CHAPTER 2

Asymptotic Behavior of Solutions of First-Order Neutral Impulsive Differential Equations with Positive and Negative Coefficients

2.1 Introduction

This chapter deals with the asymptotic behavior of solutions of first order neutral impulsive differential equations with positive and negative coefficients of the form

$$\frac{d}{dt} [x(t) - R(t)x(\alpha t)] + \frac{P(t)}{t} x(\beta t) - \frac{Q(t)}{t} x(\gamma t) = 0, \quad t \geq t_0, \quad t \neq t_k$$

(2.1.1)

$$x(t_k) = b_k x(t_k^-) + (1 - b_k) \left( \int_{\beta t_k}^{t_k} \frac{P(s/\beta)}{s} x(s) ds - \int_{\gamma t_k}^{t_k} \frac{Q(s/\gamma)}{s} x(s) ds \right), \quad k = 1, 2, \ldots.$$

(2.1.2)

where $0 < \alpha, \beta \leq \gamma < 1$, $0 < t_0 < t_1 < \cdots < t_k < \cdots$ are fixed points with $\lim_{k \to \infty} t_k = \infty$, $P(t), Q(t) \in C([t_0, \infty), [0, +\infty])$ and $R(t) \in PC([t_0, \infty), R)$, and $b_k, k = 1, 2, 3, \ldots$ are constants, $PC([t_0, \infty), R) = \{ f : [(t_0, \infty) \to R | f \text{ is continuous for } t_0 \leq t \leq t_1, t_k < t \leq t_{k+1} \text{ and } f(t_k^+) \text{ and } f(t_k^-) \text{ exist with } \lim_{t \to t_k^-} f(t) = f(t_k^-), \quad k = 1, 2, \ldots \}$.

With the system (2.1.1)-(2.1.2) one associates an initial condition of the form

$$x_{t_0} = \phi(\eta), \quad \eta \in [\rho, 1]$$

(2.1.3)

where $\rho = \min_{k} \{ \alpha, \beta \}$ $x_{t_0} = x(\eta t_0)$ for $\rho \leq \eta \leq 1$ and $\phi \in PC([\rho, 1], R) = \{ \phi : [\rho, 1] \to R | \phi \text{ is continuous everywhere except at a finite number of points} \}$.
$\eta$, and $\phi(\eta^+)$ and $\phi(\eta^-) = \lim_{\eta \to \eta^+} \phi(\eta)$ exist with $\phi(\eta^-) = \phi(\eta)$}.

A function $x(t)$ is said to be a solution of (2.1.1) and (2.1.2) satisfying the initial value condition (2.1.3) if,

(i) $x(t) = \phi(t/t_0)$ for $\rho t_0 \leq t \leq t_0$, $x(t)$ is continuous for $t \geq t_0$ and $t \neq t_k$, $k = 1, 2, 3, \ldots$.

(ii) $x(t) - R(t)x(\alpha t)$ is continuously differentiable for $t > t_0$ and $t \neq t_k$, $k = 1, 2, 3, \ldots$ and satisfies (2.1.1).

(iii) $x(t_k^+)$ and $x(t_k^-)$ exist with $x(t_k^-) = x(t_k)$ and satisfy (2.1.2).

The study of asymptotic behavior of solution of first order impulsive differential equations has attracted a good bit of attention in the last few years. The asymptotic behavior of solutions of various impulsive differential equations has been studied extensively in the literature, see for example [29, 53, 86, 97]. In [71, 86], they studied the asymptotic behavior of solutions for a nonlinear first order impulsive differential equations with constant delays. It is noted that through the impulse in (2.1.2) is a special form of the impulsive term form, the method gives in this chapter will mark this impulse term form. We also note that when all $b_k = 1$, $k = 1, 2, 3, \ldots$, the system (2.1.1)-(2.1.2) reduces to a delay differential equation without impulses and with $Q(t) = 0$, whose oscillatory behavior of solutions has been studied in [27].

In section 2.2, the study establishes sufficient conditions for every solution of (2.1.1) and (2.1.2) tends to a constant as $t \to \infty$. In section 2.3, the study presents an example to illustrate the results. The results obtained in this chapter improve and extend some of the existing results presented in [29].

