III.i. Introduction:

In Chapter-II, we have studied the nuclear surface characteristics in the limit of a semi-infinite nuclear matter under the framework of the energy density formalism. As it has been discussed in Chapter-I&II, the energy density resulting from the semiclassical $\hbar$-expansion upto second order contains gradient terms in density upto second order only and the variational density resulting from the solution of Euler-Lagrange (EL) equation falls-off too quickly in the outer surface. Any improvement to this behavior of the density can only be achieved at the cost of the surface energy. Thus, it is essential to include higher order gradient terms in the energy density to reproduce the correct energies and the correct behavior of the density in the outer part of the surface. The work of Brack et.al.[16], centelles et.al.[39] shows that with the help of fourth order semiclassical approximation (upto $\hbar^4$ terms) the difficulties faced with the second order case can be overcome. However, the variational density resulting from the EL equation for this $\hbar^4$-approximation, has an unphysical $r^{-6}$ behavior in the extreme outer part of the surface. In nuclear calculations, such
as, some moments of particle density $\rho$ and HI interaction potentials where the shape of the density at the outer part of the surface plays a dominant role, this unphysical $r^{-\delta}$ behavior of the density will show some divergence in the results.

In view of this, in Chapter-II, an energy density containing a simple fourth order gradient term has been constructed, as given in (II.4), which provides the same simplicity as in the second order case and gives correct energies and correct behavior of the surface in its outer part. It has been shown that such an energy density can, in fact, be constructed if one approximates the kinetic energy density containing fourth order gradient term as given in (II.30) and uses the correct second order approximation for the spin-orbit density $J$. Using this energy density, the nuclear surface characteristics have been studied for the cases of symmetric and asymmetric SINM. From the exponential behavior of the variational density, resulting from the EL equation in both interior and outer parts of the surface, the density has been parameterised into an $F-\nu$ shape, as given in (II.70). These parameterised densities are described by the three parameters, namely, the central density $\rho_0$, the surface thickness $a$ and the skewness parameter $\nu$. Analytical expressions for the various surface characteristics have also been obtained for these
parameterised densities, where the expressions become functions of the parameters $\rho_0$, $a$ and $\nu$ and the parameters of the interaction used. However, when one extends the calculations to the region of finite nuclei with these trial densities of F-\nu shape, the three parameters $\rho_0$, $a$ and $\nu$ turn out to be dependent on the size of the nucleus and it becomes essential to know their variation with mass number $A$.

The variation of the central density $\rho_0$ with the mass number $A$ (central compression) has been studied in terms of leptodermous expansion of the energy by Pearson[55], Tondeur et al.[56] and Brack et al.[16] in an attempt to extend the droplet model results of Myers and Swiatecki[23,24] to higher orders in $A^{-4/3}$. In the work of Brack et al.[16], the energy density functional resulting from the semiclassical $\hbar^4$-approximation has been used for the leptodermous expansion of the total energy. However, they have not considered the variation of $a$ and $\nu$ and have taken the corresponding SINM values for them. On the other hand, the central compression has been studied by Treiner and Krivine [49,50] by obtaining the variational density from an exact numerical solution of the EL equation resulting from the energy density functional containing only second order derivative terms. These studies of the central compression in both the second and
fourth order energy density approximations of Treiner and Krivine, and Brack et al. respectively, show a bending in the light nuclei region for Skyrme interactions with compressibility K around 200MeV. It is also established in both the cases that an $A^{-1/3}$ expansion of the central compression analogous to the leptodermous expansion of the energy is not possible. As regard to the contributions of the terms of various orders in the leptodermous expansion of the energy $E$, Brack et al. have pointed out that the terms of order $A^{-1/3}$ and lower contribute less than 1 Mev for heavy nuclei. However, as will shown in this chapter, these lower order terms although are of little importance so far as the total energy is concerned, they play a significant role in the variation of central density and surface thickness with mass number A.

