Chapter 4

A Log Log Law for Subsequences of Delayed Random Sums

4.1 Introduction

Let \( \{X_n, n \geq 1\} \) be a sequence of i.i.d. r.v.s with a common d.f \( F \) and let
\[
S_n = \sum_{k=1}^{n} X_k, \forall n \geq 1. \quad \text{Set} \quad T_n = \sum_{i=n+1}^{n+a_n} X_i = S_{n+a_n} - S_n, \quad \text{where} \quad a_n, \forall n \geq 1, \quad \text{be a}
\]
non-decreasing function of a positive integers of \( n \) such that \( 0 < a_n \leq n \), for all \( n \) and \( \frac{a_n}{n} \sim b_n \), where \( b_n \) is non-increasing. Let \( \gamma_n = \left\{ \log \frac{n}{a_n} + \log \log n \right\} \). The sequence \( \{T_n, n \geq 1\} \) is called a (forward) delayed sum sequence (see Lai [37]).

Now parallel to the delayed sums \( \{T_n\} \), we introduce delayed random sums as \( M_{N_n} = \sum_{j=n+1}^{n+N_n} X_j = S_{n+N_n} - S_n \), where \( \{N_n, n \geq 1\} \) be a sequence of positive integer r.v.s independent of \( \{X_n, n \geq 1\} \) such that \( \left| \frac{N_n}{e^{n^\delta}} - 1 \right| \to 0 \) a.s. as \( n \to \infty \), where \( 0 < \delta < 1 \).

Let \( \{\tau_k, k \geq 1\} \) be a strictly increasing sequence of positive integers such
that \( \tau_k^{-1}\tau_{k+1} \to r(\geq 1) \) as \( k \to \infty \). Kruglov [36] has established that, if there exists subsequence \( \{a_k\} \) and \( \{b_k\} \) of real constants, \( b_k \to \infty \) as \( k \to \infty \), such that

\[
\lim_{k \to \infty} P (b_k^{-1} S_{\tau_k} - a_k \leq x) = G_\alpha(x) \quad (4.1.1)
\]

at all continuity points \( x \) of \( G_\alpha \), then \( G_\alpha \) is necessarily a semi stable law with characteristic exponent \( \alpha \), \( 0 < \alpha < 2 \). Here \( F \) is said to belong to the domain of partial attraction of a semi stable law \( G_\alpha \) and we denote the same as \( F \in DP(\alpha) \), \( 0 < \alpha < 2 \).

When \( X_n \)'s are i.i.d symmetric stable r.v.s, Chover [11] established the law of iterated logarithm (LIL) for \( \{S_n\} \), by normalizing in the power. Divanji and Vasudeva [16] extended the same to the case of \( F \in DP(\alpha) \), \( 0 < \alpha < 2 \). When \( E X_n^2 < \infty \), Allan Gut [2] established the classical LIL for geometrically fast increasing subsequences of \( \{S_n\} \). Ingrid Torrang [34] extended the same to random subsequences.

When variance is finite, Lai [37] has studied the behavior of classical LIL for properly normalized delayed sums \( \{T_{a_n}\} \), at different \( a_n \)'s. He proved that these results are entirely different from LIL for partial sums. For independent but not identically distributed strictly positive stable r.v.s Vasudeva and Divanji
[69] studied a non-trivial limit behavior of delayed sums \( \{ T_{n_k} \} \).

When r.v.s. are i.i.d. positive asymmetric stable law with index \( \alpha \), \( 0 < \alpha < 1 \), Gooty Divanji and Raviprakash [26] studied a non-trivial behaviour for properly normalized delayed random sums \( \{ M_{N_n}, n \geq 1 \} \) in chapter 3. Observations made by Allan Gut [2] motivated us to examine whether Chover’s form of LIL for subsequences of properly normalized delayed random sums, can be obtained. We answer in this chapter affirmatively.

In the sequel we assume that \( a_k = 0 \) in (4.1.1). When \( \alpha < 1 \), \( a_k \) can always be chosen to be zero. When \( \alpha > 1 \), \( a_k \) becomes \( nEX_1 \). Here we can make \( a_k = 0 \) by shifting \( EX_1 \) to zero. Consequently, the condition \( a_k = 0 \) is no condition at all when \( \alpha \neq 1, \ 0 < \alpha < 2 \). However, when \( \alpha = 1 \), this assumption restricts only to symmetric d.f.s.

In the next section, we obtain LIL in power normalization for \( \{ M_{N_{n_k}} \} \) and linear normalization for \( \{ M_{N_{n_k}} \} \) in the last section.
4.2 LIL in the power normalization for geometrically fast increasing subsequences.

Theorem 4.2.1. Let \( \{X_n, n \geq 1\} \) be a sequence of i.i.d. non-negative r.v.s. with a common d.f. \( F \in DP(\alpha), 0 < \alpha < 2 \). Let \( \{n_k, k \geq 1\} \) be an integer subsequence such that \( \limsup_{k \to \infty} \frac{n_k}{n_{k+1}} < 1 \) and let \( \{N_k\} \) be a sequence of positive integer r.v.s. independent of \( \{X_n, n \geq 1\} \) such that \( \frac{N_k}{n_k} \to 0 \) a.s. as \( k \to \infty \), where \( 0 < \delta < 1 \). Then,

\[
\limsup_{k \to \infty} \left( \frac{M_{N_k}}{B_{N_k}} \right)^{\frac{1}{\gamma_k}} = e^{\varepsilon^*} \text{ a.s.,}
\]

where \( \varepsilon^* = \inf \left\{ \varepsilon > 0 : \sum_{k=k_0}^{\infty} (\log n_k)^{-\varepsilon} < \infty \right\} \), \( M_{N_k} = S_{n_k+N_k} - S_{n_k} \) and \( \gamma_k = \left\{ \log \frac{n_k}{N_k} + \log \log n_k \right\} \) and \( B_{n_k} \) be the smallest root of the equation \( n_k(1 - F(x) + F(-x)) = 1 \). Then \( B_{n_k} = n_k^{\frac{1}{2}} l(n_k)\eta(n_k) \), where \( l \) is a function s.v. at \( \infty \) and \( \eta \) is bounded in between two positive constants.

Proof. To prove the assertion it is necessary and sufficient to prove that for all sufficiently small \( \varepsilon_1 > 0 \),

\[
P \left( M_{N_k} \geq B_{N_k} \left( \frac{n_k}{N_k} \log n_k \right)^{\frac{\varepsilon + \varepsilon_1}{\alpha}} \text{ i.o.} \right) = 0 \tag{4.2.1}
\]
and

\[
P \left( M_{n_{nk}} \geq B_{n_{nk}} \left( \frac{n_k}{N_{n_k}} \log n_k \right)^{\frac{\varepsilon - \varepsilon_1}{\alpha}} \right. \text{ i.o.} \bigg) = 1 \tag{4.2.2}
\]

From the condition \( \left| \frac{N_{n_k}}{n_k} - 1 \right| \to 0 \text{ a.s. as } k \to \infty \), implies that there exists some \( \varepsilon > 0 \) and \( \delta \in (0, 1) \), such that \( \left| \frac{N_{n_k}}{n_k} - 1 \right| < \varepsilon \), which implies

\[
u_{n_k} \leq N_{n_k} \leq \upsilon_{n_k} \text{ a.s.,} \tag{4.2.3}
\]

where \( \nu_{n_k} = C_1 n^\delta \) and \( \upsilon_{n_k} = C_2 n^\delta, C_1 = \log(1 - \varepsilon) \) and \( C_2 = \log(1 + \varepsilon) \).