2.2 Asymptotic Behavior

In this section, we obtain some sufficient conditions for the equation (2.1.1) and (2.1.2).
Theorem 2.2.1 Assume that the following conditions hold:

(A1) \( \alpha t_k \) is not an impulsive point and \( 0 < b_k \leq 1 \) for \( k = 1, 2, 3, \ldots \) with 
\[
\sum_{k=1}^{\infty} (1 - b_k) < \infty
\]

(A2) 
\[
\lim_{t \to \infty} |R(t)| = \mu < 1 \text{ and } R(t_k) = b_k R(t_k^-) \tag{2.2.1}
\]

(A3) 
\[
H(t) = P(t) - Q(\beta t/\gamma) > 0 \text{ for } t \geq t^* = \gamma t/\beta \tag{2.2.2}
\]

(A4) 
\[
\lim_{t \to \infty} \int_{\beta t}^{\gamma t} \frac{Q(s/\gamma)}{s} ds = 0 \tag{2.2.3}
\]

(A5) 
\[
\lim_{t \to \infty} \sup \left[ \int_{\beta t}^{t/\beta} \frac{H(s/\beta)}{s} ds + \frac{Q(t/\gamma)}{H(t/\beta)} \int_{\beta t}^{t} \frac{H(s/\beta^2)}{s} ds + \mu \left( 1 + \frac{H(t/\alpha \beta)}{H(t/\beta)} \right) \right] < 2 \tag{2.2.4}
\]

then every solution of (2.1.1) and (2.1.2) tends to a constant as \( t \to \infty \).

Proof. Let \( x(t) \) be any solution of (2.1.1) and (2.1.2). We shall prove that \( \lim_{t \to \infty} x(t) \) exists and is finite. From (2.1.1) and (2.2.2)
\[
\left[ x(t) - R(t)x(\alpha t) - \int_{\beta t}^{t} \frac{H(s/\beta)}{s} x(s) ds - \int_{\beta t}^{\gamma t} \frac{Q(s/\gamma)}{s} x(s) ds \right]'
+ \frac{H(t/\beta)}{t} x(t) = 0, \quad t \geq t_0, t \neq t_k. \tag{2.2.5}
\]

From (2.1.2)
\[
x(t_k) = b_k x(t_k^-) + (1 - b_k) \left[ \int_{\beta t_k}^{\gamma t_k} \frac{Q(s/\gamma)}{s} x(s) ds \\
+ \int_{\beta t_k}^{t_k} \frac{H(s/\beta)}{s} x(s) ds \right], \quad k = 1, 2, 3, \ldots. \tag{2.2.6}
\]

From (2.2.1) and (2.2.4) we can choose \( \epsilon > 0 \) and \( \delta > 0 \) sufficiently small and \( T > t_0 \)
sufficiently large such that \( \mu + \epsilon < 1 \),

\[
\left[ \int_{\beta t}^{t/\beta} \frac{H(s/\beta)}{s} ds + \frac{Q(t/\gamma)}{H(t/\beta)} \int_{\beta t}^{t} \frac{H(s/\beta^2)}{s} ds \\
+ (\mu + \epsilon) \left( 1 + \frac{H(t/\alpha\beta)}{H(t/\beta)} \right) \right] < 2 - \delta \tag{2.2.7}
\]

and

\[ |R(t)| \leq \mu + \epsilon \text{ for } t \geq T. \tag{2.2.8} \]

Let

\[
W_1(t) = \left[ x(t) - R(t)x(\alpha t) - \int_{\beta t}^{t} \frac{H(s/\beta)}{s} x(s) ds - \int_{\beta t}^{t} \frac{Q(s/\gamma)}{s} x(s) ds \right]^2
\]

\[
W_2(t) = \int_{\beta t}^{t} \frac{H(s/\beta^2)}{s} \int_{s}^{t} \frac{Q(u/\gamma)}{\gamma} x^2(u) du ds
\]

\[
W_3(t) = \int_{\beta t}^{t} \frac{H(s/\beta^2)}{s} \int_{s}^{t} \frac{Q(u/\gamma)}{\gamma} x^2(u) du ds + (\mu + \epsilon) \int_{\beta t}^{t} \frac{H(s/\alpha\beta)}{s} x^2(s) ds.
\]

