CHAPTER 7

New Oscillation Criteria for Forced Second Order Impulsive Delay Differential Equations with Mixed Nonlinearities

7.1 Introduction

This chapter deals with the oscillation criteria for forced second order impulsive delay differential equations with mixed nonlinearities of the form

\[
\begin{align*}
[r(t)(x'(t))^\gamma]' + p(t)(x'(t))^\gamma + & \sum_{i=1}^{n} p_i(t)x^\gamma(t-\tau_i) \\
& + \sum_{i=1}^{n} q_i(t)|x(t-\tau_i)|^{\alpha_i} \text{sgn } x(t-\tau_i) = e(t), \quad t \neq \tau_k
\end{align*}
\]  

(7.1.1)

\[
x(\tau_k^+) = a_k x(\tau_k), \quad x'(\tau_k^+) = b_k x'(\tau_k).
\]  

(7.1.2)

where \{\tau_k\} denotes the impulsive moments sequence with

\[0 \leq t_0 = \tau_0 < \tau_1 < \cdots < \tau_k < \cdots, \quad \lim_{k \to \infty} \tau_k = \infty, \quad \text{and}\]

\[x(\tau_k) = x(\tau_k^-) = \lim_{t \to \tau_k^-} x(t), \quad x(\tau_k^+) = \lim_{t \to \tau_k^+} x(t)\]

\[
x'(\tau_k^-) = \lim_{h \to 0^-} \left[ \frac{x(\tau_k + h) - x(\tau_k)}{h} \right], \quad x'(\tau_k^+) = \lim_{h \to 0^+} \left[ \frac{x(\tau_k + h) - x(\tau_k^+)}{h} \right].
\]

In what follows we assume that the following conditions hold:

(A1) \( r \in C^{1}([t_0, \infty), (0, \infty]), p, p_i, q_i, e \in C([t_0, \infty), \mathbb{R}), i = 1, 2, \ldots, n \) with \( r(t) \) is positive, non decreasing and differentiable, \( \gamma \) is a ratio of odd positive integers, \( \alpha_i > 0, i = 1, 2, \ldots, n \) and \( \alpha_i > \gamma \) for \( i = 1, 2, \ldots, m \), \( \alpha_i < \gamma \) for \( i = m + 1, m + 2, \ldots, n \).

(A2) \( b_k \geq a_k > 0 \) are constants, \( k = 1, 2, \ldots \) and \( \tau_i \geq 0 \).
Let $J \subset \mathbb{R}$ be an interval, we define

$$PC(J, \mathbb{R}) = \{ x: J \rightarrow \mathbb{R} : x(t) \text{ is piecewise left continuous and has discontinuity of first kind at } \tau_k's \}. $$

By a solution of (7.1.1) we mean a nontrivial real valued function $x \in PC([t_0, \infty), \mathbb{R})$, which has the property that $r(x')' \in PC([t_0, \infty), \mathbb{R})$ such that (7.1.1) is satisfied for all $t \geq t_0$.

The study of oscillatory behavior of second order impulsive forced delay differential equations which are particular cases of (7.1.1) has been already studied. Compared to second order ordinary differential equations [16, 19, 39, 44, 67, 77, 78, 79, 84], the oscillatory behavior of impulsive second order differential equations received less attention even though such equations have many applications. The new oscillation criteria for forced second order differential and impulsive differential equations with mixed nonlinearities are studied by many authors one can refer [1, 32, 59, 79, 84].

In section 7.2, we establish some new oscillation criteria for all solutions of (7.1.1) and (7.1.2). Our results extend those obtained in [59]. In section 7.3, some examples are presented to illustrate the results.

### 7.2 Oscillation Results

In this section, we present some results concerning the oscillatory behavior of solution of (7.1.1) and (7.1.2).

We begin with the following notation.

Let $k(s) = \max\{j, t_0 < \tau_j < s\}$, $r_j = \max\{r(t); t \in [c_j, d_j]\}$, $j = 1, 2$. For two constants $c_j, d_j \notin \{\tau_k\} (j = 1, 2)$ with $c_1 < d_1 \leq c_2 < d_2$ and a function $\psi \in C([c, d], \mathbb{R})$, we define an operator $\Omega : C([c, d], \mathbb{R}) \rightarrow \mathbb{R}$ by

$$\Omega^d_c[\psi] = \begin{cases} 0 & \text{for } k(c) = k(d) \\ \psi(\tau_{k(c)}+1)\phi(c) + \sum_{\ell=k(c)+2} \psi(\tau_\ell)\epsilon(\tau_\ell) & \text{for } k(c) < k(d) \end{cases}$$
where
\[
\phi(c) = \frac{b_k^{\gamma(c)} - a_k^{\gamma(c)} + 1}{a_k^{\gamma(c)}(\tau_k(c) + 1 - c)^\gamma}, \quad \epsilon(\tau_l) = \frac{b_l^{\gamma} - a_l^{\gamma}}{a_l^{\gamma}(\tau_l - \tau_{l-1})^\gamma}.
\]

Following Philos [67], we say that a continuous function \(H(t, s)\) belongs to a function class \(D_{c,d}\) denoted by \(H \in D_{c,d}\) if
\[
H(0, 0) = 0 = H(0, 0),
\]
\[
H(t, s) > 0, \quad H(t, t) = 0 = H(t, t),
\]
\[
\frac{\partial H_1(t, s)}{\partial t} = h_1(t, s)H_1(t, s),
\]
\[
\frac{\partial H_2(t, s)}{\partial t} = -h_2(t, s)H_2(t, s).
\]

To prove our main results, we begin with the following lemma due to Sun and Wong [79].