We know that \( M_{n_{nk}} = \sum_{j=n_k+1}^{n_k+N_{n_k}} X_j = S_{n_k+N_{n_k}} - S_{n_k} \). Since \( \left\{ N_{n_k}, k \geq 1 \right\} \) is a sequence positive valued r.v.s. independent of \( \{X_n, n \geq 1\} \), by (4.2.3) we have

\( S_{\nu_{n_k}} \leq S_{n_{nk}} \leq S_{\upsilon_{n_k}} \text{ a.s.} \) which implies \( \nu_{n_k} \leq M_{n_{nk}} \leq \upsilon_{n_k} \text{ a.s.} \) and hence (4.2.1) and (4.2.2) hold whenever,

\[
P \left( M_{\nu_{n_k}} \geq B_{n_{nk}} \left( \frac{n_k}{N_{n_k}} \log n_k \right)^{\frac{\varepsilon + \varepsilon_1}{\alpha}} \right. \text{ i.o.} \bigg) = 0, \tag{4.2.4}
\]

where \( M_{\nu_{n_k}} = S_{n_k+N_{n_k}} - S_{n_k} \)

and

\[
P \left( M_{\upsilon_{n_k}} \geq B_{n_{nk}} \left( \frac{n_k}{N_{n_k}} \log n_k \right)^{\frac{\varepsilon - \varepsilon_1}{\alpha}} \right. \text{ i.o.} \bigg) = 1, \tag{4.2.5}
\]

where \( M_{\upsilon_{n_k}} = S_{n_k+N_{n_k}} - S_{n_k} \).

Since \( B_{n_k} \) monotonically increases with \( B_{n_k} = \frac{1}{n_k} l(n_k) \eta(n_k) \), where, \( l \) is
s.v. at infinity and $\eta$ is bounded in between two positive constants. By (4.2.3), we have for some constant $C_3 (> 0)$, \( \frac{B_{n_k}}{B_{n_k}} \to C_3 \) as \( n \to \infty \), which implies, there exists some $\lambda > 0$, such that,

\[
(C_3 - \lambda) \leq \frac{B_{n_k}}{B_{n_k}} \leq (C_3 + \lambda).
\]

and to prove (4.2.4) and (4.2.5), it is sufficient to prove that,

\[
P\left( M_{v_n k} \geq (C_3 - \lambda)B_{n_k} \left( \log n_k \right)^{\frac{\varepsilon_1}{\alpha}} \text{ i.o.} \right) = 0 \quad (4.2.7)
\]

and

\[
P\left( M_{v_n k} \geq (C_3 + \lambda)B_{n_k} \left( \log n_k \right)^{\frac{\varepsilon_1}{\alpha}} \text{ i.o.} \right) = 1 \quad (4.2.8)
\]

Let $A_k = \left\{ M_{v_n k} \geq (C_3 - \lambda)B_{n_k} \left( \log n_k \right)^{\frac{\varepsilon_1}{\alpha}} \right\}$

and $A_{k+1} = \left\{ M_{v_n k+1} \geq (C_3 - \lambda)B_{n_k+1} \left( \log n_{k+1} \right)^{\frac{\varepsilon_1}{\alpha}} \right\}$

or $A_{k+1}^c = \left\{ M_{v_n k+1} \leq (C_3 - \lambda)B_{n_k+1} \left( \log n_{k+1} \right)^{\frac{\varepsilon_1}{\alpha}} \right\}$

In view of (1.3.13), one can observe that, the condition (2) of Heyde [31] is satisfied by $x_k = (C_3 - \lambda)B_{n_k} \left( \log n_k \right)^{\frac{\varepsilon_1}{\alpha}}$. Following proof of Lemma (1.3.7) of Vasudeva and Divanji [68], we can show that $\frac{M_{v_n k}}{x_{n k}} \to 0$ as $k \to \infty$. Also, since, $1 \leq \limsup_{k \to \infty} \frac{y_{u_k}}{y_{n_k}} \leq \left(1 + \varepsilon\right)^{\frac{1}{2}}$, for some $\varepsilon > 0$ then by theorem in Heyde [31], we can find $C_4 (> 0)$ and $k_1$ such that for all $k \geq k_1$,

\[
P(A_k) \leq C_4 n_k P \left( X_1 \geq x_{n_k} \right).
\]
Using (1.3.13), we can find some constant $C_5 (> 0)$ and a $k_2 (\geq k_1)$ such that for all $k \geq k_2$,

$$P(A_k) \leq C_5 n_k \frac{L(x_{n_k}) \theta(x_{n_k})}{x_{n_k}^{\alpha}} = C_5 n_k \frac{L(B_{n_k}) \theta(B_{n_k})}{B_{n_k}^{\alpha} (\log n_k)^{(\epsilon^* + \epsilon_1)}} \frac{L(x_{n_k}) \theta(x_{n_k})}{L(B_{n_k}) \theta(B_{n_k})}$$

Applying Lemma (1.3.3) with $\delta = \frac{\epsilon_1}{2}$ and using the boundedness of $\theta$, one can find some constants $C_6 (> C_5)$ and $k_3 (\geq k_2)$ such that for all $k \geq k_3$,

$$P(A_k) \leq \frac{C_6}{(\log n_k)^{(\epsilon^* + \frac{\epsilon_1}{2})}}$$

(4.2.9)

and using the property of $\epsilon^*$ and condition $\limsup_{k \to \infty} \frac{n_k}{n_{k+1}} < 1$, we get $P(A_k) \to 0$ as $k \to \infty$.

We have,

$$\left\{ A_k \cap A_{k+1}^c \right\} = \left\{ M_{vn_k} \geq (C_3 - \lambda) B_{n_k} (\log n_k)^{\frac{\epsilon^* + \epsilon_1}{\alpha}}, M_{vn_{k+1}} \leq (C_3 - \lambda) B_{n_{k+1}} (\log n_{k+1})^{\frac{\epsilon^* + \epsilon_1}{\alpha}} \right\}.$$ 

Observe that,

$$\left\{ A_k \cap A_{k+1}^c \right\} \subset \left\{ M_{vn_k} \geq (C_3 - \lambda) B_{n_k} (\log n_k)^{\frac{\epsilon^* + \epsilon_1}{\alpha}}, M_{vn_k} \leq (C_3 - \lambda) B_{n_{k+1}} (\log n_{k+1})^{\frac{\epsilon^* + \epsilon_1}{\alpha}} \right\}.$$ 

Which implies,

$$\left\{ A_k \cap A_{k+1}^c \right\} \subset \left\{ (C_3 - \lambda) B_{n_k} (\log n_k)^{\frac{\epsilon^* + \epsilon_1}{\alpha}} \leq M_{vn_k} \leq (C_3 - \lambda) B_{n_{k+1}} (\log n_{k+1})^{\frac{\epsilon^* + \epsilon_1}{\alpha}} \right\}.$$
Hence,

\[ P\left(A_k \cap A_{k+1}^c\right) \]
\[ \leq P\left((C_3 - \lambda)B_{n_k}^{-1} \left(\log n_k\right)\frac{\varepsilon^* + \frac{1}{2}}{\alpha} \leq M_{v_{n_k}} \leq (C_3 - \lambda)B_{n_{k+1}}^{-1} \left(\log n_{k+1}\right)\frac{\varepsilon^* + \frac{1}{2}}{\alpha}\right) \]
\[ \leq P\left(M_{v_{n_k}} \geq \left(\frac{C_3 - \lambda)B_{n_{k+1}}^{-1} \left(\log n_{k+1}\right)\frac{\varepsilon^* + \frac{1}{2}}{\alpha}}{v_{n_k}^\frac{1}{\alpha}}\right) \right) - P\left(M_{v_{n_k}} \geq \left(\frac{C_3 - \lambda)B_{n_{k+1}}^{-1} \left(\log n_{k+1}\right)\frac{\varepsilon^* + \frac{1}{2}}{\alpha}}{v_{n_k}^\frac{1}{\alpha}}\right) \right) \]

From the definition of \(B_n\), one can find some constants \(C_7 > 0\) and \(k_4 > 0\) such that for all \(k \geq k_4\), \(B_{n_{k+1}}^{-1}B_{n_k} \leq C_7(> 0)\) and following similar arguments of (4.2.9), we can find some constants \(C_8 > 0\) and \(k_5(\geq k_4)\) such that for all \(k \geq k_5\), we have,

\[ P\left(A_k \cap A_{k+1}^c\right) \leq \frac{C_8}{\left(\log n_k\right)^{\varepsilon^* + \frac{1}{2}}} \]

From the definition of \(\varepsilon^*\), we have \(\varepsilon^* + \frac{\varepsilon^*}{2} > \varepsilon^*\) and hence by taking \(\varepsilon^* + \frac{\varepsilon^*}{2}\) for \(\varepsilon(> \varepsilon^*)\), then it follows that,

\[ \sum_{k \geq k_5} P\left(A_k \cap A_{k+1}^c\right) \leq C_8 \sum_{k \geq k_5} \frac{1}{\left(\log n_k\right)^{\varepsilon^* + \frac{\varepsilon^*}{2}}} < \infty \]

and by Nielsen Lemma (1.3.5), \(P(A_k \text{ i.o.}) = 0\), which implies the proof of (4.2.7). Consequently the proof of (4.2.1) follows from (4.2.7) and (4.2.4).

