Chapter 1

Definitions, Notations and Some Known Results

1.1 Definitions

Let \((\Omega, \mathcal{F}, P)\) be a probability space, where \(\Omega\) be a non-empty set, \(\mathcal{F}\) be a \(\sigma\)-field of subsets of \(\Omega\) and \(P\) be a probability measure defined on \(\mathcal{F}\). Herein all the random variables (r.v.s) are defined on \((\Omega, \mathcal{F}, P)\).

By a distribution function (d.f) \(F\), we mean a non-decreasing, right continuous function satisfying \(F(-\infty) = 0\) and \(F(+\infty) = 1\). For a r.v \(X\), d.f. \(F\) is defined by \(F(x) = P(X \leq x)\).

A sequence of r.v.s \(\{X_n\}\) with corresponding d.f.s \(\{F_n\}\) is said to converge in distribution (weakly or in law) to a r.v \(X\) with d.f. \(F\) if and only if, \(F_n(x) \rightarrow F(x)\), as \(n \rightarrow \infty\), at all continuity points of \(F\). Such a convergence is expressed through \(X_n \xrightarrow{L} X\) or \(F_n \xrightarrow{W} F\).

A sequence of r.v.s \(\{X_n\}\) is said to converge in probability to a r.v \(X\), if for every \(\varepsilon > 0\), \(\lim_{n \rightarrow \infty} P(|X_n - X| > \varepsilon) = 0\). This is denoted by \(X_n \xrightarrow{P} X\).

A sequence of r.v.s \(\{X_n\}\) is said to converge almost surely to a r.v \(X\) if and
only if $P\left(\lim_{n \to \infty} X_n = X\right) = 1$. We denote this by $X_n \xrightarrow{a.s.} X$ (or $X_n \to X$ a.s.).

A positive valued function $L(x)$ is said to be slowly varying (s.v.) at infinity if and only if for each $t > 0$, $\lim_{x \to \infty} \frac{L(tx)}{L(x)} = 1$. Any s.v. function $L$ has the representation, $L(x) = a(x) \exp \left[ \int_0^x \frac{\varepsilon(y)}{y} dy \right]$, where $a(x) \to C > 0$ and $\varepsilon(x) \to 0$, as $x \to \infty$. This representation of a s.v. function is known as Karamata’s representation. [See Feller [20], Vol II, page 274]

A d.f. $F$ is said to be infinitely divisible if for every integer $n$ there exists a d.f $F_n(x)$ such that $F = F_n^{*(n)}$. Here $F_n^{*(n)}$ is the $n$-fold convolution of the function $F_n$. In other words a d.f $F(x)$ with characteristic function (c.f) $f(t)$ is called infinitely divisible if for every positive integer $n$ there exists a c.f $f_n(t)$ such that $f(t) = (f_n(t))^n$.

A function $f(t)$ is an infinitely divisible c.f, if and only if it admits the representation

$$f(t) = \exp \left\{ i\gamma t + \int_{-\infty}^{\infty} \left( e^{itx} - 1 - \frac{itx}{1 + x^2} \right) \frac{1 + x^2}{x^2} dG(x) \right\}$$

where $\gamma$ is a real constant, $G(x)$ is a bounded non-decreasing function and the function under the integral sign is equal to $-\frac{t^2}{2}$ at the point $x=0$.

A d.f. $F$ is said to be stable if and only if, for every $b_1 > 0, b_2 > 0, a_1$ and $a_2$ real, there exists $b > 0$ and a real, such that the relation

$$F\left( \frac{x - a_1}{b_1} \right) * F\left( \frac{x - a_2}{b_2} \right) = F\left( \frac{x - a}{b} \right)$$
holds. The characteristic function \( f(t) \) of a stable distribution has the representation,

\[
\log f(t) = i\gamma t - C|t|^\alpha \left( 1 + i\beta \frac{t}{|t|} \omega(t, \alpha) \right)
\]

where

\[
\omega(t, \alpha) = \begin{cases} 
\tan \left( \frac{\pi \alpha}{2} \right), & \text{if } \alpha \neq 1. \\
\frac{2}{\pi} \log |t|, & \text{if } \alpha = 1.
\end{cases}
\]

and \( \alpha, \beta, \gamma \) and \( C \) are real constants with \( C \geq 0, |\beta| \leq 1, 0 < \alpha \leq 2 \). Here \( \alpha \) is called the characteristic exponent. The parameters \( \gamma \) and \( C \) merely determine location and scale and hence without loss of generality we assume that \( \gamma = 0 \) and \( C=1 \).