**Lemma 7.2.1** Let \((\alpha_1, \alpha_2, \ldots, \alpha_n)\) be an \(n\)-tuple satisfying
\[
\alpha_1 > \alpha_2 > \cdots > \alpha_m > \gamma > \alpha_{m+1} > \cdots > \alpha_n > 0.
\]
Then, there exists an \(n\)-tuple \((\eta_1, \eta_2, \ldots, \eta_n)\) with
\[
0 < \eta_i < 1
\]
and also either
\[
\sum_{i=1}^{n} \eta_i < 1
\]
or
\[
\sum_{i=1}^{n} \eta_i = 1.
\]

The proof of Lemma 7.2.1 can be established from [79, Lemma 1], where case \(\gamma = 1\) was treated.

**Theorem 7.2.1** Suppose that for any \(T \geq 0\), there exists \(c_j, d_j, \delta_j \notin \{\tau_k\}, j = 1, 2\) such that
\[
T \leq c_1 < \delta_1 < d_1 \leq c_2 < \delta_2 < d_2,
\]
and
\[
p_i(t), q_i(t) \geq 0, t \in [c_1 - \tau_i, d_1] \cup [c_2 - \tau_i, d_2], i = 1, 2, \ldots, n
\]
\[
e(t) \leq 0, \quad t \in [c_1 - \tau_i, d_1]
\]
\[
e(t) \geq 0, \quad t \in [c_2 - \tau_i, d_2]
\]

(7.2.4)
and if there exists $H_j \in D_{c_j,d_j}$, $j = 1, 2$ such that

$$
\frac{1}{H_j(\delta_j,c_j)} \int_{c_j}^{\delta_j} H_j(t,c_j) \left[ Q_j(t) - \frac{r(t)}{\gamma + 1} \left( h_{j_1}(t,c_j) - \frac{p(t)}{r(t)} \right) \right] dt \\
+ \frac{1}{H_j(d_j,\delta_j)} \int_{\delta_j}^{d_j} H_j(d_j,t) \left[ Q_j(t) - \frac{r(t)}{\gamma + 1} \left( h_{j_2}(d_j,t) + \frac{p(t)}{r(t)} \right) \right] \gamma + 1 dt
$$

$$
> \wedge (H_j,c_j,d_j)
$$

(7.2.5)

where

$$
Q_j(t) = \sum_{i=1}^{n} p_i(t) \left( \frac{t - c_j}{t - c_j + \tau_i} \right)^{\gamma} + \eta_0^{-\eta_0} |e(t)|^{\eta_0} \prod_{i=1}^{n} (\eta_i q_i(t))^{\eta_i} \left( \frac{t - c_j}{t - c_j + \tau_i} \right)^{\alpha_i \eta_i},
$$

$$
\eta_0 = 1 - \sum_{i=1}^{n} \eta_i \text{ and } \wedge (H_j,c_j,d_j) = \frac{r_j}{H_j(\delta_j,c_j)} \Omega^2 Q_{c_i} [H_j(\ldots,c_j)] + \frac{r_j}{H_j(d_j,\delta_j)} \Omega^2 Q_{d_i} [H_j(\ldots)]
$$

(7.2.6)

and $\eta_1, \eta_2, \ldots, \eta_n$ are positive constants satisfying (7.2.1) and (7.2.2) of Lemma 7.2.1.

Then, every solution of equation (7.1.1) and (7.1.2) of Lemma 7.1.2 is oscillatory.

**Proof.** Let $x(t)$ be a solution of (7.1.1) and (7.1.2). Suppose $x(t)$ does not have any zero in $[c_1,d_1] \cup [c_2,d_2]$. Without loss of generality, we may assume that $x(t) > 0$ for $t \geq t_0 - \tau > 0$ where $t_0$ depends on the solution $x(t)$ and $\tau = \max \{ \tau_i \}, i = 1, 2, \ldots, n$.

When $x(t)$ is eventually negative, the proof follows the same argument by using the interval $[c_2,d_2]$ instead of $[c_1,d_1]$. Choose $c_1, d_1 \geq t_0$ such that $p_i(t), q_i(t) \geq 0$ and $e(t) \leq 0$ for $t \in [c_1 - \tau_i, d_1]$ and $i = 1, 2, \ldots, n$. From (7.1.1), we have,

$$
[r(t)(x'(t))^\gamma]' + p(t)(x'(t))^\gamma \leq 0
$$

(7.2.7)

$$
[(x'(t))^\gamma \exp \left( \int_{c_1 - \tau_i}^{t} r'(s) + p(s) \frac{ds}{r(s)} \right)']' \leq 0 \text{ is nonincreasing on } [c_1 - \tau_i, d_1].
$$