To prove (4.2.8), we prove that for some \(d > 0\),

\[ P\left(M_{v_{n_k}} \geq (C_3 + \lambda)B_{n_k}^{-1} \left(\frac{n_k}{N_{n_k}}\log n_k\right)^{\frac{\varepsilon^* - \frac{1}{2}}{\alpha}} \text{ i.o.}\right) \geq d > 0. \]
This is done by Extended Borel Cantelli Lemma (1.3.2).

The condition, \( \limsup_{k \to \infty} \frac{n_k}{n_{k+1}} < 1 \), implies that, there exists a \( \lambda < 1 \) and \( k_6 \) such that for all \( k \geq k_6 \),

\[
\frac{n_k}{n_{k+1}} \leq \lambda. \tag{4.2.11}
\]

Now define the events,

\[
D_k = \left\{ (C_3 + \lambda)B_{n_k} (\log n_k)^{\frac{\bar{\varepsilon} - \varepsilon_1}{\alpha}} \leq M_{n_k} \leq 2(C_3 + \lambda)B_{n_k} (\log n_k)^{\frac{\bar{\varepsilon} - \varepsilon_1}{\alpha}} \right\}.
\]

Then,

\[
P(D_k) = P\left(M_{n_k} \geq 2(C_3 + \lambda)B_{n_k} (\log n_k)^{\frac{\bar{\varepsilon} - \varepsilon_1}{\alpha}} \right) - P\left(M_{n_k} \geq (C_3 + \lambda)B_{n_k} (\log n_k)^{\frac{\bar{\varepsilon} - \varepsilon_1}{\alpha}} \right).
\]

Consider,

\[
P\left(M_{n_k} \geq 2(C_3 + \lambda)B_{n_k} (\log n_k)^{\frac{\bar{\varepsilon} - \varepsilon_1}{\alpha}} \right) = P\left(M_{n_k} \geq y_{n_k} \right),
\]

where \( y_{n_k} = 2(C_3 + \lambda)B_{n_k} (\log n_k)^{\frac{\bar{\varepsilon} - \varepsilon_1}{\alpha}} \).

In view of (1.3.13), we can observe that, condition (2) of Heyde [31] is satisfied by \( y_{n_k} \) and we notice that, \( \frac{S_{n_k}}{y_{n_k}} \overset{P}{\to} 0 \) as \( k \to \infty \). Following the proof of Lemma (1.3.7) of Vasudeva and Divanjii [68], we can show that, \( \frac{M_{n_k}}{y_{n_k}} \overset{P}{\to} 0 \) as \( k \to \infty \). Also since, \( 1 \leq \limsup_{k \to \infty} \frac{y_{n_k}}{y_{n_k}} \leq (1 + \varepsilon)^{\frac{1}{\alpha}} \), by Theorem in Heyde [31], we can find some constants \( C_{y}(>0) \) and \( k_1 \) such that, for all \( k \geq k_1 \),

\[
P\left(M_{n_k} \geq y_{n_k} \right) \approx C_{y} n_k P\left(X_1 \geq y_{n_k} \right).
\]
Again using (1.3.13),

\[
P \left( M_{\alpha n_k} \geq y_{n_k} \right) \simeq C_9 n_k \frac{L \left( y_{n_k} \right)}{y_{n_k}^\alpha} \frac{L \left( B_{n_k} \right)}{B_{n_k}^\alpha} \frac{L \left( y_{n_k} \right)}{L \left( B_{n_k} \right)} \simeq \frac{C_9 n_k}{B_{n_k}^\alpha \left( \log n_k \right)^{\epsilon^* - \epsilon_1}}
\]

By (1.3.13) and properties of s.v. function, we know that,

\[
n \left( 1 - F(B_n) \right) + F(-B_n) \to C(>0) \text{ as } n \to \infty. \tag{4.2.12}
\]

Using (4.2.12), we can find some constant \( C_{10} (>0) \), such that,

\[
P \left( M_{\alpha n_k} \geq y_{n_k} \right) \geq C_{10} \frac{L \left( y_{n_k} \right)}{L \left( B_{n_k} \right)} \frac{1}{\left( \log n_k \right)^{\epsilon^* - \epsilon_1}} \simeq \frac{C_{10} \left( \log n_k \right)^{\epsilon^* - \epsilon_1}}{B_{n_k}^\alpha \left( \log n_k \right)^{\epsilon^* - \epsilon_1}}.
\]

By Lemma (1.3.3), with \( \epsilon = \frac{-\epsilon_1}{2} \), we can find some constants, \( C_{11} (>0) \) and \( k_2 \) such that, for all \( k \geq k_2 \),

\[
P \left( M_{\alpha n_k} \geq y_{n_k} \right) \geq C_{11} \frac{\left( \log n_k \right)^{-\frac{-\epsilon_1}{2}}}{L \left( B_{n_k} \right)} \frac{1}{\left( \log n_k \right)^{\epsilon^* - \epsilon_1}} \simeq \frac{C_{11} \left( \log n_k \right)^{\epsilon^* - \epsilon_1}}{\left( \log n_k \right)^{\epsilon^* - \epsilon_1}}. \tag{4.2.13}
\]

Similarly following the above process, we find some constant \( C_{12} (>0) \) such that,

\[
P \left( M_{\alpha n_k} \geq \left( C_3 + \lambda \right) B_{n_k} \left( \log n_k \right)^{\epsilon^* - \epsilon_1} \right) \geq \frac{C_{12} \left( \log n_k \right)^{\epsilon^* - \epsilon_1}}{\left( \log n_k \right)^{\epsilon^* - \epsilon_1}}. \tag{4.2.14}
\]

Substituting (4.2.13) and (4.2.14) in (4.2.11), we can find some constants \( C_{13} (>0) \) and \( k_3 (>0) \) such that, for all \( k \geq k_3 \),

\[
P \left( D_k \right) \geq \frac{C_{13} \left( \log n_k \right)^{\epsilon^* - \epsilon_1}}{\left( \log n_k \right)^{\epsilon^* - \epsilon_1}}.
\]
where \( n_k \) satisfies the condition, \( \limsup_{k \to \infty} \frac{n_k}{n_{k+1}} < 1 \). This implies by the property of \( \varepsilon^* \) and \( n_k \), we have,

\[
\sum_{k \geq k_3} P(D_k) \geq C_{13} \sum_{k \geq k_3} \frac{1}{(\log n_k)^{(\varepsilon^- - \frac{\alpha}{2})}} > C_{14} \sum_{k \geq k_3} \frac{1}{(\log k)^{(\varepsilon^- - \frac{\alpha}{2})}},
\]

(4.2.15)

for some \( C_{14}(> C_{13}) \) and hence, \( \sum_{k \geq k_3} P(D_k) = \infty \).