A stable r.v. is positive valued (negative valued) whenever \( 0 < \alpha < 1 \) and \( \beta = -1 \) (\( \beta = +1 \)) in the characteristic function representation. A stable r.v. with \( \alpha = 2 \) is a normal r.v.

A d.f. \( F \) is said to belong to the domain of attraction of a stable law with index \( \alpha, 0 < \alpha < 2 \) denoted by \( F \in DA(\alpha) \), \( 0 < \alpha < 2 \), if there exist sequences \( A_n \in \mathbb{R} \) and \( B_n > 0 \), \( B_n \to \infty \) as \( n \to \infty \) such that,

\[
\frac{S_n - A_n}{B_n} \xrightarrow{d} Y_\alpha
\]

where \( Y_\alpha \) is a stable r.v, with d.f. \( G_\alpha \). Here \( B_n \approx n^{\frac{1}{\alpha}} L(n) \), where \( L(.) \) is slowly varying (s.v) at \( \infty \). When \( 0 < \alpha < 1 \) and \( \beta = -1 \), the d.f. \( F \) is said to belong to the domain of attraction of a positive stable law with index \( \alpha, 0 < \alpha < 1 \). When \( L(.) = C > 0 \), the d.f. \( F \) is said to belong to the domain of normal attraction.
of a stable law with index $\alpha$, $0 < \alpha < 2$, denoted by $F \in \text{DNA}(\alpha)$, $0 < \alpha < 2$. [See Gnedenko and Kolmogorov [22], page 162]. Some times it can happen that the sequence $\{Z_n\}$ does not converge, for any choice of the constants $\{A_n\}$ and $\{B_n\}$, but for some subsequence $\{n_k\}$, $\{Z_{n_k}\}$ converges to a non-degenerate law. Here $F$ is said to belong to the domain of partial attraction of the respective limit law. The limit law will always be an infinitely divisible law. [See Gnedenko and Kolmogorov [22]].

A d.f. $G$ is said to be semi stable, if it is either normal or the characteristic function $f(t)$ of $G$ is of the form

$$\log f(t) = i\gamma t + \int_{-\infty}^{\infty} \left( e^{itx} - 1 - \frac{itx}{1+x^2} \right) dH(x)$$

with spectral function

$$H(-x) = \begin{cases} x^{-\alpha} \theta_1(\log x), & x > 0. \\ -x^{-\alpha} \theta_2(\log x), & x < 0. \end{cases}$$

where $0 < \alpha < 2$ and $\theta_i$ are periodic functions with common periods such that for all $x$ and all $h \geq 0$,

$$e^{\alpha h}\theta_i(x-h) - e^{-\alpha h}\theta_i(x+h) \geq 0, \quad d_i \geq \theta_i(x) \geq C_i, \quad i = 1, 2; \quad C_1 + C_2 > 0$$

A d.f. $F$ is said to belong to the domain of partial attraction of a semistable law with index $\alpha$, $0 < \alpha \leq 2$, denoted by $F \in \text{DP}(\alpha)$, $0 < \alpha \leq 2$, if there exists a sequence $(n_k)$ satisfying,

1. $n_k < n_{k+1}, k \geq 1$, 
2. $n_k$
2. \( \frac{n_{k+1}}{n_k} \to r(\geq 1) \)

and sequences \((A_{n_k})\) of real constants and \((B_{n_k})\) of positive constants

\( (B_{n_k} \to \infty \text{ as } k \to \infty) \) such that,

\[
\frac{S_{n_k} - A_{n_k}}{B_{n_k}} \xrightarrow{w} Y^*_\alpha
\]

where \(Y^*_\alpha\) is semi-stable r.v. with d.f. \(G^*_\alpha\).

