Chapter 6

Existence of Moments for Number of Boundary Crossings related to the Laws of Iterated Logarithm

6.1 Introduction

Let \( \{X_n, n \geq 1\} \) be a sequence of i.i.d. r.v.s. with a common d.f. \( F \). Let \( S_n = \sum_{k=1}^{n} X_k, n \geq 1 \), for \( n \geq 1 \). Let \( \{b_n, n \geq 1\} \) be any sequence of real numbers. We say that \( \{b_n, n \geq 1\} \) belongs to the upper class if \( P(S_n > b_n \text{ i.o.}) = 0 \) and to lower class if \( P(S_m > b_n \text{ i.o.}) = 1 \). Suppose that \( \{b_n, n \geq 1\} \) belongs to the upper class. It would be interesting to find the boundary crossing probability \( P(S_n \geq b_n \text{ for some } n \geq m > 0) \). Such boundary crossing probabilities have a statistical applications in power one tests of one-sided hypothesis and in confidence sequences for the unknown parameters of parametric families of...
distributions. See for example Lai and Lan [38], Robbins [55] and Robbins and Siegmund [54].

In brief, if \( b(n, \varepsilon) = (1 + \varepsilon) (2n \log \log n)^{1/2} \), for \( n \geq 3 \), then by classic law of iterated logarithm due to Hartman and Wintner [30] asserts that with probability one, the inequality \( |S_n - n\mu| \geq \sigma b(n, \varepsilon) \) will hold for finitely many \( n \)-values when \( \varepsilon > 0 \) and for infinitely many \( n \)-values when \( \varepsilon < 0 \). Let \( \{Y_n(\varepsilon), n \geq 3\} \) be a sequence of indicator random variables defined by,

\[
Y_n(\varepsilon) = \begin{cases} 
1, & \text{if } |S_n - n\mu| \geq \sigma b(n, \varepsilon) \\
0, & \text{otherwise}
\end{cases}
\]

Let \( \{N_m(\varepsilon), m \geq 3\} \) be the corresponding sequence of partial sums, i.e. \( N_m(\varepsilon) = \sum_{n=3}^{m} Y_n(\varepsilon) \), for \( m \geq 3 \). If \( N_\infty(\varepsilon) = \sum_{n=3}^{\infty} Y_n(\varepsilon) \), then the LIL asserts that \( P(N_\infty(\varepsilon) < \infty) = 1 \) for \( \varepsilon > 0 \), while \( P(N_\infty(\varepsilon) = \infty) = 1 \) for \( \varepsilon < 0 \). Which says that if \( \varepsilon \) is positive so that \( N_\infty(\varepsilon) \) is a proper r.v. or equivalently \( N_\infty(\varepsilon) \) has proper distribution. Observe that \( E\left(N_\infty^0\right) = P(N_\infty(\varepsilon) < \infty) = 1 \), The zero\(^{th}\) moment of \( N_\infty(\varepsilon) \) exits and hence the question arises whether \( N_\infty(\varepsilon) \) possesses any moments of positive order. In this chapter we study the moments of these boundary crossing random variables related to LIL considered in chapter 3, chapter 4 and increments of stable subordinators.
A similar study of the moments of $N_{\infty}(\varepsilon)$ is made for LIL results in the process, we first present a theorem of Slivka and Savero [59], in a more generalized form, so that we can appeal to this theorem as and when needed.

Let $\{\xi_n, n \geq 1\}$ be a sequence of r.v.s and $\{A_n, n \geq 1\}$ be a sequence of sets on the real line $\mathbb{R}$ such that $P(\xi_n \in A_n \text{ i.o.}) = 0$. Let $\{I(A_n), n \geq 1\}$ be a sequence of indicator r.v.s which are defined by,

$$I(A_n) = \begin{cases} 
1, & \text{if } \xi_n \in A_n \\
0, & \text{otherwise}
\end{cases}$$

and let $\{N_m, m \geq 1\}$ be the corresponding sequence of partial sum i.e. $N_m = \sum_{n=1}^{m} I(A_n)$. If $N_{\infty} = \sum_{n=1}^{\infty} I(A_n)$, we know that $P(N_{\infty} < \infty) = 1$ or $N_{\infty}$ is a proper r.v.