The motivation of the present chapter is two-fold. First, we examine the crucial importance of the fourth order gradient terms in the energy density functional and then the limitations of the leptodermous expansion of the energy on the variations of $\rho_o$ and $a$ with the size of the nucleus have been studied. For this purpose two energy densities, one containing derivative terms up to second order and the other containing fourth order derivative too, have been chosen. These two energy densities give the same energy per
particle in nuclear matter and same surface energy in semi-infinite nuclear matter. A further simplification is also introduced by approximating the surface energy density in a suitable way so that the EL equation in SINM leads to a density of pure Fermi shape. The surface energy densities, thus obtained, are consistent with their EL equations in SINM and involve only the SINM parameters. The motivation behind such an approximation is that since the EL equation for the energy density in SINM leads to an F-1 density distribution, it is appropriate to assume an F-1 density distribution for the nucleus in order to study the departure of the central density \( \rho_o \) and surface thickness \( a \), from their respective SINM values in terms of the ratio \( a/R \), where \( R \) defines the half density radius of the nucleus.

In section-ii of this chapter, we give the detail of the leptodermous expansion of the total energy of a spherical nucleus with \( N=Z \) and no Coulomb interaction in the energy density formalism. Simple relations are also obtained for the total energy and the mean square radius of the nucleus without assuming any specific form for the density profile. In section-iii, we present the formalism for the appropriate approximation of the surface energy density so that the resulting EL equation in an SINM leads to an F-1 density distribution. For an F-1 density
profile, simple analytical formulae are obtained for the total energy of a spherical nucleus and also for the departure of its central density and surface thickness from their respective SIMN values using the leptodermous expansion of section-ii. The results of the calculations for the two energy densities are presented in section-iv. The influence of the various characteristics of the energy density functional, such as, the central density and the nuclear matter incompressibility on these results are also discussed in the same section.

III.ii. The Leptodermous Expansion:

The concept of a leptodermous system has been discussed extensively by Myers and Swiatecki in their studies of nuclear properties in the Droplet Model[23,24]. Due to the saturating properties of the nuclear force, the nuclear density is uniform over the bulk of the nucleus and the total energy has a stable minimum when considered as a function of this bulk density. Because of this, the radius of a nucleus is almost proportional to \( A^{1/3} \) while its surface thickness (the narrow region within which the density drops from 90% to 10% of its value in the central region) is almost constant. The ratio of these two quantities which is of the order of \( A^{-1/3} \) can be used as an expansion parameter for any nuclear property which is independent of shell
effects. In this section, we outline the leptodermous expansion for a spherical system with \( N=Z \) and no Coulomb interaction. The leptodermous expansion for deformed systems can be found in the work of Brack et al.\[16\].

The usual approximation to the density \( \rho \), for a spherical system within the semiclassical approach, is

\[
\rho(r) = \rho_0 g(r), 
\]

...(III.1)

where \( \rho_0 \) is the central density and

\[
g(r) \rightarrow 0, \text{ as } r \rightarrow \infty 

g(r) \rightarrow 1, \text{ as } r \rightarrow 0 \]  ...(III.2)

The particle number \( A \) of the spherical nuclei with density distribution in (III.1), can be written as

\[
A = 4\pi \rho_0 \int_0^\infty g(r) r^2 dr = \frac{4\pi \rho_0}{3} \int_0^\infty \frac{dg}{dr} r^3 dr. 
\]

...(III.3)

Now changing the variable from \( r \) to \( z \), defined as \( z=r-R \), where \( R \) defines a radius for the nucleus, such that the density profile \( g(z) \) drops sharply within a narrow region about \( z=0 \). With this change, equation (III.3) can be written as

\[
A = -\frac{4\pi \rho_0}{3} \int_{-R}^\infty \frac{dg}{dz}(z+R)^3 dz. 
\]

...(III.4)

Since the function \( \frac{dg}{dz} \) will be peaked around \( z=0 \) and goes to zero within a very small region about \( z=0 \) compared to the nuclear
dimension $R$, the lower limit $-R$ in the above integral can be extended to $-\infty$ and the error introduced in this process is of the order $e^{-R/\alpha}$, which is practically unimportant for not too small nuclei and can be neglected. Since the standard value of the surface thickness $\alpha$ is around 0.5 fm, the extension of the lower limit from $-R$ to $-\infty$ does not incur a significant error even up to $R=2.5$ fm. However, as one goes to lighter nuclei with $R<2.5$ fm, the error gradually increases and the system no longer remains a leptodermous one rather it becomes a pachydermous one. Neglecting this exponential contribution within the limit of the leptodermous system, the expression in (III.4) becomes

$$A = \frac{4\pi \rho \alpha}{3} \left( R^3 + 3R^2 C_1 + 3R C_2 + C_3 \right), \quad \text{(III.5)}$$

where the coefficients $C_n$ are defined as

$$C_n = -\int_{-\infty}^{\infty} \frac{dg}{dz} z^n \, dz, \text{ for } n=1,2,3. \quad \text{(III.6)}$$