Therefore for $c_1 - \tau_i < s < t \leq d_1$, we have,

$$
x(t) - x(c_1 - \tau_i) \geq \frac{x'(t) \exp \left( \int_{c_1 - \tau_i}^{t} r'(s) + p(s) \frac{ds}{r(s)} \right)^{1/\gamma}}{\exp \left( \int_{c_1 - \tau_i}^{c_1 - \tau_i} r'(s) + p(s) \frac{ds}{r(s)} \right)^{1/\gamma}} (t - c_1 + \tau_i)
$$
or \( x(t) \geq x'(t)(t - c_1 + \tau_i) \) for some \( \zeta \in (c_1 - \tau_i, t) \) where \( t \in (c_1 - \tau_i, d_1] \). Noting that \( x(c_1 - \tau_i) > 0 \) and \( r(t) \) is non-decreasing we have,

\[
\frac{1}{t - c_1 + \tau_i} \geq \frac{x'(t)}{x(t)}, \quad t \in (c_1 - \tau_i, d_1].
\] (7.2.8)

Integrating (7.2.8) from \( t - \tau_i \) to \( t \) for \( t > c_1 \), we obtain

\[
\frac{x(t - \tau_i)}{x(t)} \geq \frac{t - c_1}{t - c_1 + \tau_i}, \quad t \in (c_1, d_1].
\] (7.2.9)

Define

\[
w(t) = -\frac{r(t)(x'(t))^\gamma}{(x(t))^\gamma}, \quad t \in [c_1, d_1].
\] (7.2.10)

Then, for \( t \in [c_1, d_1] \) and \( t \neq \tau_k \), we have,

\[
w'(t) = \frac{p(t)(x'(t))^\gamma}{x^\gamma(t)} + \sum_{i=1}^{n} \frac{p_i(t)x^\gamma(t - \tau_i)}{x^\gamma(t)} + \sum_{i=1}^{n} \frac{q_i(t)x^{\alpha_i}(t - \tau_i)}{x^\gamma(t)}
\]

\[
- \frac{e(t)}{x^\gamma(t)} + \gamma |\frac{w^{\gamma+1}}{(r(t))^{1/\gamma}}|
\]

\[
\geq -\frac{p(t)(w(t))}{r(t)} + \sum_{i=1}^{n} p_i(t) \left( \frac{t - c_1}{t - c_1 + \tau_i} \right)^\gamma + \sum_{i=1}^{n} q_i(t)x^{\alpha_i}(t - \tau_i)
\]

\[
- \frac{e(t)}{x^\gamma(t)} + \gamma |\frac{w^{\gamma+1}}{(r(t))^{1/\gamma}}|.
\] (7.2.11)

Recall the arithmetic-geometric mean inequality

\[
\sum_{i=0}^{n} \eta_i u_i \geq \prod_{i=0}^{n} u_i^{\eta_i}, \quad u_i \geq 0
\]

where \( \eta_0 = 1 - \sum_{i=1}^{n} \eta_i \) and \( \eta_i > 0, \ i = 1, 2, 3, \ldots, n \) are chosen to given \( \alpha_1, \alpha_2, \ldots, \alpha_n \) as in Lemma 7.2.1 satisfying (7.2.1) and (7.2.2) for \( t \in [c_1, d_1], \)

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\[
|e(t)| + \sum_{i=1}^{n} q_i(t)x^{\alpha_i}(t - \tau_i) = \eta_0(\eta_0^{-1}|e(t)|) + \sum_{i=1}^{n} \eta_i(\eta_i^{-1}q_i(t))x^{\alpha_i}(t - \tau_i)
\]

\[
\prod_{i=1}^{n} x^{\alpha_i \eta_i}(t)
\]

\[
(\eta_0^{-1}|e(t)|)^{\eta_0} \prod_{i=1}^{n} (\eta_i^{-1}q_i(t))^{\eta_i} x^{\alpha_i \eta_i}(t - \tau_i)
\]

\[
= (\eta_0^{-1}|e(t)|)^{\eta_0} \prod_{i=1}^{n} (\eta_i^{-1}q_i(t))^{\eta_i} \left( \frac{t - c_1}{t - c_1 + \tau_i} \right)^{\alpha_i \eta_i}.
\]

Then (7.2.11) becomes,

\[
w'(t) \geq - \frac{p(t)(w(t))}{r(t)} + \sum_{i=1}^{n} p_i(t) \left( \frac{t - c_1}{t - c_1 + \tau_i} \right)^\gamma 
\]

\[
+ \prod_{i=1}^{n} (\eta_i^{-1}q_i(t))^{\eta_i} \left( \frac{t - c_1}{t - c_1 + \tau_i} \right)^{\alpha_i \eta_i} (\eta_0^{-1}|e(t)|)^{\eta_0} + \frac{\gamma |w^{\gamma+1}(t)|}{(r(t))^{1/\gamma}}
\]

\[
= Q_1(t) - \frac{p(t)(w(t))}{r(t)} + \frac{\gamma |w^{\gamma+1}(t)|}{(r(t))^{1/\gamma}}, \quad t \in [c_1, d_1], \quad t \neq \tau_k.
\]  

(7.2.12)

For \( t = \tau_k, \ k = 1, 2, \ldots \), from (7.2.10) we have,

\[
w(\tau_k^+) = - \frac{r(\tau_k^+)(x'(\tau_k^+))^{\gamma}}{(x(\tau_k^+))^{\gamma}} = \frac{b_\gamma}{a_\gamma^\gamma} w(\tau_k).
\]  

(7.2.13)

Notice that whether or not impulsive moments in \([c_1, \delta_1]\) and \([\delta_1, d_1]\) we discuss the following cases, as, \( k(c_1) < k(\delta_1) < k(d_1) \); \( k(c_1) = k(\delta_1) < k(d_1) \),

\( k(c_1) < k(\delta_1) = k(d_1) \) and \( k(c_1) = k(\delta_1) = k(d_1) \).