A real valued increasing process \(\{X(t), t > 0\}\) with stationary independent increments is called a subordinator. At any fixed \(t\), the characteristic function of \(X(t)\) can be given by,

\[
\log f_t(u) = t \left( \int_0^\infty \left( e^{iux} - 1 \right) dH(x) \right), \ t \in [0, \infty)
\]

where \(H(x)\) is a spectral function which satisfies \(\int_0^\infty \frac{x}{1+x}dH(x) < \infty\), when \(H(x) = -\frac{C}{x^\alpha}, \ C > 0, \ 0 < \alpha < 1\), then the subordinator becomes a stable subordinator with exponent \(\alpha\).

Let \(\{\xi_n(p)\}\) be a sequence of p-dimensional r.v.s and \(X_p = (x_1, x_2, \ldots, x_p)\) be a point in the p-dimensional Euclidean space. For any \(\varepsilon > 0\) define,

\[
N_p(\varepsilon) = \left\{ (x_1 - \varepsilon, x_1 + \varepsilon), (x_2 - \varepsilon, x_2 + \varepsilon), \ldots, x_p, \ldots, (x_p - \varepsilon, x_p + \varepsilon) \right\}
\]

Then \(X_p\) is said to be an almost sure limit point of the sequence \(\{\xi_n(p)\}\), if and only if, for every \(\varepsilon > 0\), \(P\left(\xi_n(p) \in N_p(\varepsilon) \text{ i.o} \right) = 1\).
1.2 Notations

The following notations may be noted as they appear frequently in the thesis.

r.v. (s) : random variable (s)
d.f. (s) : distribution function (s)
c.f. (s) : characteristic function (s)
i.i.d. : independent and identically distributed
p.d.f. : probability density function
i.d. : infinitely divisible
lim sup : limit superior
lim inf : limit inferior
LLN : Laws of large numbers
CLT : Central limit theorem
LIL : Law of Iterated Logarithm
DA : Domain of attraction
DNA : Domain of normal attraction
DPA : Domain of partial attraction
s.v. : slowly varying function
i.o. : infinitely often
a.s. : almost surely

\( \sim \) : \( f(x) \sim g(x) \), as \( x \to x_0 \), means \( \lim_{x \to x_0} \frac{f(x)}{g(x)} = 1 \)

\( X \overset{d}{=} Y \) : \( X \) and \( Y \) have the same distribution

\( \mathbb{R} \) : Set of real numbers

\( I \) : Set of integer

\( \overset{d}{\rightarrow} \) : Convergence in distribution

\( \overset{w}{\rightarrow} \) : Convergence weakly

\( \overset{a.s}{\rightarrow} \) : Convergence almost surely

\( \overset{P}{\rightarrow} \) : Convergence in probability

\( [x] \) : Integer part of \( x \)

\( O \) : \( f(.) = O(g(.)) \) means \( \lim_{x \to \infty} \frac{f(x)}{g(x)} < \infty \)

\( o \) : \( f(.) = o(g(.)) \) means \( \lim_{x \to \infty} \frac{f(x)}{g(x)} = 0 \)

\( T^{(1)}(x) \) : derivative \( \frac{dT(x)}{dx} \) of a function \( T(x) \)

\( \varepsilon, \eta, \theta, C, k, m, n \) : with or without a subscript or superscript denote positive constants, not necessarily the same at each occurrence, with \( k, m \) and \( n \) confined to be integers
1.3 Some known results

**Lemma 1.3.1.** (Borel-Cantelli Lemma)[Breiman, L [9], Lemma 3.14, page 41]

Let \( \{A_n\} \) be a sequence of events in a probability space.

(i) If \( \sum_{n=1}^{\infty} P(A_n) < \infty \) then \( P(A_n \ i.o) = 0 \).

(ii) If \( \sum_{n=1}^{\infty} P(A_n) = \infty \) and if \( A_1, A_2,... \) are mutually independent events then

\[ P(A_n \ i.o) = 1. \]

**Lemma 1.3.2.** [Extended Borel-Cantelli Lemma, Spitzer, F [62], Lemma, page 317]

Let \( \{E_n\} \) be a sequence of events in a common probability space. If

(i) \( \sum_{n=1}^{\infty} P(E_n) = \infty \) and (ii) \( \liminf_{n \to \infty} \frac{\sum_{k=1}^{n} \sum_{s=1}^{n} P(E_k \cap E_s)}{\left( \sum_{k=1}^{n} P(E_k) \right)^2} \leq C(>0) \), then

\[ P(E_k \ i.o) \geq C^{-1}. \]

