-1}, B_n x_{2n-1} = y_{2n}, T_q y_{2n} = x_{2n} \quad \text{for } n = 1, 2, 3, \ldots .
\]
Now we have

\[
\mathcal{M}_1(x_{2n+1}, x_{2n}, x_{2n}, q_t) = \mathcal{M}_1(S_n A_n x_{2n}, T_n B_n x_{2n-1}, T_n B_n x_{2n-1}, q_t)
\]

\[
\geq \min\{ \mathcal{M}_1(x_{2n}, x_{2n-1}, x_{2n-1}, x_{2n}, t), \mathcal{M}_1(x_{2n}, S_n A_n x_{2n}, S_n A_n x_{2n}, t), \mathcal{M}_1(x_{2n}, T_n B_n x_{2n-1}, T_n B_n x_{2n-1}, t), \mathcal{M}_1(A_n x_{2n}, B_n x_{2n-1}, B_n x_{2n-1}, t), \mathcal{M}_1(x_{2n}, T_n B_n x_{2n-1}, T_n B_n x_{2n-1}, 2t), \mathcal{M}_1(S_n A_n x_{2n}, x_{2n-1}, 2t) \} \}
\]

\[
= \min\{ \mathcal{M}_1(x_{2n}, x_{2n-1}, x_{2n-1}, t), \mathcal{M}_1(x_{2n}, x_{2n+1}, x_{2n+1}, t), \mathcal{M}_1(x_{2n-1}, x_{2n}, x_{2n}, t), \mathcal{M}_2(y_{2n+1}, y_{2n}, y_{2n}, t), \mathcal{M}_1(x_{2n}, x_{2n}, x_{2n}, 2t), \mathcal{M}_1(x_{2n-1}, x_{2n-1}, x_{2n-1}, 2t) \} \}
\]

\[
\geq \min\{ \mathcal{M}_1(x_{2n}, x_{2n}, x_{2n}, t), \mathcal{M}_1(x_{2n}, x_{2n+1}, x_{2n+1}, t), \mathcal{M}_1(x_{2n-1}, x_{2n}, x_{2n}, t), \mathcal{M}_1(y_{2n+1}, y_{2n-1}, y_{2n}, t), \mathcal{M}_1(x_{2n+1}, x_{2n}, x_{2n}, t) \} \}
\]

\[
= \min\{ \mathcal{M}_1(y_{2n+1}, y_{2n}, y_{2n}, t), \mathcal{M}_1(y_{2n+1}, y_{2n}, y_{2n}, t) \} \}
\]

Now

\[
\mathcal{M}_1(y_{2n+1}, y_{2n}, y_{2n}, q_t) = \mathcal{M}_1(B_n S_n y_{2n-1}, A_n T_n y_{2n}, A_n T_n y_{2n}, q_t)
\]

\[
\geq \min\{ \mathcal{M}_1(y_{2n+1}, y_{2n}, y_{2n}, t), \mathcal{M}_1(y_{2n+1}, B_n S_n y_{2n-1}, B_n S_n y_{2n-1}, t), \mathcal{M}_1(y_{2n}, A_n T_n y_{2n-1}, A_n T_n y_{2n}, t), \mathcal{M}_1(S_n y_{2n-1}, T_n y_{2n-1}, T_n y_{2n}, 2t), \mathcal{M}_1(A_n T_n y_{2n}, A_n T_n y_{2n}, 2t) \} \}
\]

\[
= \min\{ \mathcal{M}_1(y_{2n+1}, y_{2n}, y_{2n}, t), \mathcal{M}_1(y_{2n+1}, y_{2n}, y_{2n}, t), \mathcal{M}_1(y_{2n}, y_{2n+1}, y_{2n+1}, t), \mathcal{M}_1(x_{2n}, x_{2n}, t), \mathcal{M}_1(y_{2n}, y_{2n}, 2t) \} \}
\]

\[
= \min\{ \mathcal{M}_1(y_{2n}, y_{2n}, y_{2n}, t), \mathcal{M}_2(y_{2n}, y_{2n}, y_{2n}, t), \mathcal{M}_2(y_{2n-1}, y_{2n-1}, y_{2n-1}, t) \}
\]

\[
\geq \min\{ \mathcal{M}_2(y_{2n-1}, y_{2n}, y_{2n}, t), \mathcal{M}_2(x_{2n}, x_{2n}, x_{2n}, t) \} \quad \text{--- (5.4.25)}
\]