**Case (i):** If \( k(c_1) < k(\delta_1) < k(d_1) \), the impulsive moments in \([c_1, \delta_1]\) are \( \tau_k(c_1)+1, \tau_k(c_1)+2, \ldots, \tau_k(\delta_1) \) and in \([\delta_1, d_1]\) are \( \tau_k(\delta_1)+1, \ldots, \tau_k(d_1) \) respectively. Multiply
both sides of (7.2.12) by \(H_1(t, c_1)\) and integrating from \(c_1\) to \(\delta_1\) we have,

\[
\int_{c_1}^{\delta_1} H_1(t, c_1)Q_1(t)dt \leq \int_{c_1}^{\delta_1} H_1(t, c_1)w'(t)dt - \int_{c_1}^{\delta_1} H_1(t, c_1)\frac{\gamma|w^{\frac{\gamma+1}{\gamma}}(t)|}{r(t)^{1/\gamma}}dt \\
+ \int_{c_1}^{\delta_1} H_1(t, c_1)\frac{p(t)}{r(t)}w(t)dt \\
\leq \sum_{\ell=k(c_1)+1}^{k(\delta_1)} H_1(\tau_\ell, c_\ell)\frac{a^\gamma_\ell - b^\gamma_\ell}{a^\ell}w(\tau_\ell) + H_1(\delta_1, c_1)w(\delta_1) \\
- \int_{c_1}^{\delta_1} H_1(t, c_1)\left[h_{11}(t, c_1) - \frac{p(t)}{r(t)}\right]w(t) + \frac{\gamma|w^{\frac{\gamma+1}{\gamma}}(t)|}{r(t)^{1/\gamma}}dt
\]

by applying the inequality \(A\omega + B\omega^{\frac{\gamma+1}{\gamma}} \geq -\left(\frac{A}{\gamma+1}\right)\gamma + \left(\frac{\gamma}{B}\right)\gamma\) we have,

\[
\int_{c_1}^{\delta_1} H_1(t, c_1)Q_1(t)dt \leq \sum_{\ell=k(c_1)+1}^{k(\delta_1)} H_1(\tau_\ell, c_\ell)\frac{a^\gamma_\ell - b^\gamma_\ell}{a^\ell}\omega(\tau_\ell) + H_1(\delta_1, c_1)\omega(\delta_1) \\
+ \int_{c_1}^{\delta_1} H_1(t, c_1)\frac{r(t)}{(\gamma+1)^{\gamma+1}}\left[h_{11}(t, c_1) - \frac{p(t)}{r(t)}\right]^{\gamma+1}dt. \quad (7.2.14)
\]

On the other hand, multiplying both sides of inequality (7.2.12) by \(H_1(d_1, t)\), then integrating it from \(\delta_1\) to \(d_1\), we have,

\[
\int_{\delta_1}^{d_1} H_1(d_1, t)Q_1(t)dt \leq \sum_{\ell=k(\delta_1)+1}^{k(d_1)} H_1(d_\ell, \tau_\ell)\frac{a^\gamma_\ell - b^\gamma_\ell}{a^\ell}\omega(\tau_\ell) + H_1(d_1, \delta_1)\omega(\delta_1) \\
+ \int_{\delta_1}^{d_1} H_1(d_1, t)\frac{r(t)}{(\gamma+1)^{\gamma+1}}\left[h_{12}(d_1, t) + \frac{p(t)}{r(t)}\right]^{\gamma+1}dt. \quad (7.2.15)
\]

Equation (7.2.14) and (7.2.15) yield

\[
\frac{1}{H_1(\delta_1, c_1)} \int_{c_1}^{\delta_1} H_1(t, c_1) \left\{Q_1(t) - \frac{r(t)}{(\gamma+1)^{\gamma+1}}\left[h_{11}(t, c_1) - \frac{p(t)}{r(t)}\right]^{\gamma+1}\right\}dt \\
+ \frac{1}{H_1(d_1, \delta_1)} \int_{\delta_1}^{d_1} H_1(d_1, t) \left\{Q_1(t) - \frac{r(t)}{(\gamma+1)^{\gamma+1}}\left[h_{12}(d_1, t) + \frac{p(t)}{r(t)}\right]^{\gamma+1}\right\}dt \\
\leq \frac{1}{H_1(d_1, \delta_1)} \sum_{\ell=k(\delta_1)+1}^{k(d_1)} H_1(\tau_\ell, c_\ell)\frac{a^\gamma_\ell - b^\gamma_\ell}{a^\ell}\omega(\tau_\ell) \\
+ \frac{1}{H_1(d_1, \delta_1)} \sum_{\ell=k(\delta_1)+1}^{k(d_1)} H_1(d_\ell, \tau_\ell)\frac{a^\gamma_\ell - b^\gamma_\ell}{a^\ell}\omega(\tau_\ell) \quad (7.2.16)
\]
from (7.2.8), we observe that,

\[
\frac{x'(t)}{x(t)} \leq \frac{1}{t - c_1 + \tau_i}, \quad t \in (c_1 - \tau_i, d_1]
\]

\[
-\frac{r(t)(x'(t))^\gamma}{(x(t))^\gamma} \geq -\frac{r(\zeta)}{(t - c_1 + \tau_i)^\gamma}
\]

letting \( t \to \tau_{k(c_1)+1}^- \), it follows that

\[
\omega(\tau_{k(c_1)+1}) \geq -\frac{r(\tau_{k(c_1)+1})}{[\tau_{k(c_1)+1} - c_1 - \tau_i]^{\gamma}} \geq -\frac{r_1}{[\tau_{k(c_1)+1} - c_1]^{\gamma}} \tag{7.2.17}
\]

and on \((\tau_{\ell-1}, \tau_{\ell})\),

\[
\omega(\tau_{\ell}) \geq -\frac{r_1}{(\tau_{\ell} - \tau_{\ell-1})^{\gamma}} \quad \text{for } t = k(c_1) + 2, \ldots, k(\delta_1). \tag{7.2.18}
\]