Let \( s > (\log k)^{\eta} \), \( \eta > 1 \), we have,

\[
P(D_k \cap D_s) = P((C_3 + \lambda)B_n \log n_k)^{\varepsilon^* - \varepsilon_k} \leq M_{u_{n_k}} \leq 2(C_3 + \lambda)B_n \log n_k^{\varepsilon^* - \varepsilon_k},
\]

\[
(C_3 + \lambda)B_n \log n_k^{\varepsilon^* - \varepsilon_k} \leq M_{u_{n_k}} \leq 2(C_3 + \lambda)B_n \log n_k^{\varepsilon^* - \varepsilon_k}
\]

\[
\leq P(D_k) P \left( M_{u_{n_k}} - M_{n_{n_k}} \geq (C_3 + \lambda)B_n \log n_k^{\varepsilon^* - \varepsilon_k} - 2(C_3 + \lambda)B_n \log n_k^{\varepsilon^* - \varepsilon_k} \right),
\]

Again, following similar to (4.2.9) and Heyde’s theorem [31], we have for \( s > (\log k)^{\eta} \), \( \eta > 1 \),

\[
P(D_k \cap D_s) \leq P(D_k) \left( u_{n_k} - u_{n_k} \right) P \left( X_1 \geq (C_3 + \lambda)B_n \log n_k^{\varepsilon^* - \varepsilon_k} \right).
\]

Applying the arguments used to get the upper bound of \( P(A_n) \), we can find a constant \( C_{15}(> 0) \) and some \( k_4(> 0) \) whereby for all \( k \geq k_4 \) and \( s > (\log k)^{\eta} \), \( \eta > 1 \),

\[
P\left(D_k \cap D_s\right) \leq C_{15} P(D_k) P(D_s).
\]

(4.2.16)

Now for \((k + 1) \leq s \leq (\log k)^{\eta} \), \( \eta > 1 \), we can note that,

\[
(D_k \cap D_s) = \left\{ \begin{array}{l}
(C_3 + \lambda)B_n \log n_k^{\varepsilon^* - \varepsilon_k} \leq M_{u_{n_k}} \leq 2(C_3 + \lambda)B_n \log n_k^{\varepsilon^* - \varepsilon_k}, \\
(C_3 + \lambda)B_n \log n_k^{\varepsilon^* - \varepsilon_k} \leq M_{u_{n_k}} \leq 2(C_3 + \lambda)B_n \log n_k^{\varepsilon^* - \varepsilon_k}
\end{array} \right\}
\]

\[
\leq \left\{ \begin{array}{l}
(C_3 + \lambda)B_n \log n_k^{\varepsilon^* - \varepsilon_k} \leq M_{u_{n_k}} \leq 2(C_3 + \lambda)B_n \log n_k^{\varepsilon^* - \varepsilon_k}, \\
M_{u_{n_k}} \geq (C_3 + \lambda)B_n \log n_k^{\varepsilon^* - \varepsilon_k}
\end{array} \right\},
\]
which implies,

\[
P \left( D_k \cap D_s \right) \leq P \left( (C_3 + \lambda)B_{n_k} (\log n_k)^{\varepsilon - \varepsilon_1} \leq M_{un_k} \leq 2(C_3 + \lambda)B_{n_k} (\log n_k)^{\varepsilon - \varepsilon_1} \cap M_{un_s} \geq (C_3 + \lambda)B_{n_s} (\log n_s)^{\varepsilon - \varepsilon_1} \right).
\]

Observe that, \( M_{un_k} \) and \( M_{un_s} \) are independent, one gets,

\[
P \left( D_k \cap D_s \right) \leq P(D_k) P \left( M_{un_s} \geq (C_3 + \lambda)B_{n_s} (\log n_s)^{\varepsilon - \varepsilon_1} \right).
\]

Again in view of (1.3.13), we can observe that, the condition (2) of Hyede [28] is satisfied by \( z_{ns} = (C_3 + \lambda)B_{n_s} (\log n_s)^{1 - \varepsilon_1} \) and we notice that, \( \frac{S_{zn_s}}{z_{zn_s}} \rightarrow 0 \) as \( s \rightarrow \infty \). Following the Proof of Lemma (1.3.7) in Vasudeva and DiVanji [59], we can show that, \( \frac{M_{un_s}}{z_{zn_s}} \rightarrow 0 \) as \( s \rightarrow \infty \). Also since, \( 1 \leq \limsup_{s \rightarrow \infty} \frac{z_{un_s}}{z_{zn_s}} \leq (1 + \varepsilon)^{\frac{1}{2}} \), \( \varepsilon > 0 \), by Theorem in Heyde [28], we can find some constants \( C_{16} > 0 \) and \( k_5 \) such that, for all \( k \geq k_5 \),

\[
P \left( M_{un_s} \geq z_{ns} \right) \simeq C_{16} n_s P \left( X_1 \geq z_{nk} \right).
\]

By (1.3.13) we have,

\[
P \left( M_{un_s} \geq z_{ns} \right) \simeq C_{16} n_s \frac{L(z_{ns})}{z_{zn_s}} = C_{16} n_s \frac{L(B_{n_s})}{B_{n_s}} \frac{L(z_{ns})}{L(B_{n_s})} \frac{1}{(\log n_s)^{(1-\varepsilon_1)}}.
\]

Using (4.2.12), we get some constants \( C_{17} > C_{16} \), and \( k_6(> k_5) \) such that, for all \( k \geq k_6 \),

\[
P \left( M_{un_s} \geq z_{ns} \right) \leq C_{17} \frac{L(z_{ns})}{L(B_{n_s})} \frac{1}{(\log n_s)^{(\varepsilon - \varepsilon_1)}}.
\]

By Lemma (1.3.3), with \( \varepsilon = \frac{\varepsilon_1}{2} \), we get that,

\[
P \left( M_{un_s} \geq z_{ns} \right) \leq C_{17} \frac{L(B_{n_s})}{L(B_{n_s})} \frac{(\log n_s)^{\frac{\varepsilon_1}{2}}}{(\log n_s)^{(\varepsilon - \varepsilon_1)}} = \frac{C_{17}}{(\log n_s)^{(\varepsilon - \varepsilon_1)}}.
\]
Using the fact that $s \geq k + 1$, one can find some constants $C_{18}$ and $k_7$, such that, for all $k \geq k_7$,

$$P \left( M_{n_s} \geq z_{n_s} \right) \leq \frac{C_{18}}{(\log k) \left( \varepsilon^* - \frac{3\varepsilon_1}{2} \right)}$$

Hence, for all $k \geq k_7$,

$$P \left( D_k \cap D_s \right) \leq \frac{C_{18}}{(\log k) \left( \varepsilon^* - \frac{3\varepsilon_1}{2} \right)} P \left( D_k \right). \quad (4.2.17)$$

Note that,

$$P \left( D_k \right) \leq P \left( M_{n_k} \geq 2(C_3 + \lambda) B_{n_k} \left( \log n_k \right)^{\frac{\varepsilon^* - \varepsilon_1}{\alpha}} \right)$$

Following steps similar to the above process of (4.2.17), we can find some constants $C_{19} > 0$ and $k_8 (> 0)$, such that, for all $k \geq k_8$ we have,

$$P \left( D_k \right) \leq \frac{C_{19}}{(\log k) \left( \varepsilon^* - \frac{3\varepsilon_1}{2} \right)}, \text{ for } s \geq k + 1 \quad (4.2.18)$$

From (4.2.17) and (4.2.18), there exists some constants $C_{20} > 0$ and $k_9 (> 0)$, such that, for all $k \geq k_9$,

$$P \left( D_k \cap D_s \right) \leq \frac{C_{20}}{(\log k)^2 \left( \varepsilon^* - \frac{3\varepsilon_1}{2} \right)}.$$

Now,

$$\sum_{k=1}^{n-1} \sum_{s=k+1}^{(\log k)^n} P \left( D_k \cap D_s \right) \leq C_{20} \sum_{k=1}^{n-1} \frac{(\log k)^n}{(\log k)^2 \left( \varepsilon^* - \frac{3\varepsilon_1}{2} \right)} \leq C_{20} \sum_{k=1}^{n-1} \frac{1}{(\log k)^2 \left( \varepsilon^* - \frac{3\varepsilon_1}{2} \right) - \eta}.$$

We notice that, for $\eta > 1$, choose $\varepsilon_1$ is sufficiently very small such that,

$$2 \left( \varepsilon^* - \frac{3\varepsilon_1}{2} \right) - \eta < 1 = (\varepsilon^* - \varepsilon_2) \text{ (say), for some } \varepsilon_2 (> 0).$$

Hence,

$$\sum_{k=1}^{n-1} \sum_{s=k+1}^{(\log k)^n} P \left( D_k \cap D_s \right) \leq C_{20} \sum_{k=1}^{n-1} \frac{1}{(\log k)^2 \left( \varepsilon^* - \varepsilon_2 \right)}.$$
For \( n \geq N_1 \), we have,
\[
\sum_{k=1}^{n-1} \sum_{s=k+1}^{n} P(D_k \cap D_s) \leq C_{20} \frac{1}{(\log n)^{\epsilon^* - \epsilon_2}}. \tag{4.2.19}
\]