Using (2.2.5) and the inequality \( 2ab \leq a^2 + b^2 \) we have

\[
\frac{dW_1}{dt} \leq - \frac{H(t/\beta)}{t} \left[ 2x^2(t) - |R(t)|x^2(\alpha t) - |R(t)|x^2(t) \right]
\]

\[
- x^2(t) \int_{\beta t}^{t} \frac{H(s/\beta)}{s} ds - \int_{\beta t}^{t} \frac{H(s/\beta)}{s} x^2(s) ds
\]

\[
- x^2(t) \int_{\beta t}^{t} \frac{Q(s/\gamma)}{s} ds - \int_{\beta t}^{t} \frac{Q(s/\gamma)}{s} x^2(s) ds
\]

\[
\frac{dW_2}{dt} \leq \frac{Q(t/\gamma)}{t} x^2(t) \int_{\beta t}^{t} \frac{H(s/\beta^2)}{s} ds - \frac{H(t/\beta)}{t} \int_{\beta t}^{t} \frac{Q(s/\gamma)}{s} x^2(s) ds
\]

and

\[
\frac{dW_3}{dt} = \frac{H(t/\beta)}{t} x^2(t) \int_{\beta t}^{t} \frac{H(s/\beta^2)}{s} ds - \frac{H(t/\beta)}{t} \int_{\beta t}^{t} \frac{H(s/\beta)}{s} x^2(s) ds
\]

\[
+ (\mu + \epsilon) \frac{H(t/\alpha\beta)}{t} x^2(t) - (\mu + \epsilon) \frac{H(t/\beta)}{t} x^2(\alpha t). \]
Set $W(t) = W_1(t) + W_2(t) + W_3(t)$. From the above inequalities we obtain

$$\frac{dW}{dt} = \frac{dW_1}{dt} + \frac{dW_2}{dt} + \frac{dW_3}{dt}.$$

$$\leq - \frac{H(t/\beta)}{t} x^2(t) \left[ 2 - \int_{\beta t}^{t/\beta} \frac{H(s/\beta)}{s} ds - \int_{\beta t}^{t} \frac{Q(s/\gamma)}{s} ds 
- \frac{Q(t/\gamma)}{H(t/\beta)} \int_{\beta t}^{t} \frac{H(s/\beta^2)}{s} ds - (\mu + \epsilon) \left( 1 + \frac{H(t/\alpha \beta)}{H(t/\beta)} \right) \right].$$

From (2.2.3) and (2.2.7), we have

$$\frac{dW}{dt} \leq - \frac{H(t/\beta)}{t} x^2(t) \left[ 2 - \left( \int_{\beta t}^{t/\beta} \frac{H(s/\beta)}{s} ds + \frac{Q(t/\gamma)}{H(t/\beta)} \int_{\beta t}^{t} \frac{H(s/\beta^2)}{s} ds 
+ (\mu + \epsilon) \left( 1 + \frac{H(t/\alpha \beta)}{H(t/\beta)} \right) \right) \right]$$

$$\leq - \delta \frac{H(t/\beta)}{t} x^2(t), \quad t \neq t_k.$$  \hspace{1cm} (2.2.9)

As $t = t_k$ we have

$$W(t_k) = b_k^2 \left[ x(t_k^-) - R(t_k^-) x(\alpha t_k^-) - \int_{\beta t_k}^{t_k} \frac{H(s/\beta)}{s} x(s) ds - \int_{\beta t_k}^{t_k} \frac{Q(s/\gamma)}{s} x(s) ds \right]^2$$

$$+ \int_{\beta t_k}^{t_k} \frac{H(s/\beta^2)}{s} \int_{s}^{t_k} \frac{Q(u/\gamma)}{u} x^2(u) du ds$$

$$+ \int_{\beta t_k}^{t_k} \frac{H(s/\beta^2)}{s} \int_{s}^{t_k} \frac{H(u/\beta)}{u} x^2(u) du ds + (\mu + \epsilon) \int_{\beta t_k}^{t_k} \frac{H(s/\alpha \beta)}{s} x^2(s) ds$$

$$\leq W(t_k^-).$$  \hspace{1cm} (2.2.10)