Thus we have,

\[
\sum_{\ell=k(c_1)+1}^{k(\delta_1)} \frac{a_\ell^7 - b_\ell^7}{a_\ell^7} \omega(\tau_{\ell}) H_1(\tau_{\ell}, c_1) \leq r_1 \Omega_{c_1}^{\delta_1}[H_1(., c_1)]
\]

and

\[
\sum_{\ell=k(\delta_1)+1}^{k(d_1)} \frac{a_\ell^7 - b_\ell^7}{a_\ell^7} \omega(\tau_{\ell}) H_1(d_1, \tau_{\ell}) \leq r_1 \Omega_{\delta_1}^{d_1}[H_1(d_1, .)].
\]

Therefore (7.2.16) becomes,

\[
\frac{1}{H_1(\delta_1, c_1)} \int_{c_1}^{\delta_1} H_1(t, c_1) \left\{ Q_1(t) - \frac{r(t)}{(\gamma + 1)^{\gamma+1}} \left[ h_{11}(t, c_1) - \frac{p(t)}{r(t)} \right]^{\gamma+1} \right\} dt
\]

\[
+ \frac{1}{H_1(d_1, \delta_1)} \int_{\delta_1}^{d_1} H_1(d_1, t) \left\{ Q_1(t) - \frac{r(t)}{(\gamma + 1)^{\gamma+1}} \left[ h_{12}(d_1, t) + \frac{p(t)}{r(t)} \right]^{\gamma+1} \right\} dt
\]

\[
\leq \frac{r_1 \Omega_{c_1}^{\delta_1}}{H_1(\delta_1, c_1)} H_1(., c_1) + \frac{r_1}{H_1(d_1, \delta_1)} \Omega_{c_1}^{d_1}[H_1(d_1, .)]
\]

\[
= \wedge(H_1, c_1, d_1) \tag{7.2.19}
\]

which contradicts (7.2.5) for \( j = 1 \).

**Case (ii):** If \( k(c_1) < k(\delta_1) < k(d_1) \), there is no impulsive moments in \([c_1, \delta_1]\). Then we have

\[
\int_{c_1}^{\delta_1} H_1(t, c_1) Q_1(t) dt \leq H_1(\delta_1, c_1) \omega(\delta_1)
\]

\[
+ \int_{c_1}^{\delta_1} H_1(t, c_1) \frac{r(t)}{(\gamma + 1)^{\gamma+1}} \left[ h_{11}(t, c_1) - \frac{p(t)}{r(t)} \right]^{\gamma+1} dt.
\]
Thus using $\Omega_{c_1}^{\delta_1}[H_1(\cdot,c_1)] = 0$ we obtain,

$$\frac{1}{H_1(\delta_1,c_1)} \int_{c_1}^{\delta_1} H_1(t,c_1) \left\{ Q_1(t) - \frac{r(t)}{(\gamma + 1)^{\gamma+1}} \left[ h_{11}(t,c_1) - \frac{p(t)}{r(t)} \right]^{\gamma+1} \right\} dt$$

$$+ \frac{1}{H_1(d_1,\delta_1)} \int_{\delta_1}^{d_1} H_1(d_1,t) \left\{ Q_1(t) - \frac{r(t)}{(\gamma + 1)^{\gamma+1}} \left[ h_{12}(d_1,t) + \frac{p(t)}{r(t)} \right]^{\gamma+1} \right\} dt$$

$$\leq \frac{r_1}{H_1(d_1,\delta_1)} \Omega_{\delta_1}^{d_1} H_1(\cdot,c_1)$$

$$\leq \land(H_1,c_1,d_1)$$

which is a contradiction for $j = 1$. By a similar arguments, we can easily verify for the other two cases. Hence the proof is complete. 

**Remark 7.2.1** When $a_k = b_k = 1$ and $\tau_1 = \tau_2 = \cdots = \tau_n = \tau$, $\sum_{i=1}^{n} p_i(t) = q(t)$ and $p(t) = 0$, Theorem 7.2.1 reduces to Theorem 2.3 in [84].

**Remark 7.2.2** When $\tau_1 = \tau_2 = \cdots = \tau_n = \tau$, $\gamma = 1$, $\sum_{i=1}^{n} p_i(t) = q(t)$, Theorem 7.2.1 reduces to Theorem 2.3 in [59].