Hence

\[
\mathcal{M}_1(x_{2n+1}, x_{2n}, x_{2n}, q_t) \geq \min\{ \mathcal{M}_1(x_{2n}, x_{2n}, x_{2n}, t), \mathcal{M}_2(y_{2n}, y_{2n}, y_{2n}, t) \} \quad \text{--- (5.4.26)}
\]
Similarly we have
\[ M_1(x_{2n}, x_{2n-1}, x_{2n-1}, qt) \geq \min \{ M_1(x_{2n-1}, x_{2n-2}, x_{2n-2}, t), M_2(y_{2n-1}, y_{2n-2}, y_{2n-2}, t) \} \]
\[ M_2(y_{2n}, y_{2n-1}, y_{2n-1}, qt) \geq \min \{ M_2(y_{2n-1}, y_{2n-2}, y_{2n-2}, t), M_1(x_{2n-1}, x_{2n-2}, x_{2n-2}, t) \} \quad --- (5.4.27) \]

Hence
\[ M_1(x_{2n}, x_{2n-1}, x_{2n-1}, qt) \geq \min \{ M_1(x_{2n-1}, x_{2n-2}, x_{2n-2}, t), M_2(y_{2n-1}, y_{2n-2}, y_{2n-2}, t) \} \quad --- (5.4.28) \]

from inequalities \((5.4.25)\), \((5.4.26)\), \((5.4.27)\) and \((5.4.28)\), we have
\[ M_1(x_{n+1}, x_n, x_n, qt) \geq \min \{ M_1(x_n, x_{n-1}, x_{n-1}, t), M_2(y_n, y_{n-1}, y_{n-1}, t/q) \} \]
\[ \vdots \]
\[ \geq \min \{ M_1(x_1, x_0, x_0, t/q^n), M_2(y_1, y_0, y_0, t/q^n) \} \rightarrow 1 \quad \text{as } n \rightarrow \infty \]

Thus \(\{x_n\}\) is a Cauchy sequence in \(X\). Since \((X, M, *)\) is complete, \(\{x_n\}\) converges to a point \(z\) in \(X\). Similarly we prove \(\{y_n\}\) converges to a point \(w\) in \(Y\).

Suppose \(\{A_n\}\) is continuous, then 
\[ \lim_{n \to \infty} A_n x_{2n} = A_n z = \lim_{n \to \infty} y_{2n+1} = w. \]

Now we prove \(S_n A_n z = z\).

Suppose \(S_n A_n z \neq z\).