From (4.2.15) we have for \( n \geq N_2 \),
\[
\sum_{k=1}^{\infty} P(D_k) \geq C_{12} \frac{\log k}{(\log k)^{1 - \frac{\epsilon_1}{2}}} \geq C_{21} (\log n)^{\epsilon_1}, \tag{4.2.20}
\]
for some \( C_{21} > 0 \). Note that,
\[
\sum_{k=1}^{n} \sum_{s=1}^{n} P(D_k \cap D_s) = \left( \sum_{k=1}^{n} P(D_k) \right)^2 = \sum_{k=1}^{n} P(D_k)^2.
\]

By (4.2.20), the second term of the right side, tends to zero as \( n \to \infty \). Hence, consider,
\[
2 \sum_{k=1}^{n-1} \sum_{s=k+1}^{n} P(D_k \cap D_s) = 2 \sum_{k=1}^{n-1} \sum_{s=k+1}^{n} P(D_k \cap D_s) + \sum_{k=1}^{n} P(D_k)^2.
\]

By (4.2.19) and (4.2.20), we have,
\[
\lim_{n \to \infty} 2 \sum_{k=1}^{n-1} \sum_{s=k+1}^{n} P(D_k \cap D_s) = 0 \tag{4.2.23}
\]
and by (4.2.16) and (4.2.20), we get that,
\[
\liminf_{n \to \infty} \frac{2 \sum_{k=1}^{n-1} \sum_{s=(\log k)^{\gamma_{1}+1}}^{n} P(D_k \cap D_s)}{\left( \sum_{k=1}^{n} P(D_k) \right)^2} \leq 2C_{15} > 0. \tag{4.2.24}
\]

Using (4.2.23) and (4.2.24) in (4.2.21) we have,
\[
\liminf_{n \to \infty} \frac{\sum_{k=1}^{n} \sum_{s=1}^{n} P(D_k \cap D_s)}{\left( \sum_{k=1}^{n} P(D_k) \right)^2} \geq C_{22} > 0.
\]
In view of (4.2.15) and (4.2.24), appealing to Extended Borel-Cantelli Lemma (1.3.2) and Hewit-Savege zero-one law, we get $P(D_k \text{ i.o.}) = 1$ Hence the proof of the Theorem is completed.

**Theorem 4.2.2.** Let $F \in DP(\alpha)$, $0 < \alpha < 2$. Let $\{n_k, k \geq 1\}$ be an integer subsequence such that $\liminf_{k \to \infty} \frac{n_k}{n_{k+1}} > 0$ and let $\{N_{n_k}\}$ be a sequence of positive integer r.v.s. independent of $\{X_n, n \geq 1\}$ such that, 

$$\left| e^{\frac{N_{n_k}}{n_k}} - 1 \right| \to 0 \text{ a.s as } k \to \infty, \text{ where } 0 < \delta < 1.$$  

Then,

$$\limsup_{k \to \infty} \left( \frac{M_{N_{n_k}}}{B_{N_{n_k}}} \right)^{\frac{1}{n_k}} = e^{\frac{1}{\alpha}} \text{ a.s.,}$$

where $M_{N_{n_k}} = S_{n_k + N_{n_k}} - S_{n_k}$.

**Proof.** Following similar steps of Theorem 4.2.1, using (4.2.3) and (4.2.6), it is enough to show that for any $\varepsilon_1 \in (0, 1)$,

$$P \left( M_{u_{n_k}} \geq (C_3 - \lambda) B_{u_{n_k}} \left( \log n_k \right)^{\frac{1+\varepsilon_1}{\alpha}} \text{ i.o.} \right) = 0 \quad (4.2.25)$$

and

$$P \left( M_{u_{n_k}} \geq (C_3 + \lambda) B_{u_{n_k}} \left( \log n_k \right)^{\frac{1-\varepsilon_1}{\alpha}} \text{ i.o.} \right) = 1 \quad (4.2.26)$$

We have,

$$\left( \frac{M_{u_{n_k}}}{B_{u_{n_k}}} \right)^{\frac{1}{n_k}} = \left( \frac{M_{u_{n_k}}}{B_{u_{n_k}}} \frac{B_{u_{n_k}}}{B_{n_k}} \right)^{\frac{1}{n_k}}$$
From the definition of $B_n$ and by elementary properties of a s.v. functions, we can show that,

$$\left( \frac{B_{n-k}}{B_{n-k}} \right)^{\frac{1}{\gamma_{n-k}}} \to 1 \text{ as } k \to \infty.$$  

Hence by Theorem 3.2.1 we can observe that,

$$\limsup_{k \to \infty} \left( \frac{M_{n-k}}{B_{n-k}} \right)^{\frac{1}{\gamma_{n-k}}} \leq \limsup_{n \to \infty} \left( \frac{M_{n-k}}{B_{n-k}} \right)^{\frac{1}{\gamma_{n}}} = e^{\frac{1}{\alpha}} \text{ a.s.},$$

which proves (4.2.25).

From the condition $\liminf_{k \to \infty} \frac{n_k}{n_{k+1}} > 0$, we can see that, the sequences are at most geometrically increasing which implies that, there exists $\rho > 1$ such that $n_{k+1} \leq \rho n_k$.

Define $q_j = \min \{ k : n_k > M^j \}, j = 1, 2, ..., $ where $M$ is chosen such that $\frac{\rho}{M} < 1$. Proceeding as in Allan Gut [2], we can show that $M^j < n_{q_j} < \rho M^j$ and $\frac{1}{\rho M} \leq \frac{n_{q_{j+1}}}{n_{q_j}} \leq \frac{\rho}{M} < 1$. Consequently $\{n_{q_j}\}$ satisfies the condition $\limsup_{k \to \infty} \frac{n_k}{n_{k+1}} < 1$ of Theorem 4.2.1 and also the relation $\sum_{j=1}^{\infty} \left( \log n_{q_j} \right)^{-\varepsilon_1} < \infty$ holds for all $\varepsilon_1 > 1$ (i.e. $\varepsilon^* = 0$). Now (4.2.26) follows from Theorem 4.2.1.

Hence the proof of the Theorem is completed.

\[\square\]

Remark 4.2.1. In Theorem 4.2.1, one can note that the case $\varepsilon^* = 0$ needs a separate argument, because the sequence $n_k = [\exp(\exp k)], k \geq 1$, shows that
such a sequence is possible and hence we avoid the case \( \varepsilon^* = 0 \).

### 4.3 LIL in the linear normalization for geometrically fast increasing subsequences.

In this section we consider sequence of i.i.d. positive asymmetric stable r.v.s. and obtain limit infimum results in linearly delayed random sums for geometrically fast increasing subsequences.

**Theorem 4.3.1.** Let \( \{X_n, n \geq 1\} \) be a sequence of i.i.d. positive asymmetric stable r.v.s. with index \( \alpha, 0 < \alpha < 1 \). Let \( \{n_k, k \geq 1\} \) be an integer subsequence such that \( \limsup_{k \to \infty} \frac{n_k}{n_{k+1}} < 1 \) and let \( \{N_{n_k}\} \) be a sequence of positive integer r.v.s. independent of \( \{X_n, n \geq 1\} \) such that \( \left| \frac{N_{N_{n_k}}}{n_k} - 1 \right| \to 0 \) a.s as \( k \to \infty \), where \( 0 < \delta < 1 \). a.s. Then,

\[
\liminf_{k \to \infty} \left( \frac{M_{N_{n_k}}}{\beta(N_{n_k})} \right) = \varepsilon^* \text{ a.s.,}
\]

where \( \varepsilon^* = \sup \left\{ \varepsilon_1 > 0 : \sum_{k \geq k_0} (\log n_k)^{-\varepsilon_1} < \infty \right\} \), for some \( k_0 > 0 \),

\[
\beta(N_{n_k}) = \theta \alpha N_{n_k}^{\frac{\alpha}{\alpha - 1}} \left( \log \frac{n_k}{N_{n_k}} + \log \log n_k \right) \left( \frac{\alpha - 1}{\alpha} \right), \quad \theta = (B(\alpha))^{\frac{1}{\alpha}},
\]

\[
B(\alpha) = (1 - \alpha) \alpha \left( \cos \frac{\pi \alpha}{2} \right) \left( \frac{\pi}{\alpha - 1} \right) \text{ and } M_{N_{n_k}} = \sum_{j=n_k+1}^{n_k+N_{n_k}} X_j = S_{n_k+N_{n_k}} - S_{n_k}.
\]
Proof. Equivalently, we show that, for any $\varepsilon > 0$,

$$P \left( M_{N_{nk}} \leq (\varepsilon^* - \varepsilon) \beta \left( N_{nk} \right) \text{i.o.} \right) = 0 \quad (4.3.1)$$

and

$$P \left( M_{N_{nk}} \leq (\varepsilon^* + \varepsilon) \beta \left( N_{nk} \right) \text{i.o.} \right) = 1 \quad (4.3.2)$$