**Theorem 7.2.2** Assume that for any $T > 0$, there exists $c,d,\delta \notin \{\tau_k\}$ such that $c < \delta < d$ and $p_i(t), q_i(t) \geq 0$ for $t \in [c - \tau_i, d]$, $i = 1,2,\ldots,n$, and there exists $H \in D_{a,b}$ such that

$$\frac{1}{H(\delta,c)} \int_{c}^{\delta} H(t,c) \left\{ \overline{Q}(t) - \frac{r(t)}{(\gamma + 1)^{\gamma+1}} \left[ h_1(t,c) - \frac{p(t)}{r(t)} \right]^{\gamma+1} \right\} dt$$

$$+ \frac{1}{H(d,\delta)} \int_{\delta}^{d} H(d,t) \left\{ \overline{Q}(t) - \frac{r(t)}{(\gamma + 1)^{\gamma+1}} \left[ h_2(d,t) + \frac{p(t)}{r(t)} \right]^{\gamma+1} \right\} dt$$

$$\geq \land(H,c,d)$$

(7.2.20)

where

$$\overline{Q}(t) = \left( \frac{t-c}{t-c+\tau_i} \right) \gamma \left[ \prod_{i=1}^{n} p_i(t) \right] + \left[ \prod_{i=1}^{n} (\eta_i^{-1} q_i(t)) \right] \left( \frac{t-c}{t-c+\tau_i} \right)^{\alpha_i \eta_i}$$

(7.2.21)

and $\eta_1, \eta_2, \ldots, \eta_n$ are positive constants satisfying (7.2.1) and (7.2.2) of Lemma 7.2.1, then (7.1.1) and (7.1.2) with $e(t) \equiv 0$ is oscillatory.

**Proof.** The proof is immediate from Theorem 7.2.1, if we put $e(t) \equiv 0$, $\eta_0 = 0$ and apply conditions (7.2.1) and (7.2.2) of Lemma 7.2.1.  

\[\square\]
Before stating our next result we introduce another function class say $v(t) \in C^1[c, d]$, $v(t) \neq 0$, $v^{\gamma+1}(t) > 0$ and $v(c) = v(d) = 0$.

**Theorem 7.2.3** Assume that for any $T > 0$, there exists $c_j, d_j \notin \{\tau_k\}$, $j = 1, 2$ such that $c_1 < d_1 \leq c_2 < d_2$ and (7.2.4) holds, and there exists $v_j \in E_{c_j, d_j}$ such that

$$
\int_{c_j}^{d_j} \left\{ Q_j(t)v_j^{\gamma+1}(t) - r(t) \left( v_j'(t) - \frac{p(t)v_j(t)}{(\gamma + 1)r(t)} \right)^{\gamma+1} \right\} dt > r_j\Omega_{c_j}^{d_j}[v_j^{\gamma+1}] \quad (7.2.22)
$$

for $j = 1, 2$, where, $Q_j(t)$ is defined as in (7.2.6), then (7.1.1) and (7.1.2) is oscillatory.

**Proof.** Suppose that $x(t)$ is a nonoscillatory solution of (7.1.1) and (7.1.2). Proceed as in the proof of Theorem 7.2.1 to get (7.2.12) and (7.2.13).

If $k(c_1) < k(d_1)$, there are all impulsive moments in $[c_1, d_1]$; $\tau_{k(c_1)+1}, \tau_{k(c_1)+2}, \ldots, \tau_{k(d_1)}$. Multiply both sides of (7.2.12) by $v_1^{\gamma+1}(t)$ and integrating over $[c_1, d_1]$, then using integration by parts, we obtain

$$
\sum_{\ell=k(c_1)+1}^{k(d_1)} \int_{c_1}^{d_1} \left\{ \left[ (\gamma + 1)v_1'(t) - \frac{p(t)v_1(t)}{r(t)} \right] v_1(t) \right\} dt \\
\geq \int_{c_1}^{d_1} \left\{ (\gamma + 1)v_1'(t) - \frac{p(t)v_1(t)}{r(t)} \right\} v_1(t) \omega(t) \\
+ \frac{\gamma v_1^{\gamma+1}(t)}{(r(t))^{\gamma/1/\gamma}} |\omega^{\gamma+1}(t)| dt \\
(7.2.23)
$$

by apply the inequality $A\omega + B\omega^{\gamma+1} \geq - \left( \frac{A}{\gamma + 1} \right)^{\gamma+1} \left( \frac{\gamma}{B} \right)^{\gamma}$ we get,

$$
\sum_{\ell=k(c_1)+1}^{k(d_1)} \int_{c_1}^{d_1} \left\{ Q_1(t)v_1^{\gamma+1}(t) - r(t) \left( v_1'(t) - \frac{p(t)v_1(t)}{r(t)(\gamma + 1)} \right)^{\gamma+1} \right\} dt.
$$

Thus we have,

$$
\sum_{\ell=k(c_1)+1}^{k(d_1)} \frac{a_\ell \gamma - b_\ell \gamma}{a_\ell} \omega(\tau_\ell) v_1^{\gamma+1}(\tau_\ell) \\
\geq \int_{c_1}^{d_1} \left\{ Q_1(t)v_1^{\gamma+1}(t) - r(t) \left( v_1'(t) - \frac{p(t)v_1(t)}{r(t)(\gamma + 1)} \right)^{\gamma+1} \right\} dt. \quad (7.2.24)
$$
Proceeding as in the proof of Theorem 7.2.1 and using (7.2.17) and (7.2.18), we obtain

$$\int_{c_1}^{d_1} \left[ Q_1(t)v_1^{\gamma+1}(t) - r(t) \left( v_1'(t) - \frac{p(t)v_1(t)}{r(t)(\gamma + 1)} \right)^{\gamma+1} \right] dt \leq r_1 \Omega_{c_1}^{d_1}[v_1^{\gamma+1}]$$

which contradicts our assumption (7.2.22).