This expression for $A$ in (III.5) is the so called leptodermous expansion of the mass number, where the first term is the volume contribution and the rest three terms give the contributions from the surface, which appear as moments of the function $dg/dz$ defined in (III.6). The expression in (III.5) can be inverted to obtain an expansion of $R$ in powers of $A^{-1/3}$ to be
A similar leptodermous expansion can be obtained for the mean square radius, which is defined as

$$<r^2> = \frac{1}{A} \int \rho(r) r^2 dr.$$  ...(III.8)

Using the same change of variable and the leptodermous approximation of $A$ given in (III.5), the above expression becomes

$$<r^2> = \frac{3}{5} \left( \frac{R^5 + 5R^4 C + 10R^3 C + 10R^2 C + 5RC + C}{R^3 + 3R^2 C + 3RC + C} \right).$$  ...(III.8a)

Upon dividing the numerator by the denominator and using the expansion of $R$ given in (III.7), the above expression becomes

$$<r^2> = \frac{3}{5} \frac{R^2}{r_0^{2/3}} \left[ 1 + 5(C^2 - C_1^2) \frac{A^{2/3}}{r_0^2} + 2C (C^2 - C_1^2) \frac{A^{-1}}{r_0^3} + \cdots \right].$$  ...(III.9)

It is to be noted that if the function $dg/dz$ is symmetric about $z=0$, the coefficients $C_n$ with odd $n$ vanish.

We now consider the leptodermous expansion of the total energy of a spherical system with $N=Z$ and no Coulomb interaction. In the spirit of the energy density formalism, the total energy of such a system can be written as

$$E = \int H d^3 r.$$  ...(III.10)

Here $H$ is the energy density and, as mentioned earlier, contains a
non-gradient part $\rho \delta(\rho)$ and gradient terms of both second and fourth order in the density $\rho$. In order to isolate the contributions arising from the surface region, $E$ can be written as

$$E = \delta(\rho_o)A + 4\pi \int_0^\infty \left[ H - \rho \delta(\rho_o) \right] r^2 dr,$$

...(III.11)

without any approximation or loss of generality. Here $\delta(\rho_o)$ is the energy per particle in the central region of the nucleus and the equation divides the total energy into a volume term $\delta(\rho_o)A$ and an integral whose contributions come only from a region localized in the surface. This can be easily seen since $\rho$ tends to zero in the outer part of the surface and $[H - \rho \delta(\rho_o)]$ tends to zero in the deep interior. Because of this localization, it is convenient to make a substitution $r = z + R$ in (III.11) and carrying out a leptodermous expansion to the integral in the same way as for the particle number $A$, the resulting expression becomes

$$E = \delta(\rho_o)A + 4\pi \left[ \frac{R^2 b + 2R b + b}{2} \right],$$

...(III.12)

with

$$b_n = \int_0^{\infty} [H - \rho \delta(\rho_o)] z^{n-1} dz, \quad n = 1, 2, 3$$

...(III.13)

The first term in (III.12) is the volume contribution and the second term gives the contributions from the surface region which appear as the moments of the surface energy density $[H - \rho \delta(\rho_o)]$. In obtaining these expressions, the error introduced in extending the lower limits of the integrals in (III.12&13) from $-R$ to $-\infty$
have been neglected which is practically unimportant for not too small nuclei. Now the total energy per particle can be obtained in the form

\[ \frac{E}{A} = \mathcal{E}(\rho_0) + \frac{3}{\rho_0} \left( \frac{R^2b + 2Rb \rho_0 + 2}{R^3 + 3R^2 \rho_0 + 3R \rho_0 + \rho_0} \right) \]

(III.14)

The \( A^{-1/3} \) expansion of \( E/A \) analogous to (III.7&9) is

\[ \frac{E}{A} = \mathcal{E}(\rho_0) + \frac{3}{\rho_0} \left( \frac{b}{r_0} \frac{A^{-1/3}}{r_0} + 2(b - C b) \frac{A^{-2/3}}{r_0^2} + \right. \\
\left. \left\{ 3C b - 2C b + b \right\} \frac{A^{-1}}{r_0^3} + \ldots \right) \]