**Theorem 6.1.1.** For any $\lambda \geq 1$, $EN_{\infty}^\lambda < \infty$ whenever $\sum_{n=1}^{\infty} n^{\lambda-1} P(\xi_n \in A_n) < \infty$

**Proof.** The proof is on the lines of Slivka and Savero [59] and hence omitted. \qed

### 6.2 Moments of number of boundary crossings related to delayed random sums

Here we study the existence of moments for boundary crossing r.v.s related to LIL of Theorem (3.2.1), (3.3.1) and (3.4.1) of chapter 3.
Define for any $\varepsilon > 0$,

$$
Y_n(\varepsilon) = \begin{cases} 
1, & \text{if } M_{t_n} \geq t_n^{\frac{1}{\alpha}} \left( \frac{n}{t_n} \log n \right)^{\frac{1 + \varepsilon_1}{\alpha}} \\
0, & \text{otherwise}
\end{cases}
$$

Let $N_m(\varepsilon)$ be the partial sum sequence of $Y_n(\varepsilon)$, i.e. $N_m(\varepsilon) = \sum_{n=1}^{m} Y_n(\varepsilon)$, for $m \geq 3$, observe that from (3.2.1) of Theorem (3.2.1), $N_\infty(\varepsilon)$ is a proper r.v. Here we show that all the moments in $0 < \lambda \leq 1$ are finite for this proper r.v.

**Theorem 6.2.1.** Let $\{X_n, n \geq 1\}$ be a sequence of i.i.d. positive asymmetric stable r.v.s., with common d.f. $F$, with index $\alpha$, $0 < \alpha < 1$. Let $\{t_n, n \geq 1\}$ be a sequence of positive integer r.v.s. independent of $\{X_n, n \geq 1\}$ such that

$$
\left| e^{t_n} - 1 \right| \rightarrow 0 \text{ a.s as } n \rightarrow \infty, \text{ where } 0 < \delta < 1.
$$

Then for any $\varepsilon > 0$ and for any $\lambda$, $0 < \lambda \leq 1$, $EN_\infty^\lambda < \infty$ whenever

$$
\sum_{n \geq 3} n^{\lambda - 1} P \left( M_{t_n} \geq t_n^{\frac{1}{\alpha}} \left( \frac{n}{t_n} \log n \right)^{\frac{1 + \varepsilon_1}{\alpha}} \right) < \infty
$$

where $M_{t_n} = S_{n+t_n} - S_n$.

**Proof.** First we show that, for $\lambda = 1$, $EN_\infty(\varepsilon) < \infty$ and then claim that the existence of lower moments follows from higher moments. From the Theorem (6.1.1), identifying $\xi_n$ with $M_{t_n}$, $A_n$ with $\left[ t_n^{\frac{1}{\alpha}} \left( \frac{n}{t_n} \log n \right)^{\frac{1 + \varepsilon_1}{\alpha}}, \infty \right)$ and $N_\infty$ with
we can find that for the LIL in Theorem (3.2.1), $EN_\infty(\varepsilon) < \infty$ in view of the following arguments. We know by Theorem (6.1.1), $EN_\infty(\varepsilon) < \infty$, whenever

$$
\sum_{n=3}^{\infty} P \left( M_{t_n} \geq \frac{1}{n} \left( \frac{n}{t_n} \log n \right)^{\frac{1+\varepsilon}{\alpha}} \right) < \infty.
$$

From the proof of (3.2.1) of Theorem (3.2.1) and by (3.2.11), we have

$$
\sum_{n=3}^{\infty} P \left( M_{t_n} \geq \frac{1}{n} \left( \frac{n}{t_n} \log n \right)^{\frac{1+\varepsilon}{\alpha}} \right) \leq C_1 \sum_{n=3}^{\infty} \frac{1}{n (\log n)^{1+\varepsilon_1}} < \infty,
$$

which establishes that $EN_\infty(\varepsilon) < \infty$, for $\lambda = 1$ and therefore $EN_\infty^\lambda < \infty$ for $\lambda < 1$. Hence proof of the Theorem is completed.

Define for any $\varepsilon > 0$,

$$
Y_n(\varepsilon) = \begin{cases} 
1, & \text{if } M_{t_n} \leq \frac{1}{n} \left( \frac{n}{t_n} \log n \right)^{-\varepsilon} \\
0, & \text{otherwise}
\end{cases}
$$

Let $N_m(\varepsilon)$ be the partial sum sequence of $Y_n(\varepsilon)$, i.e. $N_m(\varepsilon) = \sum_{n=1}^{m} Y_n(\varepsilon)$, for $m \geq 3$, observe that from (3.3.1) of Theorem (3.3.1), $N_\infty(\varepsilon)$ is a proper r.v.