If $k(c_1) = k(d_1)$, then $\Omega_{c_1}^{d_1}[v_1^{\gamma+1}] = 0$, and there are no impulsive moments in $[c_1, d_1]$. Similar to the proof of (7.2.24) we obtain,

$$\int_{c_1}^{d_1} \left[ Q_1(t)v_1^{\gamma+1}(t) - r(t) \left( v_1'(t) - \frac{p(t)v_1(t)}{r(t)(\gamma + 1)} \right)^{\gamma+1} \right] dt \leq 0$$

which is a contradiction, hence the proof is complete.

**Theorem 7.2.4** Assume that for any $T > 0$, there exists $c_1, d_1 \not\in \{\tau_k\}$ such that $c_1 < d_1$, and $p_i(t), q_i(t) \geq 0$ for $t \in [c_1, d_1]$ and if there exists $v \in C^1[c_1, d_1]$ such that

$$\int_{c_1}^{d_1} \left[ Q(t)v^{\gamma+1}(t) - r(t) \left( v'(t) - \frac{p(t)v(t)}{r(t)(\gamma + 1)} \right)^{\gamma+1} \right] dt > r_1 \Omega_{c_1}^{d_1}[v^{\gamma+1}]$$

where

$$Q(t) = \left( \frac{t - c}{t - c + \tau_i} \right)^\gamma \left[ \sum_{i=1}^{n} p_i(t) \right] + \left[ \prod_{i=1}^{n} (\eta_{i}^{-1} q_i(t) )^{\eta_{i}} \right] \left( \frac{t - c}{t - c + \tau_i} \right)^{\alpha_{i}\eta_{i}}$$

and $\alpha_1 > \alpha_2 > \cdots > \alpha_m > \gamma > \alpha_{m+1} > \cdots > \alpha_n > 0$ and $\eta_i > 0$, $i = 1, 2, \ldots, n$ satisfying (7.2.1) and (7.2.2) of Lemma 7.2.1. Then every solution of (7.1.1) and (7.1.2) is oscillatory.

**Proof.** The proof of the above theorem is in fact a particular version of the proof of Theorem 7.2.3 by putting $e(t) \equiv 0$, $\eta_0 = 0$ and apply the conditions (7.2.1) and (7.2.2) of Lemma 7.2.1. \qed

### 7.3 Examples

In this section, some examples are presented to illustrate our main results.
Example 7.3.1 Consider the impulsive delay differential equation

\[
[[x'(t)]'] + \cos(2t)(x'(t))^\gamma + \ell t^\gamma (t - \pi/8) + \ell t^{\gamma + 1} x^\gamma (t) \\
+ \ell_1 \sin^3(t)|x(t - \pi/8)|^{\alpha_1} \text{sgn } x(t - \pi/8) + \ell_2 \cos^3(t)|x(t)|^{\alpha_2} \text{sgn } x(t) \\
= -\cos^3(2t), \quad t \neq \tau_k
\]  

(7.3.1)

where \( k \in \mathbb{N}, t \geq t_0 > 0, \tau_{2n} = 2n\pi + \pi/5, \tau_{2n+1} = 2n\pi + 2\pi/7, \, n = 0, 1, 2, \ldots, \ell, \ell_1, \ell_2 \) are positive constants. Also note that \( r_1 = r_2 = 1 \). Now we choose \( \eta_0 = \eta_1 = \eta_2 = 1/3, \alpha_1 = \frac{57}{7}, \alpha_2 = \frac{7}{2}, e(t) = -\cos^3(2t), \tau_1 = \pi/8, \tau_2 = 0 \). For any \( T \geq 0 \), we can choose \( n \) large enough such that \( T < c_1 = 2k\pi + \pi/8 < \delta_1 = 2k\pi + \pi/6 < d_1 = c_2 = 2k\pi + \pi/4 < \delta_2 = 2k\pi + \pi/3 < d_2 = 2k\pi + \pi/2, \, k = 0, 1, 2, \ldots \). If we choose \( H_1(t, s) = H_2(t, s) = (t - s)^2 \), with \( h_{j_1}(t, s) = h_{j_2}(t, s) = 1, \, j = 1, 2 \).

Then by using the mathematical software Maple 6 with \( j = 1, \gamma = 1 \), we obtain

\[
\frac{1}{H_1(\delta_1, c_1)} \int_{c_1}^{\delta_1} H_1(t, c_1) \left[ Q_1(t) - \frac{r(t)}{(\gamma + 1)^{\gamma + 1}} \left( h_{11}(t, c_1) - \frac{p(t)}{r(t)} \right)^{\gamma + 1} \right] dt \\
+ \frac{1}{H_1(d_1, \delta_1)} \int_{\delta_1}^{d_1} H_1(d_1, t) \left[ Q_1(t) - \frac{r(t)}{(\gamma + 1)^{\gamma + 1}} \left( h_{12}(d_1, t) + \frac{p(t)}{r(t)} \right)^{\gamma + 1} \right] dt
\]

\( > \wedge (H_1(c_1, d_1)) \)

\[
\approx 0.062462\ell + 0.036392(\ell_1 \ell_2)^{1/3} - 0.063532 \approx \frac{27}{5\pi}
\]

\[
\approx 0.062462\ell + 0.036392(\ell_1 \ell_2)^{1/3} > \frac{27}{5\pi} + 0.063532
\]  