We have
\[ M_1(S_n A_n z, z, z, qt) = \lim_{n \to \infty} M_1(S_n A_n z, T_n B_n x_{2n-1}, T_n B_n x_{2n-1}, qt) \]
\[ \geq \lim_{n \to \infty} \min \{ M_1(z, x_{2n-1}, x_{2n-1}, t), M_1(z, S_n A_n z, S_n A_n z, t), M_1(x_{2n-1}, T_n B_n x_{2n-1}, T_n B_n x_{2n-1}, t), \]
\[ \quad M_2(A_n z, B_n x_{2n-1}, B_n x_{2n-1}, t), M_2(z, T_n B_n x_{2n-1}, T_n B_n x_{2n-1}, 2t), M_1(S_n A_n z, x_{2n-1}, x_{2n-1}, 2t) \} \]
\[ = \min \{ M_1(z, z, z, t), M_1(z, S_n A_n z, S_n A_n z, t), M_1(z, z, z, t), M_2(w, w, w, t), M_1(z, z, z, 2t), \]
\[ \quad M_1(S_n A_n z, z, z, 2t) \} \]
\[ > M_1(z, S_n A_n z, S_n A_n z, t) \quad \text{(since } q < 1) \]
Which is a contradiction.
Thus $S_n A_n z = z$.
Hence $S_n w = z$. (Since $A_n z = w$)
Now we prove $B_m S_n w = w$.
Suppose $B_m S_n w \neq w$.
We have
\[
\mathcal{M}(B_m S_n w, w, w, t) = \lim_{n \to \infty} \mathcal{M}(B_m S_n w, y_{2n+1}, y_{2n+1}, t)
\]
\[
= \lim_{n \to \infty} \mathcal{M}(B_m S_n w, A_n T_n y_{2n}, A_n T_n y_{2n}, t)
\]
\[
\geq \lim_{n \to \infty} \min\{ \mathcal{M}(w, y_{2n}, y_{2n}, t), \mathcal{M}(w, B_m S_n w, B_m S_n w, t), \mathcal{M}(y_{2n}, A_n T_n y_{2n}, A_n T_n y_{2n}, t),
\mathcal{M}(S_n w, T_n y_{2n}, T_n y_{2n}, t), \mathcal{M}(w, A_n T_n y_{2n}, A_n T_n y_{2n}, 2t), \mathcal{M}(B_m S_n w, y_{2n}, y_{2n}, 2t) \}
\]
\[
= \min\{ \mathcal{M}(w, w, t), \mathcal{M}(w, B_m S_n w, B_m S_n w, t), \mathcal{M}(w, w, t), \mathcal{M}(z, z, t), \mathcal{M}(w, w, 2t), \mathcal{M}(B_m S_n w, w, 2t) \}
\]
\[
> \mathcal{M}(w, B_m S_n w, B_m S_n w, t) \quad (\text{Since } q < 1) \text{ which is a contradiction.}
\]
Thus $B_m S_n w = w$.
Hence $B_m z = w$. (Since $S_n w = z$)
Similarly we prove $T_n B_m z = z$ and $A_n T_n w = w$.
Hence $T_n w = z$. (Since $B_m z = w$)
The same results hold if one of the mappings $\{B_n\}, \{S_n\}$ and $\{T_n\}$ is continuous.

**Uniqueness:** Let $z'$ be another common fixed point of $\{S_n A_n\}$ and $\{T_n B_n\}$ in $X$, $w'$ be another common fixed point of $\{B_m S_n\}$ and $\{A_n T_n\}$ in $Y$.

We have
\[
\mathcal{M}(z, z', z', t) = \mathcal{M}(S_n A_n z, T_n B_n z', T_n B_n z', t)
\]
\[
\geq \min\{ \mathcal{M}(z, z', z', t), \mathcal{M}(z, S_n A_n z, S_n A_n z, t), \mathcal{M}(z', T_n B_n z', T_n B_n z', t) \}
\]
\[ M(A \cdot z, B \cdot z', B \cdot z', t), M(T \cdot z, T \cdot z', T \cdot z', 2t), M(S \cdot A \cdot z, z', z', 2t) \]
\[ \geq \min \{ M(z, z', z', t), M(z, z, z), M(z', z', z', t), M(w, w', w', t), M(z', z', 2t) \} \]
\[ = \min \{ M(z, z', z', t), M(z, z, z), M(z', z', z', t), M(w, w', w', t), M(z, z', z', 2t) \} \]
\[ = \min \{ M(z, z', z', t), 1, 1, M(w, w', w', t), M(z, z', z', 2t) \} \]
\[ > M(w, w', w', t) \quad (\text{Since } q < 1) \]

Thus, \( z = z' \).

So the point \( z \) is the unique common fixed point of \( \{S_n A_n\} \) and \( \{T_n B_n\} \) in \( X \). Similarly we prove \( w \) is a unique common fixed point of \( \{B_n S_n\} \) and \( \{A_n T_n\} \) in \( Y \).

Remark 5.4.14: If \((X, M, *)\) and \((Y, M, *)\) are the same \( M \)-fuzzy metric spaces in the above theorem 5.4.13, then we obtain the following theorem as corollary.

**Corollary 5.4.15[31]:** Let \( S \) and \( T \) be two self mappings of a complete \( M \)-fuzzy metric space \((X, M, *)\). If there exists a number \( k \in (0, 1) \) such that

\[ M(Sx, Ty, Ty, kt) \geq \min \{ M(x, y, y, t), M(x, Sx, Sx, t), M(y, Ty, Ty, t), M(x, x, Sx, t), M(y, y, Ty, t) \} \]

for all \( x, y \in X \) and \( t > 0 \), then \( S \) and \( T \) have a unique common fixed point.