Here we show that all the moments in $\lambda \in (0, 1]$ are finite for this proper r.v.

**Theorem 6.2.2.** Let $\{X_n, n \geq 1\}$ be a sequence of i.i.d. positive asymmetric stable r.v.s., with common d.f. F, with index $\alpha$, $0 < \alpha < 1$. Let $\{t_n, n \geq 1\}$ be a sequence of positive r.v.s. independent of $\{X_n, n \geq 1\}$ such that
\[
\left| e^{\frac{t_n}{N_n}} - 1 \right| \to 0 \text{ a.s as } n \to \infty, \text{ where } 0 < \delta < 1. \text{ Let } \gamma_n = \left( \log \frac{n}{N_n} + \log \log n \right) \]
and \[
\lim \inf_{n \to \infty} \left( \frac{M_{N_n}}{N_n} \right)^{\frac{1}{n}} = 1 \text{ a.s.}
\]

Then for any \( \varepsilon > 0 \) and for any \( \lambda, 0 < \lambda \leq 1 \), \( EN_{\lambda \infty} \) \( < \infty \) whenever
\[
\sum_{n \geq 3} n^{\lambda - 1} P \left( M_{t_n} \leq t_n \left( \frac{n}{t_n} \log n \right)^{-\varepsilon} \right) = 0.
\]

**Proof.** Here also first we show that for \( \lambda = 1 \), \( EN_{\infty} (\varepsilon) < \infty \) and then claim that the existence of lower moments follows from higher moments.

From the Theorem (6.1.1), identifying \( \xi_n \) with \( M_{t_n} \), \( A_n \) with \( \left( -\infty, t_n^{\frac{1}{\varepsilon}} \left( \frac{n}{t_n} \log n \right)^{-\varepsilon} \right] \)
and \( N_{\infty} \) with \( N_{\infty} (\varepsilon) \). By Theorem (6.1.1) we have \( EN_{\infty} (\varepsilon) < \infty \), whenever
\[
\sum_{n \geq 3} P \left( M_{t_n} \leq t_n \left( \frac{n}{t_n} \log n \right)^{-\varepsilon} \right) < \infty.
\]

From the proof of (3.3.1) of Theorem (3.3.1), we can find some constant \( C_2 > 0 \) such that,
\[
\sum_{n \geq 3} P \left( M_{t_n} \leq t_n \left( \log n \right)^{-\varepsilon} \right) \leq C_2 \sum_{k \geq 1} \frac{1}{k^{b(1+\varepsilon)}} < \infty,
\]
where \( b < 1 \). Which proves that \( EN_{\infty} (\varepsilon) < \infty \), for \( \lambda = 1 \) and therefore \( EN_{\lambda \infty} \) \( < \infty \) for \( \lambda < 1 \). Hence proof of the Theorem is completed. \( \square \)
6.3 Moments of number of boundary crossings related to subsequences of upper sums limit

Here we study the existence of moments for boundary crossing r.v.s related to LIL of Theorem (4.2.1) and Theorem (4.2.2) of chapter 4.

In chapter 4, it is assumed that \( \{X_n, n \geq 1\} \) is a sequence of i.i.d. non r.v.s. which are in the domain of partial attraction of a semi stable law and the following Theorems are proved, which we club together and write as a Theorem 6.3.1.

Define for any \( \varepsilon > 0 \),

\[
Y_n(\varepsilon) = \begin{cases} 
1, & \text{if } M_{t_{n_k}} \geq B_{t_{n_k}} \left( \frac{n_k}{l_{n_k}} \log n_k \right)^{\frac{\theta + 1}{\alpha}} \\
0, & \text{otherwise}
\end{cases}
\]

Let \( N_m(\varepsilon) \) be the partial sum sequence of \( Y_n(\varepsilon) \), i.e. \( N_m(\varepsilon) = \sum_{n=1}^{m} Y_n(\varepsilon) \), for \( m \geq 3 \), observe that from (4.2.1) of Theorem (4.2.1), \( N_\infty(\varepsilon) \) is a proper r.v.

Here we show that all the moments in \( \lambda \in (0,1] \) are finite for this proper r.v.