(III.15)

The main dependence of the total energy per particle and the mean square radius on the size of the nucleus within the leptodermous approximation has been separated out in (III.15&9) respectively. However, a very smooth variation is still coming from the coefficients \( b \) and \( C \) through their dependence on how the density profile drops in the narrow region around \( z=0 \). This dependence, as well as the dependence of the coefficients \( b \) on the central density \( \rho_0 \) (note that the coefficients \( C \) are practically independent of the central density \( \rho_0 \)) must be determined for each nucleus in a self-consistent way. However, this cannot be obtained analytically for the general case.

In the following section, we present a simple approximation to the surface energy density which allows to study this variation
analytically in terms of a pure Fermi distribution for the density profile \( g(z) \) in a semi-infinite nuclear medium.

III.iii. A Simple Approximation To Surface Energy Density:

The calculation of the total energy and the mean square radius in the leptodermous approximation described in section-ii requires the complete knowledge of the coefficients \( b_n \) and \( c_n \) in a self-consistent way for the given surface energy density \([H-\rho\delta(\rho_o)]\). The energy density \( H \), in general, can be written as the sum of a non-gradient part and gradient terms in \( \rho \) of various orders,

\[
H = \rho\delta(\rho) + \text{gradient terms in } \rho. \tag{III.16}
\]

This is based on the theorem of Kohn and co-workers[10], which states that the ground state energy density of a system of fermions can be expressed in terms of the local density \( \rho \) and its derivatives. While the non-gradient part \( \rho\delta(\rho) \) can be derived by using an effective n-n interaction, the exact nature of the gradient terms in (III.16), as discussed in Chapter-I, is not exactly known for the nucleus although the semiclassical \( h \)-expansion has been extensively used with notable success. In view of this and to study the variation of the coefficients \( b_n \) and \( c_n \) with mass number \( A \), we approximate the surface energy density \([H-\rho\delta(\rho_o)]\) suitably so that the EL equation in SINM exactly leads to a pure Fermi shape (F-1) for the density \( \rho \). The motivation of
imposing a pure Fermi shape for the density \( \rho \) in SINM is that this form of density is extensively used in all nuclear calculations. Moreover, with such a density the surface properties are now described by just two parameters, namely, the central density \( \rho_0 \) and the surface thickness \( a \). For the above purpose we expand the function \( \xi(\rho) \) in (III.16) about \( \rho = \rho_{nm} \),

\[
\xi(\rho) = \xi(\rho_{nm}) + \frac{K}{18} \left( \frac{\rho - \rho_{nm}}{\rho_{nm}} \right)^2 + \ldots, \quad ...(III.17)
\]

where \( K \) is the nuclear matter incompressibility. It is to be noted here that although the contributions arising from the higher order terms in this expansion are small compared to the contributions from the leading term \( \frac{K}{18} \left( \frac{\rho - \rho_{nm}}{\rho_{nm}} \right)^2 \), their effects cannot be neglected altogether[49,50]. In view of this, we approximate the non-gradient part of \( H \) by the simple form

\[
\rho \xi(\rho) \approx \rho \left[ \xi(\rho_{nm}) + W_0 \left( \frac{\rho - \rho_{nm}}{\rho_{nm}} \right)^2 \right], \quad ...(III.18)
\]

where \( W_0 \) is a constant and takes account of the effects arising from the higher order terms of the expansion in (III.17) in an appropriate way. The motivation behind such an approximation to the surface energy density, as will be shown in section-iv, is that the crucial importance of the fourth order derivative terms in the energy density and the variation of \( \rho_0 \) and \( a \) with the size of the nucleus can be studied in a general way quite independent
of a detail knowledge of the constant $W$. A simple and accurate estimation of $W$ has been discussed in section-v(d) and its values for the sets of Skyrme-type interactions of Sharma and Nagarajan[62,63] have been given in Table-9. From the values of $W$, it can be seen that it is nearly equal to $K/18$ but the differences depend strongly on $\rho_n^{nm}(d^n\rho/d\rho_n^{nm})$ with $n \geq 3$. It is worthwhile to mention here that this approximation to $\rho_s(\rho)$ was first proposed by Skyrme[3] and have been used by Moszkowski and Coworkers[58,59] in the studies of the droplet model parameters and HI interaction potentials. This approximation has also been used in the relativistic description of SINM by Boguta and Moszkowski[60]. However, in these studies $W$ is approximated by $-8(\rho^{nm})$, thereby fixing the incompressibility $K$ at $-18\rho^{nm}$.