**Lemma 1.3.3.** [Drasin, D. and Seneta, E. [17]]

Let \( L \) be any s.v. function and let \( (x_n) \) and \( (y_n) \) be sequence of real constants tending to \( \infty \), as \( n \to \infty \). Then for any \( \delta > 0 \),

\[ \lim_{n \to \infty} y_n^\delta \frac{L(x_n y_n)}{L(x_n)} = \infty \quad \text{and} \quad \lim_{n \to \infty} y_n^\delta \frac{L(x_n y_n)}{L(x_n)} = 0. \]
Lemma 1.3.4. [Divanji, G. and Vasudeva, R. [16]]

Let $F \in \text{DP}(\alpha), \ 0 < \alpha < 2$ and let $B_n = \inf \{ x > 0 : 1 - F(x) + F(-x) \geq \frac{1}{n} \}$. Then $B_n \simeq n^{\frac{1}{\alpha}} l(n) \eta(n)$, where $l$ is a function slowly varying at $\infty$ and $\eta$ is a function such that both $\eta$ and $\frac{1}{\eta}$ are bounded.

Lemma 1.3.5. [Nielsen, O.B. [47]]

Let $(A_n)$ be a sequence of events in a common probability space. If $P(A_n) \to 0$ as $n \to \infty$ and $\sum_{n=1}^{\infty} P(A_n \cap A_{n+1}^c) < \infty$. Then $P(A_n \ i.o) = 0$.

Lemma 1.3.6. [Vasudeva, R. and Divanji, G. [67], Lemma, page 295]

Let $X_1$ be a positive stable r.v. with characteristic function,

$$E(\exp\{iuX_1\}) = \exp \left\{ -|u|^\alpha \left( 1 - \frac{iu}{|u|} \tan \left( \frac{\pi \alpha}{2} \right) \right) \right\}, \ 0 < \alpha < 1.$$ 

Then, as $x \to 0$, $P\left(X_1 \leq x \right) \simeq \frac{x^{\frac{\alpha}{2(1-\alpha)}}}{\sqrt{2\pi \alpha B(\alpha)}} \exp\left\{ -B(\alpha)x^{\frac{\alpha}{\alpha-1}} \right\}$,

where $B(\alpha) = (1 - \alpha)^{\frac{\alpha}{1-\alpha}} \left( \cos \left( \frac{\pi \alpha}{2} \right) \right)^{\frac{1}{\alpha-1}}$.

Lemma 1.3.7. [Vasudeva, R. and Divanji, G. [68]]

Let $F \in \text{DP}(\alpha), \ 0 < \alpha < 2$. Let $(x_n)$ be a monotone sequence of real numbers tending to $\infty$, as $n \to \infty$. Then $\frac{S_n}{B_n x_n} \overset{p}{\to} 0$, as $n \to \infty$, with $B_n = n^{\frac{1}{\alpha}} l(n)$, where $l$ is slowly varying at $\infty$. 

Lemma 1.3.8. [Wichura, M.J. [73]]

For each $C > 0$, $P \left( \frac{S_n}{A_n} \leq d(\alpha)C \right) = \exp \left\{ - (1 + o(1)) \left( \frac{C}{\theta_a} \right)^{\lambda} \log \log n \right\}$, where $S_n = \sum_{k=1}^{n} X_k$, $A_n = \theta_a n^{\frac{1}{\alpha}} \left( \log \log n \right)^{\frac{\alpha - 1}{\alpha}}$, $\theta_a = \frac{\alpha}{|\alpha - 1|^{\frac{1}{\alpha}}}$, $\lambda = \frac{\alpha}{\alpha - 1}$, $d(\alpha) = 1$, if $\alpha < 1$ and $-1$, if $\alpha > 1$ and $\alpha \neq 1$.

Theorem 1.3.9. [Allan Gut [2], Theorem 2.1, page 30]

Let $\{n_k\}_{k=1}^{\infty}$ be a strictly increasing subsequence of positive integers such that $\lim \inf_{k \to \infty} \frac{n_k}{n_{k+1}} > 0$. Further, let $\{X_n\}_{n=1}^{\infty}$ be i.i.d r.v.s, set $S_n = \sum_{k=1}^{n} X_k$ and suppose that $EX_1 = 0$, $EX_1^2 = \sigma^2 < \infty$. Then

$$\limsup_{k \to \infty} \left( \liminf_{k \to \infty} \frac{S_{n_k}}{\sqrt{n_k \log \log n_k}} \right) = +\sigma \sqrt{2} \left( -\sigma \sqrt{2} \right) \ a.s.$$ 

Conversely, if

$$P \left( \limsup_{k \to \infty} \frac{|S_{n_k}|}{\sqrt{n_k \log \log n_k}} < \infty \right) > 0,$$

then $EX_1^2 < \infty$ and $EX_1 = 0$.