**Theorem 6.3.1.** Let \( \{X_n, n \geq 1\} \) be a sequence of i.i.d. non-negative r.v.s., with common d.f. \( F \) and assume that \( F \) is in the domain of partial attraction of a semi stable law with index \( \alpha, 0 < \alpha < 2 \). Let \( \{n_k, k \geq 1\} \) be an integer
subsequence and \( \{ t_{n_k} \} \) be a sequence of positive integer r.v.s independent of
\( \{ X_n, n \geq 1 \} \) such that \( e^{t_{n_k}/n_k} - 1 \to 0 \) a.s as \( k \to \infty \), where \( 0 < \delta < 1 \). a.s.

\[
\limsup_{k \to \infty} \left( \frac{M_{t_{n_k}}}{B_{t_{n_k}}} \right)^{1/n_k} = e^{\theta} \quad \text{a.s.,}
\]

where \( \theta = \begin{cases} 
\varepsilon^*, & \text{if } \limsup_{k \to \infty} \frac{n_k}{n_{k+1}} < 1 \\
1, & \text{if } \liminf_{k \to \infty} \frac{n_k}{n_{k+1}} > 0,
\end{cases} \)

\( \varepsilon^* = \inf \left\{ \varepsilon > 0 : \sum_{k=k_0}^{\infty} (\log n_k)^{-\varepsilon} < \infty \right\} \), \( M_{t_{n_k}} = S_{n_k+t_{n_k}} - S_{n_k} \), \( \gamma_n = \left\{ \log \frac{n}{t_n} + \log \log n \right\} \)

and \( B_n \) be the smallest root of the equation \( n(1 - F(x) + F(-x)) = 1 \). Then

\( B_n = n^{1/\lambda} l(n) \eta(n) \), where \( l \) is a function s.v. at \( \infty \) and \( \eta \) is bounded between two positive constants.

Then for any \( \varepsilon > 0 \) and for any \( \lambda, 0 < \lambda \leq 1 \), \( EN^\lambda_{\infty} < \infty \) whenever

\[
\sum_{n \geq 3} n^{\lambda - 1} P \left( M_{t_{n_k}} \geq B_{t_{n_k}} \left( \frac{n_k}{t_{n_k}} \log n_k \right)^{\theta + \varepsilon_1} \right) < \infty.
\]

**Proof.** The above Theorem is equivalent to, for any \( \varepsilon_1 > 0 \)

\[
P \left( M_{t_{n_k}} \geq B_{t_{n_k}} \left( \frac{n_k}{t_{n_k}} \log n_k \right)^{\theta + \varepsilon_1} \right) = 0
\]

and

\[
P \left( M_{t_{n_k}} \geq B_{t_{n_k}} \left( \frac{n_k}{t_{n_k}} \log n_k \right)^{\theta - \varepsilon_1} \right) = 1
\]
From (4.2.4) and (4.2.7), we can find correspond $N_\infty(\varepsilon)$ as,

$$N_\infty(\varepsilon) = \sum_{n=3}^{\infty} I \left( M_{t_{n_k}} \geq (C_3 - \lambda) B_{t_{n_k}} \left( \log n_k \right)^{\theta+1} \right),$$

where $C_3 (> 0)$.

Identifying $b_n = (C_3 - \lambda) B_{t_{n_k}} \left( \log n_k \right)^{\theta+1}$ and appealing to Theorem (6.1.1) above, for any $\lambda$ in $(0, 1]$, we claim that $E N_\infty^\lambda(\varepsilon) < \infty$ by showing that

$$\sum_{n\geq3} n^{\lambda-1} P \left( M_{t_{n_k}} \geq B_{t_{n_k}} \left( \frac{n_k}{t_{n_k}} \log n_k \right)^{\theta+1} \right) < \infty.$$

First we show that for $\lambda = 1$, $E N_\infty(\varepsilon) < \infty$ and then claim that the existence of lower moments follows from higher moments.

By equations (4.2.4), (4.2.7) and (4.2.9) of Theorem (4.2.1) and similar arguments of Theorem (4.2.2), we can find some constants $C_1 > 0$ and $k_1 > 0$ such that, for all $k \geq k_1$,

$$E N_\infty(\varepsilon) \leq C_1 \sum_{k \geq k_1} \frac{1}{(\log n_k)^{(\theta+1)}} < \infty,$$

which implies $E N_\infty^\lambda < \infty$ for $\lambda < 1$. Hence proof of the Theorem is completed.
6.4 Moments of number of boundary crossings related to subsequences of lower sums limit

In this section we study some boundary crossings r.v.s. related to Theorem 4.3.1 and Theorem 4.3.2. Here we consider $X_n$'s are positive asymmetric stable r.v.s, with d.f. $F$, with index $\alpha$, $0 < \alpha < 1$.