The derivative terms in $H$ are now chosen in simplest form so that together with the approximation in (III.18) the EL equation in SINM yields a pure Fermi distribution for the density $\rho$. The form of the two surface energy densities, one containing gradient terms up to second order only while the other containing the fourth order gradient terms too and satisfying this condition, can now be expressed as

$$H - \rho S(\rho) = W \left[ \left( \frac{\rho - \rho_{nm}}{\rho_{nm}} \right)^2 - \left( \frac{\rho_0 - \rho_{nm}}{\rho_{nm}} \right)^2 \right] \rho$$

$$+ (B+C\rho)(\nabla \rho)^2 + \xi_4 \frac{(\nabla \rho)^4}{\rho^3}, \quad ...(III.19)$$
and

\[ H - \rho \delta(\rho_0) = \omega \left[ \left( \frac{\rho - \rho_{nm}}{\rho_{nm}} \right)^2 - \left( \frac{\rho - \rho_{nm}}{\rho_{nm}} \right)^2 \right] \rho + \xi_2 \frac{(\nabla \rho)^2}{\rho}. \quad \text{(III.20)} \]

The condition that these surface energy densities yield an F-1 density distribution in SINM, gives the following results for the parameters \( \xi_2, B, C \) and \( \xi_4 \):

\[ \xi_2 = \frac{W a_0^2}{\rho_{nm}}, \quad B = -\frac{2W a_0^2}{\rho_{nm}}, \quad C = -\frac{W a_0^2}{\rho_{nm}^2} \quad \text{and} \quad \xi_4 = \frac{W a_0^4}{3 \rho_{nm}}, \quad \text{(III.21)} \]

where \( a_{nm} \) is the surface thickness parameter in the Fermi distribution of density in an SINM. Thus, the surface energy densities given in (III.19&20) involve only three parameters \( \rho_{nm} \), \( a_{nm} \) and \( \omega_0 \).

III.iv. Variation Of Central Density And Surface Thickness In Finite Nuclei.

As discussed in section-ii of this chapter, a smoothly varying contribution to the total energy and to the mean square radius comes from the variations of the central density and the surface thickness with mass number \( A \). In order to account for these contributions, we have constructed energy densities given in (III.19&20) containing fourth and second order gradient corrections, respectively. Since the solutions of the corresponding EL equations in SINM give F-1 density distribution, it is appropriate to assume an F-1 density distribution for a
spherical nucleus with \( N=Z \) and no Coulomb interaction,

\[
\rho(r) = \frac{\rho_0}{1 + \exp[(r-R)/a]} \tag{III.22}
\]

where \( \rho_0 \) is the central density and \( a \) is the surface thickness of the nucleus. With this density distribution, the coefficients \( C_n \) in (III.6) are given by

\[
C_1 = C_3 = C_5 = 0, \\
C_2 = \frac{\pi a^2}{3} \quad \text{and} \quad C_4 = \frac{7\pi a^4}{15} \tag{III.23}
\]

and the gradient of the density \( \rho' \) becomes a function of \( \rho \) and the parameters \( \rho_0 \) and \( a \).

Using the leptodermous expansion of the particle number and the recurrence relation

\[
I_{n+1} = \rho_0 \left( I_n - \frac{a}{n} \frac{\partial I_n}{\partial R} \right) \tag{III.24}
\]

where \( I = \int \rho d^3 R \), one can evaluate the integrals for \( b_n \) in (III.13) and express them as simple functions of the central density \( \rho_0 \) and the surface thickness \( a \). The results for the two different surface energy densities in (III.19&20) are now given as

\[
b_1 = \frac{W \rho_0 a}{3} \left[ \left( 6x^2 - \frac{9a^2}{2} \right) y + \left( x^2 - \frac{a^2}{4} \right)/y + x/4y^3 \right], \tag{III.25a}
\]