From (2.2.9) and (2.2.10) we get $W(t)$ is decreasing and

$$\frac{H(t/\beta)}{t} x^2(t) \in L^1(t_0, \infty)$$  \hspace{1cm} (2.2.11)

and hence for any $h \geq 0$ we have

$$\lim_{t \to \infty} \int_{ht}^{t} \frac{H(s/\beta)}{s} x^2(s) ds = 0.$$  \hspace{1cm} (2.2.12)
On the other hand

\[ 0 \leq W_2(t) = \int_{\beta t}^{t} \left( \frac{H(s/\beta^2)}{s} \int_{s}^{t} \frac{Q(u/\gamma)}{u} x^2(u) \, du \right) \, ds \]

\[ \leq 2 \int_{\beta t}^{t} \frac{H(u/\beta)}{u} x^2(u) \, du \to 0 \text{ as } t \to \infty. \]

and

\[ 0 \leq W_3(t) = \int_{\beta t}^{t} \frac{H(s/\beta^2)}{s} \int_{s}^{t} \frac{H(u/\beta)}{u} x^2(u) \, ds \, du + \left( \mu + \epsilon \right) \int_{at}^{\alpha t} \frac{H(s/\alpha \beta)}{s} x^2(s) \, ds \]

\[ \leq 2 \int_{\beta t}^{t} \frac{H(u/\beta)}{u} x^2(u) \, du + 2 \int_{at}^{\alpha t} \frac{H(s/\beta)}{s} x^2(s) \, ds \to 0 \text{ as } t \to \infty. \]

Hence, \( \lim_{t \to \infty} W_1(t) = \lim_{t \to \infty} W(t) = \eta \), which is finite. That is,

\[ \lim_{t \to \infty} \left[ x(t) - R(t) x(\alpha t) - \int_{\beta t}^{t} \frac{H(s/\beta)}{s} x(s) \, ds - \int_{\beta t}^{t} \frac{Q(s/\gamma)}{s} x(s) \, ds \right]^2 = \eta. \quad (2.2.13) \]

Next we shall prove that

\[ \lim_{t \to \infty} \left[ x(t) - R(t) x(\alpha t) - \int_{\beta t}^{t} \frac{H(s/\beta)}{s} x(s) \, ds - \int_{\beta t}^{t} \frac{Q(s/\gamma)}{s} x(s) \, ds \right] \]

exists and is finite. Let

\[ y(t) = x(t) - R(t) x(\alpha t) - \int_{\beta t}^{t} \frac{H(s/\beta)}{s} x(s) \, ds - \int_{\beta t}^{t} \frac{Q(s/\gamma)}{s} x(s) \, ds. \]

From (2.2.5) we have \( y'(t) + \frac{H(t/\beta)}{t} x(t) = 0 \) and from (2.2.6) we have

\[ y(t_k) = b_k \left[ x(t_k^-) - R(t_k^-) x(\alpha t_k^-) - \int_{\beta t_k}^{t_k} \frac{H(s/\beta)}{s} x(s) \, ds - \int_{\beta t_k}^{t_k} \frac{Q(s/\gamma)}{s} x(s) \, ds \right] \]

\[ = b_k y(t_k^-). \]

Therefore system (2.2.5)-(2.2.6) can be rewritten as

\[ y'(t) + \frac{H(t/\beta)}{t} x(t) = 0, \quad t \geq t_0, t \neq t_k \]

\[ y(t_k) = b_k y(t_k^-) \quad k = 1, 2, \ldots. \quad (2.2.14) \]
In view of (2.2.13) we have
\[ \lim_{t \to \infty} y^2(t) = \eta. \] (2.2.15)

If \( \eta = 0 \), then \( \lim_{t \to \infty} y^2(t) = 0 \). If \( \eta > 0 \), then there exists a sufficiently large \( T_1 \) such that \( y(t) \neq 0 \) for any \( t > T_1 \). Otherwise there is a sequence \( \tau_1, \tau_2, \tau_3, \ldots, \tau_k, \ldots \) with \( \lim_{k \to \infty} \tau_k = +\infty \) such that \( y(\tau_k) = 0 \) so \( y^2(\tau_k) = 0 \) as \( k \to \infty \). This is a contradiction to \( \eta > 0 \). Therefore for \( t_k > T_1, t \in [t_k, t_{k+1}) \) we have \( y(t) > 0 \) or \( y(t) < 0 \) because \( y(t) \) is continuous on \([t_k, t_{k+1})\). Without loss of generality we assume that \( y(t) > 0 \) on \( t \in [t_k, t_{k+1}) \), it follows that \( y(t_{k+1}) > 0 \), thus \( y(t) > 0 \) on \([t_{k+1}, t_{k+2})\). By induction, we can conclude that \( y(t) > 0 \) on \([t_k, \infty)\).