From the condition $\left| \frac{N_{nk}}{e^{n_k}} - 1 \right| \to 0 \text{ a.s. as } k \to \infty$, implies that there exists some $\varepsilon > 0$ and $\delta \in (0, 1)$, such that $\left| \frac{N_{nk}}{e^{n_k}} - 1 \right| < \varepsilon$, which implies

$$u_{nk} \leq N_{nk} \leq v_{nk} \text{ a.s.,} \quad (4.3.3)$$

where $u_{nk} = C_1 n_k^\delta$ and $v_{nk} = C_2 n_k^\delta$, $C_1 = \log(1 - \varepsilon)$ and $C_2 = \log(1 + \varepsilon)$.

Consequently, as $X_n$’s are i.i.d. positive asymmetric stable r.v.s., we have,

$$M_{u_{nk}} \leq M_{N_{nk}} \leq M_{v_{nk}} \text{ a.s.,} \quad (4.3.4)$$

By the condition, $\left| \frac{N_{nk}}{e^{n_k}} - 1 \right| \to 0 \text{ a.s as } k \to \infty$, and from (4.3.3), we can observe that, $\frac{\beta \left( N_{nk} \right)}{\beta \left( n_k \right)} \to C_3 \text{ a.s., as } k \to \infty$, for some constant $C_3(> 0)$, which implies that, there exits some $\delta_1 > 0$, such that,

$$(C_3 - \delta_1) \leq \frac{\beta \left( N_{nk} \right)}{\beta \left( n_k \right)} \leq (C_3 + \delta_1). \quad (4.3.5)$$

Using (4.3.4) and (4.3.5), it is enough to prove (4.3.1) and (4.3.2) as,

$$P \left( M_{u_{nk}} \leq (\varepsilon^* - \varepsilon) \left( C_3 + \delta_1 \right) \beta \left( n_k \right) \text{i.o.} \right) = 0, \quad (4.3.6)$$
where \( M_{n_k} = S_{n_k + \eta n_k} - S_{n_k} \) and

\[
P \left( M_{n_k} \leq (\epsilon^* + \epsilon) (C_3 - \delta_1) \beta \left( n_k \right) i.o. \right) = 1, \tag{4.3.7}
\]

where \( M_{n_k} = S_{n_k + \eta n_k} - S_{n_k} \).

The condition \( \limsup_{k \to \infty} \frac{n_k}{n_{k+1}} < 1 \) implies that, there exists some constant \( b > 1 \) such that, for some \( k \) large,

\[
n_{k+1} \geq bn_k. \tag{4.3.8}
\]

From the fact that, \( X_n \)'s are positive asymmetric stable r.v.s. with exponent \( \alpha, \ 0 < \alpha < 1 \), we have \( \frac{M_{n_k}}{u_{n_k}^\alpha} \) and \( X_1 \) are identically distributed and hence,

\[
P \left( M_{n_k} \leq (\epsilon^* - \epsilon) (C_3 + \delta_1) \beta \left( n_k \right) \right) = P \left( X_1 \leq \frac{(\epsilon^* - \epsilon) (C_3 + \delta_1) \beta \left( n_k \right)}{u_{n_k}^\alpha} \right).
\]

In view of definition of \( \beta \left( N_{n_k} \right) \), we can notice that, there exits some constant \( C_4 > 0 \) such that,

\[
\frac{(\epsilon^* - \epsilon) (C_3 + \delta_1) \beta \left( n_k \right)}{u_{n_k}^\alpha} = C_4 \theta_{\alpha} (\epsilon^* - \epsilon) (\log \log n_k) \frac{\alpha}{\alpha - 1}.
\]
Taking $x$ as $C_4 \theta_\alpha (\varepsilon^* - \varepsilon) (\log \log n_k)^{\alpha-1}$ in Lemma 1.3.6, we can find some constants $k_1 (> 0)$ and $C_5 (> 0)$ such that,

\[
P \left( X_1 \leq \frac{\varepsilon^* - \varepsilon (1 + \delta_1 ) \beta (n_k)}{u_{n_k}^{\frac{1}{2}}} \right) \sim \frac{C_5}{(\log \log n_k)^{\frac{3}{2}}} \exp \left\{ - (\varepsilon^* - \varepsilon)^{\frac{\alpha}{\alpha - 1}} \log \log n_k \right\}
\]

\[
\sim \frac{C_5}{(\log n_k)^{\frac{1}{2}} (\log n_k)^{\frac{\alpha}{\alpha - 1}}}
\]

\[
\leq \frac{C_5}{(\log n_k)^{\frac{\alpha}{\alpha - 1}}}
\]

From the fact that, $0 < (\varepsilon^* - \varepsilon) < \varepsilon^*$ for some $\varepsilon$ sufficiently small and by the definition of $\varepsilon^*$ and also by (4.3.8), we have, for some $k_1 (> 0)$,

\[
\sum_{k \geq k_1} P \left( M_{\varepsilon n_k} \leq (\varepsilon^* + \varepsilon) (C_3 + \delta_1 ) \beta (n_k) \right) \leq C_5 \sum_{k \geq k_1} \frac{1}{(\log n_k)^{\frac{\alpha}{\alpha - 1}}} < \infty.
\]

So that (4.3.6) follows by Borel - Cantelli Lemma. Consequently, (4.3.1) follows from (4.3.6).

To prove (4.3.7), we prove that, for some $d > 0$,

\[
P \left( M_{\varepsilon n_k} \leq (\varepsilon^* + \varepsilon) (C_3 - \delta_1 ) \beta (n_k) \ i.o. \right) \geq d > 0, \quad (4.3.9)
\]

Define the events,

\[
D_k = \left\{ M_{\varepsilon n_k} \leq (\varepsilon^* + \varepsilon) (C_3 - \delta_1 ) \beta (n_k) \right\}
\]

Then we have,

\[
P (D_k) = P \left( M_{\varepsilon n_k} \leq (\varepsilon^* + \varepsilon) (C_3 - \delta_1 ) \beta (n_k) \right)
\]
From the fact that, $X_n$’s are positive asymmetric stable r.v.s. with exponent $\alpha$, $0 < \alpha < 1$, we have $\frac{M_{n_k}}{v_{n_k}^{\frac{1}{\alpha}}}$ and $X_1$ are identically distributed and hence,

$$P(D_k) = P \left( X_1 \leq \frac{(\varepsilon^* + \varepsilon) (C_3 - \delta_1) \beta(n_k)}{v_{n_k}^{\frac{1}{\alpha}}} \right). \quad (4.3.10)$$

Again by the definition of $\beta \left( N_{n_k} \right)$, we observe that, there exits some constant $C_6 > 0$ such that,

$$\frac{(\varepsilon^* + \varepsilon) (C_3 - \delta_1) \beta(n_k)}{v_{n_k}^{\frac{1}{\alpha}}} = C_6 \theta_{\alpha} (\varepsilon^* + \varepsilon) (\log \log n_k)^{\frac{\alpha - 1}{\alpha}} \quad \text{in Lemma 1.3.6, we can find some constants } k_2(>0) \text{ and } C_7(>0) \text{ such that,}$$

$$P(D_k) = P \left( X_1 \leq \frac{(\varepsilon^* + \varepsilon) (C_3 - \delta_1) \beta(n_k)}{v_{n_k}^{\frac{1}{\alpha}}} \right) \sim \frac{C_7}{(\log \log n_k)^{\frac{1}{2}}} \exp \left\{ - (\varepsilon^* + \varepsilon)^{\frac{\alpha}{\alpha - 1}} \log \log n_k \right\}$$

$$\sim \frac{C_7}{(\log \log n_k)^{\frac{1}{2}}} (\log n_k)^{(\varepsilon^* + \varepsilon)\left(\frac{\alpha}{\alpha - 1}\right)}$$

$$\geq \frac{C_7}{(\log n_k)^{(\varepsilon^* + \varepsilon)\left(\frac{\alpha}{\alpha - 1}\right)}} \quad (4.3.11)$$