(7.3.2)

for \( j = 1, \gamma = 1/3 \), we obtain

\[
\approx 0.130375\ell + 0.051917(\ell_1 \ell_2)^{1/3} - 0.101661 > 0.110528
\]

\[
\approx 0.130375\ell + 0.051917(\ell_1 \ell_2)^{1/3} > 0.212189.
\]  

(7.3.3)

In a similar way, with \( j = 2, \gamma = 1 \) we obtain

\[
\approx 0.130375\ell + 0.051917(\ell_1 \ell_2)^{1/3} - 0.101661 > 0.110528
\]

\[
\approx 0.130375\ell + 0.051917(\ell_1 \ell_2)^{1/3} > 0.212189.
\]  

(7.3.4)
Consider the impulsive delay differential equation

\[
\frac{1}{H_2(\delta_2, c_2)} \int_{c_2}^{d_2} H_2(t, c_2) \left[ Q_2(t) - \frac{r(t)}{(\gamma + 1)^{\gamma + 1}} \left( h_{21}(t, c_2) - \frac{p(t)}{r(t)} \right) \right] dt \\
+ \frac{1}{H_2(d_2, \delta_2)} \int_{\delta_2}^{d_2} H_2(d_2, t) \left[ Q_2(t) - \frac{r(t)}{(\gamma + 1)^{\gamma + 1}} \left( h_{22}(d_2, t) + \frac{p(t)}{r(t)} \right) \right] dt \\
\approx \left( H_2(c_2, d_2) \right) \\
\approx 0.459012 \ell + 0.084876(\ell_1 \ell_2)^{1/3} - 0.046669 > \frac{18}{7\pi} \\
\approx 0.459012 \ell + 0.084876(\ell_1 \ell_2)^{1/3} > \frac{18}{7\pi} + 0.046669 \\
\text{(7.3.4)}
\]

for \( j = 2, \gamma = 1/3 \) we obtain

\[
\approx 0.507616 \ell + 0.132303(\ell_1 \ell_2)^{1/3} - 0.117349 > 0.055109 \\
\approx 0.507616 \ell + 0.132303(\ell_1 \ell_2)^{1/3} > 0.172458. \\
\text{(7.3.5)}
\]

So, if we choose the constants \( \ell, \ell_1, \ell_2 \) such that (7.3.2) - (7.3.5), then by Theorem 7.2.1 equation (7.3.1) is oscillatory.

**Example 7.3.2** Consider the impulsive delay differential equation

\[
[[x'(t)]']' + \cos(2t)(x'(t))^\gamma + a_0[t^\gamma(t - \pi/8) + t^{\gamma + 1} x^\gamma(t)] \\
+ a_1\sin^3(t) |x(t - \pi/8)|^{5\gamma/2} \sgn x(t - \pi/8) + a_2\cos^3(t)|x(t)|^{\gamma/2} \sgn x(t) \\
= -\cos^3(2t), \ t \neq \tau_k \\
\quad x(\tau_k^+) = \frac{1}{2} x(\tau_k), \ x'(\tau_k^+) = \frac{3}{4} x'(\tau_k), \ \tau_k = 2k\pi + \pi/5 \\
\text{(7.3.6)}
\]

where \( k \in \mathbb{N}, t \geq t_0 > 0, a_0, a_1, a_2 \) are positive constants. We can choose \( n \) large enough \( c_1 = 2n\pi + \pi/8, d_1 = 2n\pi + \pi/4, c_2 = 2n\pi + \pi/4, d_2 = 2n\pi + \pi/2, n = 1, 2, \ldots. \)

If we take \( v_1(t) = t, v_2(t) = t^3 \) then by using the mathematical software Maple 6, we obtain for \( j = 1, \gamma = 1, \)

\[
\int_{c_1}^{d_1} \left\{ Q_j(t)v_j^{\gamma + 1}(t) - r(t) \left( v_j'(t) - \frac{p(t)v_j(t)}{(\gamma + 1)r(t)} \right)^{\gamma + 1} \right\} dt > r_j\Omega_{c_j}^d[v_j^{\gamma + 1}] \\
\approx a_0(0.091592) + 0.0193399(a_1a_2)^{1/3} - 0.320095 > 0.837758 \\
\approx a_0(0.091592) + 0.0193399(a_1a_2)^{1/3} > 1.157853. \\
\text{(7.3.7)}
\]
In a similar way, for $j = 2, \gamma = 1$ we obtain

\[
\approx a_0(9.216034) + (0.818256)(a_1a_2)^{1/3} > 3.730294 \tag{7.3.8}
\]

by computing, $r_1\Omega_0^d[v_1^{\gamma+1}] = 0.837758$, $r_2\Omega_0^d[v_2^{\gamma+1}] = 0$.

In a similar fashion, for $j = 1, \gamma = 1/3$ we obtain

\[
\approx a_0(0.217692) + (0.0553222)(a_1a_2)^{1/3} - 0.317956 > 0.837758
\]

\[
\approx a_0(0.217692) + (0.0553222)(a_1a_2)^{1/3} > 1.155714 \tag{7.3.9}
\]

and for $j = 2, \gamma = 1/3$,

\[
\approx a_0(4.38933) + (0.681575)(a_1a_2)^{1/3} > 7.783846. \tag{7.3.10}
\]

So, if we choose the constants $a_0, a_1, a_2$ such that (7.3.7) - (7.3.10), then by Theorem 7.2.3 equation (7.3.6) is oscillatory.