Define for any $\varepsilon > 0$,

$$Y_k(\varepsilon) = \begin{cases} 
1, & \text{if } M_{tn_k} \leq (\theta \pm \varepsilon) \beta(t_{nk}) \\
0, & \text{otherwise}
\end{cases}$$

where

$$\theta = \begin{cases} 
\varepsilon^*, & \text{if } \{n_k\} \text{ is at least geometrically fast} \\
1, & \text{if } \{n_k\} \text{ is at most geometrically fast},
\end{cases}$$

$$M_{tn_k} = S_{tn_k} - S_{n_k}$$

and $\varepsilon^* = \sup \left\{ \varepsilon_1 > 0 : \sum_{k \geq k_0} (\log n_k)^{-\varepsilon_1} \frac{1}{\alpha - 1} < \infty \right\}$, for some $k_0 > 0$ and $\{t_n, n \geq 1\}$ be a sequence of positive r.v.s. independent of $\{X_n, n \geq 1\}$ such that $\left| \frac{t_n}{n^\delta} - 1 \right| \to 0$ a.s as $n \to \infty$, where $0 < \delta < 1$. For any $\varepsilon > 0$, we have from (4.3.1),

$$P \left( M_{tn_k} \leq (\varepsilon^* - \varepsilon) \beta(t_{nk}) \right. \left. \text{i.o.} \right) = 0 \quad (6.4.1)$$
Define $N_\infty(\varepsilon) = \sum_{k=1}^{\infty} Y_k(\varepsilon)$ and observe that $N_\infty(\varepsilon)$ is a proper r.v. in view of (6.4.1). Let $\{N_{m_k}(\varepsilon)\}$ be the corresponding sequence of partial sums. i.e. $N_{m_k}(\varepsilon) = \sum_{k=1}^{m_k} Y_k(\varepsilon)$, where $\{m_k, k \geq 1\}$ is a subsequence of integer sequence.

If $N_\infty(\varepsilon) = \sum_{k=1}^{\infty} Y_k(\varepsilon)$, we know that $P(N_\infty(\varepsilon) < \infty) = 1$ or $N_\infty(\varepsilon)$ is a proper r.v. of number of boundary crossings and hence it is interesting to study the existence of moments for this boundary crossings r.v.s. and obtain moments of this proper r.v. $N_\infty(\varepsilon)$ as Corollary to Theorem 4.3.1 and Theorem 4.3.2.

**Corollary 6.4.1.** For $\varepsilon > 0$ and for any $\eta$, $0 < \eta \leq 1$, $EN_\infty^{\eta} < \infty$, if

$$\sum_{k=1}^{\infty} n_k^{\eta-1} P \left( M_{t_{n_k}} \leq (\theta \pm \varepsilon) \beta \left( t_{n_k} \right) \right) < \infty.$$

**Proof.** First we show that for $\eta = 1$, $EN_\infty < \infty$ and then claim that the existence of lower moments follows from that of the higher moments.

Observe that

$$EN_\infty(\varepsilon) = \sum_{k \geq 1} P \left( M_{t_{n_k}} \leq (\theta \pm \varepsilon) \beta \left( t_{n_k} \right) \right).$$

Following similar arguments of the proof of (4.3.1) and (4.3.21) we can find some constant $C_i > 0$ and some $k_i > 0$ such that, for all $k \geq k_i$,

$$EN_\infty(\varepsilon) \leq C_i \sum_{k \geq k_i} \frac{1}{(\log n_k) \left( \theta \pm \varepsilon \right)}.$$
and by the definition of $\theta$, yields $EN_\infty(\varepsilon) < \infty$ for $\eta = 1$. Consequently we have $EN_\infty^n < \infty$ for $0 < \eta \leq 1$. Thus the proof of the corollary is completed. \qed

6.5 Moments of number of boundary crossings for increments of stable subordinators

Here we study the existence of moments for boundary crossing r.v.s. related to LIL of increments of stable sub subordinators.