\[
b_2 = \frac{W \rho_0 a^2}{6} \left[ 3x^2 y + x^3/4 + 11x/12y^2 \right], \tag{III.25b}
\]
for the fourth order surface energy density and

\[
b_1 = \frac{W_0 a}{3} \left( \frac{6x^2}{2} - \frac{9}{2} x^3 y + \frac{3x}{2y} \right),
\]

\[
b_2 = \frac{W_0 a}{6} \left[ 3x^2 y + 3x \right],
\]

\[
b_3 = -\frac{W_0 a}{3} \left[ \frac{2\pi^2}{3} \left( x^2 - \frac{3x^3}{4} \right) y + \frac{\pi^2 x y}{2} \right],
\]

for the second order energy density. In these relations 

\[ x = \rho_0 / \rho_{nm} \]

and 

\[ y = a / a_{nm}. \]

Substitution of these expressions for the coefficients \( b_n \) and \( c_n \) into (III.15) allows one to express the total energy in powers of \( A^{1/3} \),

\[
E = \xi_v A + \xi_s A^{2/3} + \xi_c A^{1/3} + \xi_g + O(A^{-1/3}),
\]

where

\[
\xi_v = \frac{W_0 (-2x + x^2)}, \quad \xi_s = \frac{3b_1 / \rho_o r_o}, \quad \xi_c = \frac{6(b_2 - C b_1) / \rho_o r_o^2}, \quad \xi_g = \frac{3[(3C_2 - 2C_3) b_1 - 2C_3 b_1 + b_1] / \rho_o r_o^3},
\]

are the volume, surface, curvature and gaussian curvature contributions, respectively. The contribution \( O(A^{-1/3}) \) contains
terms of order $A^{-1/3}$ and lower. Brack et al. [16] have pointed out that the contribution of order $A^{-1/3}$ and lower is less than 1MeV for heavy nuclei and hence the expansion in (III.27) up to the $A^0$-term is sufficient to represent the total energy. However, it will be shown that although these lower order terms are practically unimportant so far as the total energy is concerned, they play a crucial role in the study of the variation of the central density and the surface thickness with the size of the nucleus. In connection to these studies in section-v(b), it will be shown that the contribution from these lower order terms in the variation of $\rho_0$ and $a$ are vital in the mass region $A \leq 100$.

In order to show the relative importance of these lower order terms in the variation of $\rho_0$ and $a$, we expand the total energy, as given in (III.12), in a Taylor series about $\rho_o = \rho_{nm}$ and $a = a_{nm}$ subject to the constraint for the particle number $A$ in (III.5) (without making further expansion in powers of $A^{-1/3}$),

$$E = E(\rho_{nm}, a_{nm}) + \varepsilon E(\rho_{nm}, a_{nm}) + \lambda E(\rho_{nm}, a_{nm})$$

$$+ \frac{1}{2} \varepsilon^2 E(\rho_{nm}, a_{nm}) + \frac{1}{2} \lambda^2 E(\rho_{nm}, a_{nm}) + \varepsilon \lambda E(\rho_{nm}, a_{nm}) + \cdots,$$

...(III.29)

where $\varepsilon = \frac{\rho_o - \rho_{nm}}{\rho_{nm}}$ and $\lambda = \frac{a - a_{nm}}{a_{nm}}$.  

...(III.30)
The dots and primes on $E$ indicate differentiations with respect to central density $\rho_0$ and surface thickness $a$, respectively. Since the departures $(\rho_0 - \rho_{nm})/\rho_{nm}$ and $(a - a_{nm})/a_{nm}$ are expected to be small, it is quite reasonable to restrict the expansion up to quadratic terms in $\varepsilon$ and $\lambda$. The minimization $\partial E/\partial \varepsilon = 0$ and $\partial E/\partial \lambda = 0$, then gives

$$\varepsilon = \frac{E^{\prime\prime} - 2E^{\prime}E - E^2}{E^2} \quad \ldots \quad (III.31)$$

$$\lambda = \frac{E^{\prime\prime} - 2E^{\prime}E - E^2}{E^2} \quad \ldots \quad (III.32)$$