Theorem 1.3.10. [Allan Gut [2], Theorem 2.2, page 31]

Let $\{n_k\}_{k=1}^{\infty}$ be a strictly increasing subsequence of positive integers such that $\limsup_{k \to \infty} \frac{n_k}{n_{k+1}} < 1$. Let $\{X_n\}_{n=1}^{\infty}$ be i.i.d r.v.s and $S_n = \sum_{k=1}^{n} X_k$. Suppose that $EX_1 = 0$, $EX_1^2 = \sigma^2 < \infty$. Then

$$\limsup_{k \to \infty} \left( \liminf_{k \to \infty} \frac{S_{n_k}}{\sqrt{n_k \log \log n_k}} \right) = +\sigma \varepsilon^* \left( -\sigma \varepsilon^* \right) \ a.s.$$
where $\varepsilon^* = \inf \left\{ \varepsilon > 0; \sum_{k=3}^{\infty} (\log n_k)^{-\frac{\varepsilon^2}{2}} < \infty \right\}$. In particular, if $\varepsilon^* = 0$, then,

$$\left( \frac{S_{nk}}{\sqrt{n_k \log \log n_k}} \right) \to 0 \quad \text{a.s., as } k \to \infty.$$

Conversely, suppose that $\varepsilon^* > 0$. If

$$P \left( \limsup_{k \to \infty} \frac{|S_{nk}|}{\sqrt{n_k \log \log n_k}} < \infty \right) > 0,$$

then $EX_1^2 < \infty$ and $EX_1 = 0$.

**Theorem 1.3.11.** [Allan Gut and Spataru, A. [4], Theorem 1, page 1871]

Suppose that $EX = 0$, that $EX^2 = \sigma^2$, that $E \left[ X^2 \times (\log^+ \log^+ |X|)^{1+\delta} \right] < \infty$ for some $\delta > 0$ and that $a_n = O \left( \sqrt{n (\log \log n)^{-\gamma}} \right)$ for some $\gamma > \frac{1}{2}$. Then

$$\lim_{\varepsilon \downarrow \sigma \sqrt{2}} \sqrt{\varepsilon^2 - 2\sigma^2} \sum_{n \geq 3} \frac{1}{n} P \left( |S_n| \geq \varepsilon \sqrt{n \log \log n + a_n} \right) = \sigma \sqrt{2}.$$

**Theorem 1.3.12.** [Chover, J. [11], Theorem, page 441]

Let $\{X_n, n \geq 1\}$ be mutually independent r.v.s, identically distributed according to the symmetric stable distribution with exponent $\alpha$, $0 < \alpha < 2$. Then

$$\limsup_{n \to \infty} \left| n - \frac{1}{\alpha} S_n \right| (\log \log n)^{-1} = \frac{1}{e^2} \quad \text{a.s.}$$

**Theorem 1.3.13.** [Divanji, G. and Vasudeva, R. [16]]

Let $F \in \text{DP} (\alpha), \ 0 < \alpha < 2$. Then there exists a s.v. function $L$ and a function $\theta$ bounded in between two positive numbers $b_1, b_2$, $0 < b_1 \leq b_2 < \infty$, such that

$$\lim_{x \to \infty} \frac{x^\alpha \left( 1 - F(x) + F(-x) \right)}{L(x) \theta(x)} = 1.$$
Theorem 1.3.14. [Divanji, G. and Vasudeva, R. [16]]

Let $F \in DP(\alpha)$, $0 < \alpha < 2$. Then $\limsup_{n \to \infty} \frac{S_n}{B_n} \log \log n = e^\alpha$ a.s.