Let $\{X(t), 0 \leq t < \infty\}$ be a completely asymmetric stable process or stable subordinators defined on a common probability space $(\Omega, \mathcal{F}, P)$. Let $a_t$, $t > 0$ be a non-negative valued function of $t$ such that (i) $0 < a_t \leq t$ (ii) $a_t \to \infty$, as $t \to \infty$ and (iii) $\frac{a_t}{t} \downarrow 0$, as $t \to \infty$. Let $Y(t) = X(t + a_t) - X(t)$, $t > 0$ and $Y(0) = 0$. Define $\beta(t) = \theta_\alpha a_t^{\frac{1}{\alpha}} \left(\log \frac{t}{a_t} + \log \log t\right)^{\frac{\alpha - 1}{\alpha}}$, where $\theta_\alpha = (B(\alpha))^{\frac{1 - \alpha}{\alpha}}$

and $B(\alpha) = (1 - \alpha)\alpha^{\alpha-1} \left(\cos\left(\frac{\alpha\pi}{2}\right)\right)^{\frac{1}{\alpha-1}}$, $0 < \alpha < 1$. Observe that the process has the property that $t^{\frac{1}{\alpha}}X(t)$ and $X(1)$ are identically distributed.

Theorem 6.5.1. Let $a_t$, $t > 0$, be a non-decreasing function of $t$ such that

(i) $0 < a_t \leq t$ (ii) $a_t \to \infty$, as $t \to \infty$ and (iii) $\frac{a_t}{t} \downarrow 0$ as $t \to \infty$. Let $(t_k)$ be an
increasing sequence of positive integers such that

\[
\eta = \begin{cases} 
\varepsilon^*, & \text{if } \limsup_{k \to \infty} \frac{t_{k+1} - t_k}{a_{t_k}} < 1 \\
1, & \text{if } \liminf_{k \to \infty} \frac{t_{k+1} - t_k}{a_{t_k}} > 1
\end{cases}
\]

Then

\[
\liminf_{k \to \infty} \frac{Y(t_k)}{\beta(t_k)} = \eta \text{ a.s.}
\]

where \(\varepsilon^* = \sup \left\{ \varepsilon > 0 : \sum_{k=k_0}^{\infty} \left( g(t_k) \right)^{-\varepsilon^{-\lambda}} < \infty \right\}, \quad g(t) = \left( \frac{t_{k+1}}{a_{t_k}} \log t_k \right)
\)

and \(\lambda = \frac{\alpha}{\alpha - 1} < 0, \quad 0 < \alpha < 1\).

Then for any \(\varepsilon > 0\) and for any \(b, 0 < b \leq 1\), \(EN^b_\infty < \infty\) whenever

\[
\sum_{k \geq 3} t_{k}^{b-1} P (Y(t_k) \leq (\eta - \varepsilon_1) \beta(t_k)) < \infty.
\]

**Proof.** The above Theorem is equivalent to, for any \(\varepsilon_1 > 0\)

\[
P (Y(t_k) \leq (\eta + \varepsilon_1) \beta(t_k) \text{ i.o.}) = 1 \quad (6.5.1)
\]

and

\[
P (Y(t_k) \leq (\eta - \varepsilon_1) \beta(t_k) \text{ i.o.}) = 0 \quad (6.5.2)
\]

Correspondingly, \(N_\infty(\varepsilon)\) can be defined as,

\[
N_\infty(\varepsilon) = \sum_{k=3}^{\infty} I (Y(t_k) \leq (\eta - \varepsilon_1) \beta(t_k))
\]

Identifying \(b_n = \beta(t)\) and appealing to Theorem (6.1.1) above, for any \(b\) in \((0, 1]\),

we claim that \(EN^b_\infty(\varepsilon) < \infty\) by showing that
\[
\sum_{k=3}^{\infty} t_k^{b-1} P(Y(t_k) \leq (\eta - \varepsilon_1) \beta(t_k)) < \infty.
\]

First we show that for \( b = 1 \), \( EN_\infty(\varepsilon) < \infty \) and then claim that the existence of lower moments follows from higher moments.