With the help of the above expressions for $\varepsilon$ and $\lambda$, the total energy in (III.29) under the quadratic approximation becomes

$$E = E(\rho_{nm}, a_{nm}) + \frac{1}{2} \left[ \varepsilon E(\rho_{nm}, a_{nm}) + \lambda E(\rho_{nm}, a_{nm}) \right]. \quad \ldots \quad (III.33)$$

iv(a). Relations for $E(\rho_{nm}, a_{nm})$ and its various derivatives:

In making the Taylor series expansion of total energy, we have used the expression of $E$ as given in (III.12) with the constraint that the particle number in (III.5) be conserved. For the F-1 density distribution the expression for the total number of particles becomes

$$A = \frac{4\pi \rho_0}{3} \left( R^3 + \pi^2 a^2 R \right). \quad \ldots \quad (III.34)$$
In order that $A$ be conserved under the variation of $\rho_0$ and $a$, the following conditions must be satisfied,

$$\frac{dA}{d\rho_0} = 0 \quad \text{and} \quad \frac{dA}{da} = 0.$$  \hspace{1cm} (III.35)

From these conditions and the expression of $A$ in (III.34), the following expressions can be obtained,

$$R = -\frac{R}{3} \left[ \frac{1 + \alpha^2}{1 + \alpha^2 / 3} \right], \hspace{1cm} \text{(III.36a)}$$

$$R = \frac{2}{3} R \left[ \frac{1 + \alpha^2}{1 + \alpha^2 / 3} - \frac{1}{3} \left( \frac{1 + \alpha^2}{1 + \alpha^2 / 3} \right)^3 \right], \hspace{1cm} \text{(III.36b)}$$

$$R = -\frac{2}{3} R \left[ \frac{\alpha^2}{1 + \alpha^2 / 3} \right], \hspace{1cm} \text{(III.36c)}$$

$$R = -\frac{2}{3} R \left[ \frac{\alpha^2}{1 + \alpha^2 / 3} - \frac{4}{3} \frac{\alpha^4}{(1 + \alpha^2 / 3)^2} - \frac{4}{3} \frac{\alpha^4}{(1 + \alpha^2 / 3)^3} \right], \hspace{1cm} \text{(III.36d)}$$

$$R = \frac{2}{9} R \left[ \frac{\alpha^2 (1 + \alpha^2)}{(1 + \alpha^2 / 3)^2} - \frac{2 \alpha^2 (1 + \alpha^2)}{(1 + \alpha^2 / 3)^3} \right], \hspace{1cm} \text{(III.36e)}$$

where $\alpha = a/R$, $R = \rho_0 \partial R/\partial \rho_0$, $R = a \partial R/\partial a$ and so on. Note that these quantities are to be evaluated at $a = a_{nm}$ and $\rho = \rho_{nm}$. Using these expressions and the values of $b_1 \cdot b_2 \cdot b_3$ and their various derivatives evaluated at $x = 1$ and $y = 1$, the total energy $E(\rho_{nm}, a_{nm})$ and its various derivatives can be written analytically as follows,

$$\frac{E(\rho_{nm}, a_{nm})}{A} = 8(\rho_{nm}) + \frac{W f \left( \frac{5}{2} + \frac{25}{6} + \frac{5\pi^2 - 6}{6} \right)}{1 + \pi^2 \alpha^2} \hspace{1cm} \text{(III.37a)}$$
\[
\frac{\dot{E}(\rho_{nm}, a_{nm})}{A} = W_0 \left\{ -\frac{\alpha}{(1+\pi^2 \alpha^2/3)} \left[ \frac{5}{3} + \frac{25}{18} \alpha \right] + \frac{\alpha^2}{(1+\pi^2 \alpha^2)} \left[ \frac{32}{3} - 2\alpha \right] \right\} 
\]

\ldots \text{(III.37b)}

\[
\frac{\dot{E}(\rho_{nm}, a_{nm})}{A} = W_0 \left\{ 2 - \frac{\alpha}{(1+\pi^2 \alpha^2)} \left[ \frac{29}{2} - \frac{39}{2} \alpha + \frac{29\pi^2 + 6\alpha}{6} \right] 
+ \frac{\alpha}{(1+\pi^2 \alpha^2/3)} \left[ \frac{10}{3} - \frac{13}{3} \alpha + \frac{1}{1+\pi^2 \alpha^2/3} \left( \frac{5}{9} - \frac{30+25\alpha}{27(1+\pi^2 \alpha^2/3)} \right) \right] \right\} 
\]