From the fact that, $0 < (\varepsilon^* + \varepsilon) < \varepsilon^*$ for some $\varepsilon$ sufficiently small and by the definition of $\varepsilon^*$ we have,

$$\sum_{k \geq k_2} P \left( 0 < M_{n_k} \leq (\varepsilon^* + \varepsilon) (1 - \delta_1) \beta(n_k) \right) \geq C_7 \sum_{k \geq k_2} \frac{1}{(\log n_k)^{(\varepsilon^* + \varepsilon)\left(\frac{\alpha}{\alpha - 1}\right)}} = \infty.$$
and hence \( P(D_k) = \infty \).

Now let \( s > t_k \), where \( t_k = k + \tau (\log k) \), for some \( \tau > 1 \). Then,

\[
P\left( D_k \cap D_s \right) = P\left( 0 < M_{v_{nk}} \leq (\varepsilon^* + \varepsilon) (C_3 - \delta_1) \beta(n_k) \right) \cap 0 < M_{v_{ns}} \leq (\varepsilon^* + \varepsilon) (C_3 - \delta_1) \beta(n_s) \).
\]

Observe that,

\[
\left\{ 0 < M_{v_{nk}} \leq (\varepsilon^* + \varepsilon) (C_3 - \delta_1) \beta(n_k), \ 0 < M_{v_{ns}} \leq (\varepsilon^* + \varepsilon) (C_3 - \delta_1) \beta(n_s) \right\}
\]

\[
\subseteq \left\{ 0 < M_{v_{nk}} \leq (\varepsilon^* + \varepsilon) (C_3 - \delta_1) \beta(n_k), \ 0 < M_{v_{ns}} - M_{v_{nk}} \leq (\varepsilon^* + \varepsilon) (C_3 - \delta_1) \beta(n_s) \right\}
\]

and hence,

\[
P\left( D_k \cap D_s \right) \leq P\left( 0 < M_{v_{nk}} \leq (\varepsilon^* + \varepsilon) (C_3 - \delta_1) \beta(n_k) \right) \cap 0 < M_{v_{ns}} - M_{v_{nk}} \leq (\varepsilon^* + \varepsilon) (C_3 - \delta_1) \beta(n_s) \).
\]

As \( M_{v_{nk}} \) and \( M_{v_{ns}} - M_{v_{nk}} \) are independent, we get,

\[
P\left( D_k \cap D_s \right) \leq P(D_k) P\left( M_{v_{ns}} - M_{v_{nk}} \leq (\varepsilon^* + \varepsilon) (C_3 - \delta_1) \beta(n_s) \right)
\]

(4.3.12)

Again using the fact that, \( X_n \)’s are positive asymmetric stable r.v.s. with exponent \( \alpha \), \( 0 < \alpha < 1 \), we have, \( \frac{M_{v_{ns}} - M_{v_{nk}}}{v_{ns} - v_{nk}} \) and \( X_1 \) are identically distributed and hence,

\[
P\left( M_{v_{ns}} - M_{v_{nk}} \leq (\varepsilon^* + \varepsilon) (C_3 - \delta_1) \beta(n_s) \right) = P\left( X_1 \leq \frac{(\varepsilon^* + \varepsilon) (C_3 - \delta_1) \beta(n_k)}{v_{ns} - v_{nk}} \right).
\]

Following the steps similar to those used to get (4.3.11), we can find some
constant $C_8(>0)$ and a $k_3(>0)$ and such that, for all $k \geq k_3$,

$$P \left( M_{vn_s} - M_{vn_k} \leq (\varepsilon^* + \varepsilon) (1 - \delta_i) \beta (n_s) \right) \leq \frac{C_8}{(\log n_s)^{\left(\varepsilon^* + \varepsilon\right) \left(\frac{\alpha}{\varepsilon} - 1\right)}}$$

Notice that, there exists some constant $C_9(>C_8)$ such that,

$$P \left( M_{vn_s} - M_{vn_k} \leq (\varepsilon^* + \varepsilon) (1 - \delta_i) \beta (n_s) \right) \leq C_9 P (D_s).$$

From (4.3.12) we have,

$$P \left( D_k \cap D_s \right) \leq C_9 P (D_k) P (D_s), \quad (4.3.13)$$

for some $s > t_k$, where $t_k = k + \tau (\log k)$, for some $\tau > 1$.

Now for $(k + 1) \leq s \leq t_k$, where $t_k = k + \tau (\log k)$, for some $\tau > 1$, observe that $M_{vn_s}$ and $M_{vn_k}$ are independent and using the inequality $s \geq (k + 1)$ we have, $P (D_k \cap D_s) \leq P (D_k)$ which implies,

$$\sum_{k=1}^{n} \sum_{s=k+1}^{t_k} P \left( D_k \cap D_s \right) \leq \sum_{k=1}^{n} \sum_{s=k+1}^{t_k} P (D_k) \leq \sum_{k=1}^{n} \tau (\log k) P (D_k) \quad (4.3.14)$$

Using Lemma 1.3.6 in (4.3.10) and hence from (4.3.14), we can find some constant $C_{10}(>0)$ such that,

$$\sum_{k=1}^{n} \sum_{s=k+1}^{t_k} P \left( D_k \cap D_s \right) \leq C_{10} \sum_{k=1}^{n} \frac{\log k}{(\log n_k)^{\left(\varepsilon^* + \varepsilon\right) \left(\frac{\alpha}{\varepsilon} - 1\right)}} \quad (4.3.15)$$

From (4.3.8) we have, $n_k \geq b^k n_0$, where $b > 1$ and $n_0$ is fixed and hence,

$$\sum_{k=1}^{n} \sum_{s=k+1}^{t_k} P \left( D_k \cap D_s \right) \leq C_{10} \sum_{k=1}^{n} \frac{\log k}{k^{\left(\varepsilon^* + \varepsilon\right) \left(\frac{\alpha}{\varepsilon} - 1\right)}} \quad (4.3.16)$$

Using (4.3.8) in (4.3.11), we can find some constant $C_{11}(>0)$ and a $k_4(>0)$ such that, for all $k \geq k_4$,

$$\sum_{k=k_4}^{n} P (D_k) \geq C_{11} \sum_{k=k_4}^{n} \frac{1}{k^{\left(\varepsilon^* + \varepsilon\right) \left(\frac{\alpha}{\varepsilon} - 1\right)}}.$$
From the definition of $\varepsilon^*$ we have $\varepsilon^* \geq 1$, which implies $(\varepsilon^* + \varepsilon)^{\left(\frac{n}{\varepsilon^* + \varepsilon}\right)}$ and hence,

$$k^{(\varepsilon^* + \varepsilon)^{\left(\frac{n}{\varepsilon^* + \varepsilon}\right)}} < k$$

or

$$\sum_{k=k_4}^{n} \frac{1}{k^{(\varepsilon^* + \varepsilon)^{\left(\frac{n}{\varepsilon^* + \varepsilon}\right)}}} > \sum_{k=k_4}^{n} \frac{1}{k}$$

Therefore,

$$\sum_{k=k_4}^{n} P(D_k) \geq C_{11} \sum_{k=k_4}^{n} \frac{1}{k^{(\varepsilon^* + \varepsilon)^{\left(\frac{n}{\varepsilon^* + \varepsilon}\right)}}} \geq C_{11} \sum_{k=k_4}^{n} \frac{1}{k} \sim \log n \quad \text{ (4.3.17)}$$

From (4.3.16) and (4.3.17), there exists $C_{12} (> 0)$ such that,

$$\sum_{k=1}^{n} \sum_{s=k+1}^{t_k} P(D_k \cap D_s) \leq C_9 \sum_{k=1}^{n} \sum_{s=1}^{t_k} P(D_k) P(D_s) \leq C_9 \left( \sum_{k=1}^{n} P(D_k) \right)^2 \leq C_9 \left( \sum_{k=1}^{n} \frac{1}{k^{(\varepsilon^* + \varepsilon)^{\left(\frac{n}{\varepsilon^* + \varepsilon}\right)}}} \right) \log n \leq C_{10} \leq C_{12} (> 0), \quad \text{ (4.3.18)}$$

it holds for $(k+1) \leq s \leq t_k$, $t_k = k + \tau (\log k)$, for some $\tau > 1$.