Theorem 1.3.15. [Heyde, C.C. [31], Theorem, page 1576]

Suppose that $\{X_n, n \geq 1\}$ be a sequence of i.i.d r.v.s which do not belong to the domain of partial attraction of the normal distribution, i.e., those for which

$$\liminf_{u \to \infty} \frac{u^2 P(|X_1| > u)}{\int_{|x| \leq u} x^2 dF(x)} > 0,$$

where $F(x) = P(X_1 \leq x)$. Let $S_n = \sum_{i=1}^n X_i$ and $(y_n)$ be a monotone sequence of positive numbers such that $y_n \to \infty$, as $n \to \infty$, and $\frac{S_n}{y_n} \to 0$. For any $\alpha > 1$, let $\liminf_{u \to \infty} \frac{P(|X_1| > \alpha y_n)}{P(|X| > y_n)} > 0$.

Then $0 < \liminf_{n \to \infty} \frac{P(|S_n| > y_n)}{P(\max_{k \leq n} |X_k| > y_n)} \leq \limsup_{n \to \infty} \frac{P(|S_n| > y_n)}{P(\max_{k \leq n} |X_k| > y_n)} < \infty$.

Theorem 1.3.16. [Kruglov, V.M. [36], page 685]

Let $\{X_n, n \geq 1\}$ be a sequence of i.i.d r.v.s with common d.f. $F$. Set $S_n = \sum_{j=1}^n X_j$, $n \geq 1$ and Let $\{n_k, k \geq 1\}$ be a strictly increasing subsequence of positive integers such that $\frac{n_{k+1}}{n_k} \to r, (\geq 1)$, as $k \to \infty$, if there exist sequences $(a_k)$ and $(b_k)$ of real constants, $b_k \to \infty$, as $k \to \infty$, such that

$$\lim_{k \to \infty} P\left(\frac{S_{n_k}}{b_k} - a_k \leq x\right) = G_\alpha(x)$$

at all continuity points $x$ of $G_\alpha$, then $G_\alpha$ is necessarily a semistable d.f. with characteristic exponent $\alpha$, $0 < \alpha \leq 2$. Here $F$ is said to belong to the domain of partial attraction of $G_\alpha$. 
Theorem 1.3.17. [Lai, T.L. [37]), Theorem 1, page 434]

Let \( \{X_n, n \geq 1\} \) be a sequence of independent random variables such that \( EX_n = 0, \ E X_n^2 = \sigma_n^2 \) and \( \lim_{n \to \infty} \sigma_n^2 = \sigma^2 > 0 \). Let \( 0 < \alpha < 1 \). Suppose, for \( j \geq j_0 \), there exists \( \gamma_j \geq 0 \) such that \( \gamma_j = o \left( \frac{\alpha}{j} \left( \log j \right)^{-\frac{1}{2}} \right) \) and

\[
\exp \left\{ t^2 \sigma_j^2 \left( 1 - \frac{|t| \gamma_j}{2} \right) \right\} \leq E \exp (tX_j) \leq \exp \left\{ t^2 \sigma_j^2 \left( 1 + \frac{|t| \gamma_j}{2} \right) \right\}
\]

whenever \( |t| \gamma_j \leq 1 \). Then

\[
\limsup_{n \to \infty} \frac{X_n + \ldots + X_{n+[\alpha n]}}{2(1 - \alpha)n^\alpha \log n} = \sigma \quad \text{a.e.}
\]

\[
\liminf_{n \to \infty} \frac{X_n + \ldots + X_{n+[\alpha n]}}{2(1 - \alpha)n^\alpha \log n} = -\sigma \quad \text{a.e.}
\]

\[
\limsup_{n \to \infty} \max_{0 \leq j \leq n^\alpha} \frac{|X_n + \ldots + X_{n+j}|}{2(1 - \alpha)n^\alpha \log n} = \sigma \quad \text{a.e.}
\]

Theorem 1.3.18. [Vasudeva, R. and Divanji, G. [70], Lemma 5, page 67]

Let \( S_n = \sum_{k=1}^{n} X_k \) and \( T_n = S_{n+a_n} - S_n \), where \( \{a_n\} \) is non-decreasing sequence of positive integers. Let \( \gamma_n = \left( \log \frac{n}{a_n} + \log \log n \right)^{-1} \). Then

\[
\limsup_{n \to \infty} \frac{T_n}{(a_n^2)^{\gamma_n}} = e^{\alpha} \quad \text{a.s.}
\]