From (2.2.15) we have
\[ \lim_{t \to \infty} y(t) = \lim_{t \to \infty} \left[ x(t) - R(t)x(\alpha t) - \int_{\beta t}^t \frac{H(s/\beta)}{s} x(s)ds - \int_{\beta t}^{\gamma t} \frac{Q(s/\gamma)}{s} x(s)ds \right] = \lambda \] (2.2.16)
where \( \lambda = \sqrt{\eta} \) and is finite.

In view of (2.2.14) we have
\[ \int_{\beta t}^t \frac{H(s/\beta)}{s} x(s)ds = y(\beta t) - y(t) + \sum_{\beta t < t_k < t} [y(t_k) - y(t_{k-1})] \]
\[ = y(\beta t) - y(t) - \sum_{\beta t < t_k < t} [(1 - b_k)y(t_{k-1})]. \]

Let \( t \to \infty, \sum_{k=1}^{\infty} (1 - b_k) < \infty \), we have
\[ \lim_{t \to \infty} \int_{\beta t}^t \frac{H(s/\beta)}{s} x(s)ds = 0. \] (2.2.17)

From (2.2.16), we have
\[ \lim_{t \to \infty} \left[ x(t) - R(t)x(\alpha t) - \int_{\beta t}^{\gamma t} \frac{Q(s/\gamma)}{s} x(s)ds \right] = \lambda. \] (2.2.18)

Next we shall prove that \( \lim_{t \to \infty} [x(t) - R(t)x(\alpha t)] \) exists and is finite. To prove this, first prove that \( |x(t)| \) is bounded. If \( |x(t)| \) is unbounded then there is a sequence \( \{s_n\} \) such that \( s_n \to \infty, |x(s_n^-)| \to \infty \) as \( n \to \infty \) and \( |x(s_n^-)| = \sup_{t_0 \leq t \leq s_n} |x(t)| \) where if \( s_n \) is not an impulsive point then \( x(s_n^-) = x(s_n) \). Thus, we have
\[ \left| x(s_n^-) - R(s_n^-)x(\alpha s_n) - \int_{\beta s_n}^{\gamma s_n} \frac{Q(s/\gamma)}{s} x(s)ds \right| \]
\[ \geq |x(s_n^-)| \left( 1 - (\mu + \epsilon) - \int_{\beta s_n}^{\gamma s_n} \frac{Q(s/\gamma)}{s}ds \right) \to \infty \] as \( n \to \infty \).
which contradicts (2.2.18). So $|x(t)|$ is bounded. Also by (2.2.3) we obtain

$$0 \leq \left| \int_{\beta t}^{\gamma t} \frac{Q(s/\gamma)}{s} x(s) \, ds \right| \leq \int_{\beta t}^{\gamma t} \frac{Q(s/\gamma)}{s} |x(s)| \, ds \to 0 \text{ as } t \to \infty$$

which together with (2.2.18) gives

$$\lim_{t \to \infty} [x(t) - R(t) x(\alpha t)] = \lambda.$$  \hspace{1cm} (2.2.19)

Next we prove that $\lim_{t \to \infty} x(t)$ exists and finite.

If $\mu = 0$, then $\lim_{t \to \infty} x(t) = \lambda$ which is finite.

If $0 < \mu < 1$, then $R(t)$ is eventually positive or negative and also we can find a sufficiently large $T_2$ such that $|R(t)| < 1$ for $t > T_2$. Set $\limsup_{t \to \infty} x(t) = L$, $\liminf_{t \to \infty} x(t) = l$, then we can choose two sequences $\{a_n\}$ and $\{b_n\}$ such that $a_n \to \infty$, $b_n \to \infty$ as $n \to \infty$ and $\lim_{n \to \infty} x(a_n) = L$, $\lim_{n \to \infty} x(b_n) = l$. Since $|R(t)| < 1$ for $t > T_2$, we have the following two possible cases.

**Case 1.** If $0 < R(t) < 1$ for $t > T_2$, then

$$\lim_{n \to \infty} [x(a_n) - R(a_n) x(\alpha a_n)] \geq L + \mu l$$

and

$$\lim_{n \to \infty} [x(b_n) - R(b_n) x(\alpha b_n)] \leq l + \mu L.$$

Therefore $L + \mu l \leq l + \mu L$. i.e., $L \leq l$. But $L \geq l$, it follows that $L = l$. Hence $L = l = \lambda/(1 + \mu)$ which shows that $\lim_{t \to \infty} x(t)$ exists and finite.