From (4.3.13), we have, $s > t_k$, $t_k = k + \tau (\log k)$, for some $\tau > 1$,

$$\sum_{k=1}^{n} \sum_{s=t_k}^{n} P(D_k \cap D_s) \leq C_9 \sum_{k=1}^{n} \sum_{s=t_k}^{n} P(D_k) P(D_s) \leq C_9 \left( \sum_{k=1}^{n} P(D_k) \right) \left( \sum_{s=1}^{n} P(D_s) \right) \leq C_9 \left( \sum_{k=1}^{n} P(D_k) \right)^2 \leq C_9 \left( \sum_{k=1}^{n} P(D_k) \right)^2 \quad \text{ (4.3.19)}$$
Observe that,
\[
\sum_{k=1}^{n} \sum_{s=t_k}^{n} P(D_k \cap D_s) \sim 2 \sum_{k=1}^{n} \sum_{s=k+1}^{t_k} P(D_k \cap D_s)
\]
\[
\sim 2 \sum_{k=1}^{n-1} \left( \sum_{s=k+1}^{t_k} P(D_k \cap D_s) + \sum_{s=t_k+1}^{n} P(D_k \cap D_s) \right)
\]
\[
\sim 2 \sum_{k=1}^{n-1} \sum_{s=k+1}^{t_k} P(D_k \cap D_s) + 2 \sum_{k=1}^{n-1} \sum_{s=t_k+1}^{n} P(D_k \cap D_s)
\]
(4.3.20)

This implies,
\[
\frac{\sum_{k=1}^{n} \sum_{s=1}^{t_k} P(D_k \cap D_s)}{\left( \sum_{k=1}^{n} P(D_k) \right)^2} \sim \frac{2 \sum_{k=1}^{n-1} \sum_{s=k+1}^{t_k} P(D_k \cap D_s)}{\left( \sum_{k=1}^{n} P(D_k) \right)^2} + \frac{2 \sum_{k=1}^{n-1} \sum_{s=t_k+1}^{n} P(D_k \cap D_s)}{\left( \sum_{k=1}^{n} P(D_k) \right)^2}
\]
From (4.3.18), (4.3.19) and (4.3.20), one can find some constant \( C_{13} (> 0) \) such that,
\[
\frac{\sum_{k=1}^{n} \sum_{s=1}^{t_k} P(D_k \cap D_s)}{\left( \sum_{k=1}^{n} P(D_k) \right)^2} \geq C_{13} (> 0).
\]
In view of the series \( \sum_{k \geq 2} P(D_k) = \infty \), appealing to Extended Borel Cantelli Lemma (1.3.2) and Hewitt - Sevage zero-one law, proof of (4.3.7) follows from \( P(D_k \text{ i.o.}) = 1 \) and consequently proof of (4.3.2) follows from (4.3.7). Hence proof of the theorem is completed.

\[\square\]

**Theorem 4.3.2.** Let \( \{X_n, n \geq 1\} \) be a sequence of i.i.d. positive asymmetric stable r.v.s. with index \( \alpha, 0 < \alpha < 1 \). Let \( \{n_k, k \geq 1\} \) be an integer subsequence such that \( \lim \inf_{k \to \infty} \frac{n_k}{n_{k+1}} > 0 \) and let \( \{N_{n_k}, k \geq 1\} \) be a sequence of positive integer
r.v.s. independent of \( \{X_n, n \geq 1\} \) such that
\[
\left| e^{\frac{N_{n_k}}{n_k}} - 1 \right| \to 0 \text{ a.s as } k \to \infty,
\]
where \( 0 < \delta < 1 \). Then,
\[
\liminf_{k \to \infty} \left( \frac{M_{n_k}}{\beta \left( N_{n_k} \right)} \right) = 1 \text{ a.s.},
\]

Proof. To prove the theorem, it is enough to show that, for any \( \varepsilon \in (0, 1) \),
\[
P \left( M_{N_{n_k}} \leq (1 - \varepsilon) \beta \left( N_{n_k} \right) \text{ i.o.} \right) = 0 \tag{4.3.21}
\]
and
\[
P \left( M_{N_{n_k}} \leq (1 + \varepsilon) \beta \left( N_{n_k} \right) \text{ i.o.} \right) = 1 \tag{4.3.22}
\]

Following similar steps of Theorem 4.3.1, it is sufficient to prove (4.3.21) and (4.3.22) as,
\[
P \left( M_{u_{n_k}} \leq (1 - \varepsilon) \left( C_3 + \delta_1 \right) \beta \left( n_k \right) \text{ i.o.} \right) = 0, \tag{4.3.23}
\]
where \( M_{u_{n_k}} = S_{n_k + u_{n_k}} - S_{n_k} \) and
\[
P \left( M_{v_{n_k}} \leq (1 + \varepsilon) \left( C_3 - \delta_1 \right) \beta \left( n_k \right) \text{ i.o.} \right) = 1, \tag{4.3.24}
\]
where \( M_{v_{n_k}} = S_{n_k + v_{n_k}} - S_{n_k} \).

The condition \( \liminf_{k \to \infty} \frac{n_k}{n_{k+1}} > 0 \) implies that, there exits some constant \( \rho > 1 \) such that,
\[
n_{k+1} < \frac{n_k}{\rho}, \text{ for some } k \text{ large.} \tag{4.3.25}
\]
This in turn implies that, \(\{M_{\nu n_k}, k \geq 1\}\) is a sequence of mutually independent r.v.s.

To prove (4.3.23), it is sufficient to show (4.3.6) for \(\epsilon^* = 1\). For any \(\epsilon \in (0, 1)\), by Theorem 3.2 of Vidyalaxmi et. al. [71] we claim that,

\[
\lim \inf_{k \to \infty} \left( \frac{M_{\nu n_k \beta(n_k)}}{\beta(n_k)} \right) \geq \lim \inf_{k \to \infty} \left( \frac{M_{n_k \beta(n_k)}}{\beta(n_k)} \right) = 1 \text{ a.s.,}
\]

which establishes (4.3.23), for \(\epsilon^* = 1\) and consequently proof of (4.3.21) follows from (4.3.23), for \(\epsilon^* = 1\).

Now to establish (4.3.24), for \(\epsilon^* = 1\), we proceed as in Allan Gut [2]. Define, \(m_j = \min\{k : n_k > M^j\}, j = 1, 2, ..., \) and \(M\) is chosen such that \(\frac{1}{\rho^2 M} < 1\).

Hence using (4.3.25), we get that, \(M^j < n_{m_j} \leq \frac{1}{\rho^2 M}, j=1,2,.... \) Consequently \(\{n_{m_j}\}\) satisfies the condition \(\lim \sup_{k \to \infty} \frac{n_{m_j-1}}{n_{m_j}} < 1\) of Theorem 4.3.1 with

\[
\sum_{k \geq k_0} \left( \log n_{m_j} \right)^{-\varepsilon_1} \frac{\alpha}{\varepsilon_1} \text{ for all } \varepsilon < 1 \text{ (i.e. } \varepsilon^* \geq 1).\]

Hence (4.3.24) follows from Theorem 4.3.1, for \(\epsilon^* = 1\). Consequently (4.3.22) follows from (4.3.24) and hence proof of Theorem is completed.

Section 4.2 communicated to the journal with title “A log log law for subsequences of delayed random sums” by Gooty Divanji and Raviprakash, K. N. [29].
Section 4.3 forms the content of “Limit infimum results for subsequences of delayed random sums and related boundary crossing problem”, by Gooty Divanji and Raviprakash, K. N. [27].