We now complete the proof by showing that, for any \( \varepsilon_1 \in (0, 1) \),

\[
P\left(Y(t_k) \leq (\eta - \varepsilon_1) \beta(t_k) \text{ i.o.}\right) = 0.\]

By the definition of \( \eta \), we have \( t_{k+1} \leq t_k + a_{t_k} \), for large \( k \) and from Mijnheer [46], we can find a \( k_1 \) such that for all \( k \geq k_1 \),

\[
P\left(Y(t_k) \leq (\eta - \varepsilon_1) \beta(t_k) \text{ i.o.}\right) = P\left(X(t_k + a_{t_k}) - X(t_k) \leq (\eta - \varepsilon_1) \beta(t_k) \text{ i.o.}\right).
\]

Hence in order to prove (6.5.2), it is enough if we show that

\[
P\left(X(t_k + a_{t_k}) - X(t_k) \leq (\eta - \varepsilon_1) \beta(t_k) \text{ i.o.}\right) = 0 \quad (6.5.3)
\]

We know that \( t^{-\frac{1}{\alpha}} X(t) \overset{d}{=} X(1) \) which implies

\[
P\left(X(t_k + a_{t_k}) - X(t_k) \leq (\eta - \varepsilon_1) \beta(t_k)\right) = P\left( X(1) \leq \frac{(\varepsilon - \varepsilon_1) \beta(t_k)}{(a_{t_k})^{\frac{1}{\alpha}}} \right)
\]

\[
(6.5.4)
\]

and

\[
\frac{(\eta - \varepsilon_1) \beta(t_k)}{(a_{t_k})^{\frac{1}{\alpha}}} \approx (\eta - \varepsilon_1) \theta_{\alpha} \left( \log \left( \frac{t_k}{a_{t_k} \log t_k} \right) \right)^{\frac{\alpha-1}{\alpha}}
\]
By taking \( x = (\eta - \epsilon_1) \theta_\alpha (\log g(t_k))^{\frac{\alpha - 1}{\alpha}} \), where \( g(t_k) = \frac{a}{\alpha t_k} \log t_k \), in the above lemma, one can find a \( k_2 > 0 \) and \( C_1 > 0 \) such that for all \( k \geq k_2 \),

\[
P \left( X(1) \leq \frac{(\eta - \epsilon_1) \beta(t_k)}{(a t_k)^{\frac{1}{\alpha}}} \right) \leq \frac{C_1}{\left( \log g(t_k) \right)^{\frac{1}{2}}} \exp \left\{ (\eta - \epsilon_1)^{\frac{\alpha}{\alpha - 1}} \log g(t_k) \right\}.
\]

Observe that using properties of \( \{a_t\} \) one can find some constant \( C_2 > C_1 \) and \( k_3 \) such that for all \( k \geq k_3 \), we have,

\[
P \left( X(1) \leq \frac{(\eta - \epsilon_1) \beta(t_k)}{(a t_k)^{\frac{1}{\alpha}}} \right) \leq \frac{C_2}{\left( g(t_k) \right)^{\frac{\alpha}{\alpha - 1}}}.
\]

Notice that \( \epsilon^* = \sup \left\{ \epsilon > 0 : \sum_{k=k_0}^{\infty} (g(t_k))^{-\epsilon - \lambda} < \infty \right\} \) and \( \lambda = \frac{\alpha}{\alpha - 1} < 0 \), \( 0 < \alpha < 1 \) which yields \( \epsilon^* \geq 1 \). Since \( \epsilon_1 \in (0, 1) \), choose \( \epsilon_1 \) sufficiently small, one can find \( k_4 \geq k_5 \) such that for all \( k \geq k_4 \),

\[
\sum_{k=k_4}^{\infty} P \left( X(1) \leq \frac{(\eta - \epsilon_1) \beta(t_k)}{(a t_k)^{\frac{1}{\alpha}}} \right) \leq \sum_{k=k_4}^{\infty} \frac{C_2}{\left( g(t_k) \right)^{(\eta - \epsilon_1) - \lambda}} < \infty,
\]

where \( \lambda = \frac{\alpha}{\alpha - 1} < 0 \), \( 0 < \alpha < 1 \),

which implies \( E N^b_\infty < \infty \) for \( b < 1 \). Hence proof of the Theorem is completed.

\( \square \)

Section 6.2 communicated to journal with title “A log log law for delayed random sums” by Gooty Divanji and Raviprakash, K. N.
Section 6.3 communicated to journal with title “A log log law for subsequences of delayed random sums” by Gooty Divanji and Raviprakash, K. N. [29].

Section 6.4 forms the content of “Limit infimum results for subsequences of delayed random sums and related boundary crossing problem”, by Gooty Divanji and Raviprakash, K. N. [27].

Section 6.5 forms the content of “Almost sure limit inferior for increments of stable subordinators”, by Vidyalaxmi, K., Raviprakash, K.N. and Gooty Divanji [71].