**Case 2.** If $-1 < R(t) < 0$ for $t > T_2$, then

$$L = \lim_{n \to \infty} x(a_n) = \lim_{n \to \infty} [x(a_n) - R(a_n) x(\alpha a_n) + R(a_n) x(\alpha a_n)] = \lambda + \mu L.$$

Similarly

$$l = \lim_{n \to \infty} x(b_n) = \lim_{n \to \infty} [x(b_n) - R(b_n) x(\alpha b_n) + R(b_n) x(\alpha b_n)] = \lambda + \mu l.$$

Therefore $L = l = \lambda/(1 - \mu)$. This shows that $\lim_{t \to \infty} x(t)$ exists and finite.
**Theorem 2.2.2** Assume that the conditions in Theorem 2.2.1 hold, then every oscillatory solution of (2.2.1) and (2.2.2) tends to zero as $t \to \infty$.

**Theorem 2.2.3** The conditions in Theorem 2.2.1 together with

$$\int_{t_0}^\infty \frac{H(s/\beta)}{s} \, ds = \infty \quad (2.2.20)$$

imply that every solution of (2.2.1) and (2.2.2) tends to zero as $t \to \infty$.

**Proof.** By Theorem 2.2.2, we only have to prove that every nonoscillatory solution of (2.2.1) tends to zero as $t \to \infty$. Let $x(t)$ be an eventually positive solution of (2.2.1). We shall prove $\lim_{t \to \infty} x(t) = 0$.

By Theorem 2.2.1 we rewrite (2.2.1) and (2.2.2) in the form $y'(t) + \frac{H(t/\beta)}{t} x(t) = 0$. Integrating from $t_0$ to $t$ on both sides we get

$$\int_{t_0}^{t} \frac{H(s/\beta)}{s} x(s) \, ds = y(t_0) - y(t) - \sum_{t_0 < t_k < t} (1 - b_k) y(t_k).$$

Using $\sum_{k=1}^{\infty} (1 - b_k) < \infty$ and (2.2.18) we get $\int_{t_0}^{\infty} \frac{H(s/\beta)}{s} x(s) \, ds < \infty$ which together with (2.2.20), we get $\lim_{t \to \infty} x(t) = 0$. \qed

**Remark 2.2.1** When $Q(t) = 0$ the results of this paper reduce to that of [29].

**2.3 Example**

In this section, we present an example to illustrate the results obtained in the previous section.

**Example 2.3.1** Consider the following impulsive differential equation

$$\frac{d}{dt} [x(t) - R(t) x(t/e)] + \frac{1}{2t(ln t - 1)} x(t/2e) - \frac{1}{4t(ln 2t - 1)} x(t/e) = 0,$$

$t \geq t_0 = 3, \ t \neq k$

$$x(k) = \left( \frac{k^2 - 1}{k^2} \right) x(k^-) + \left( \frac{1}{k^2} \right) \left( \int_{k/2e}^{k} \frac{1}{2sln 2s} x(s) \, ds - \int_{k/e}^{k} \frac{1}{4sln 2s} x(s) \, ds \right)$$

for $k = 4, 5, 6, \ldots$. \quad (2.3.1)
Here \( R(t) = \frac{k}{4(k-1)^2} [t], \ t \in [k-1,k), \)

\[
R(k) = \frac{k^2 - 1}{k^2} R(k^-), \sum_{k=1}^{\infty} (1 - b_k) = \sum_{k=1}^{\infty} \left( \frac{1}{k^2} \right) < \infty,
\]

\( H(t) = P(t) - Q(\beta t/\gamma) = \frac{1}{4(ln t - 1)} > 0, \)

\[
\lim_{t \to \infty} \sup \left[ \int_{t/\beta}^{t} \frac{H(s/\beta)}{s} ds + \frac{Q(t/\gamma)}{H(t/\beta)} \int_{t/\beta}^{t} \frac{H(s/\beta^2)}{s} ds + \mu \left( 1 + \frac{H(t/\alpha \beta)}{H(t/\beta)} \right) \right] < 2.
\]

Thus all the conditions of Theorem 2.2.1 are satisfied and therefore every solution of the equation (2.3.1) tends to a constant as \( t \to \infty \).