\ldots \text{(III.37c)}

\[
\frac{\dot{E}(\rho_{nm}, a_{nm})}{A} = \frac{W_0 \alpha^2}{(1+\pi^2 \alpha^2)} \left[ \frac{25}{6} + \frac{5\pi^2 - 6}{3} \alpha - \frac{8\pi^2 \alpha}{(1+\pi^2 \alpha^2/3)} \left[ \frac{5}{9} + \frac{25}{18} \alpha \right] \right] 
\]

\ldots \text{(III.37d)}

\[
\frac{\dot{E}(\rho_{nm}, a_{nm})}{A} = \frac{W_0 \alpha}{(1+\pi^2 \alpha^2)} \left[ \frac{9}{2} + \frac{23}{2} \alpha + \frac{19\pi^2 + 6\alpha}{6} \right] - \frac{2\pi^2 \alpha^2}{(1+\pi^2 \alpha^2/3)} \left[ \frac{5}{3} + \frac{25}{6} \alpha \right] 
+ \frac{20\pi^4 \alpha^4}{3(1+\pi^2 \alpha^2/3)^2} - \frac{8\pi^4 \alpha^4}{(1+\pi^2 \alpha^2/3)^3} \left[ \frac{5}{9} - \frac{25\alpha^3}{162} \right] \right\} \text{ (III.37e)}
\]

\[
\frac{\dot{E}(\rho_{nm}, a_{nm})}{A} = \frac{W_0 \alpha}{(1+\pi^2 \alpha^2)} \left[ - \frac{7}{2} + \frac{97}{6} \alpha - \frac{7\pi^2 + 18\alpha^2}{6} \right] - \frac{64\pi^2 \alpha^3}{9 (1+\pi^2 \alpha^2/3)} 
- \frac{W_0 \alpha^2}{(1+\pi^2 \alpha^2/3)} \left[ \frac{25}{18} - \frac{25\pi^2 \alpha^2}{27(1+\pi^2 \alpha^2/3)} - \frac{4\pi^2 \alpha}{(1+\pi^2 \alpha^2/3)^2} \left( \frac{5\pi^2 \alpha}{27} - \frac{25}{54} \right) \right] 
\]

\ldots \text{(III.37f)}

for the fourth order energy density given in (III.19) and

\[
\frac{E(\rho_{nm}, a_{nm})}{A} = S(\rho_{nm}) + \frac{W_0 \left( 3\alpha + 6\alpha^2 + \pi^2 \alpha^3 \right)}{(1+\pi^2 \alpha^2)} \text{ (III.38a)}
\]
\[
\frac{\dot{E}(\rho_{nm}, a_{nm})}{A} = W_0 \left[ -\frac{2\alpha(1+\alpha)}{(1+\pi^2 \alpha^2/3)} + \frac{12\alpha^2}{(1+\pi^2 \alpha^2/3)} \right] \quad \text{(III.38b)}
\]

\[
\frac{\dot{E}(\rho_{nm}, a_{nm})}{A} = W_0 \left[ 2 - \frac{\alpha}{(1+\pi^2 \alpha^2/3)} \left( 15 - 18\alpha + 5\pi^2 \alpha^2 \right) \right.
+ \frac{\alpha}{(1+\pi^2 \alpha^2/3)} \left\{ 4(1-\alpha) + \frac{(1+\pi^2 \alpha^2/3)}{3} - \frac{4(1+\alpha)}{3(1+\pi^2 \alpha^2/3)} \right\} \] \quad \text{(III.38c)}

\[
\frac{\dot{E}(\rho_{nm}, a_{nm})}{A} = \frac{W_0 \alpha^2}{(1+\pi^2 \alpha^2/3)} \left[ 6 + 2\pi^2 \alpha - \frac{4\pi^2 \alpha}{(1+\pi^2 \alpha^2/3)}(1+\alpha) \right] \quad \text{(III.38d)}
\]

\[
\frac{\ddot{E}(\rho_{nm}, a_{nm})}{A} = \frac{W_0 \alpha}{(1+\pi^2 \alpha^2/3)} \left[ 3 + 6\alpha + 3\pi^2 \alpha^2 - \frac{4\pi^2 \alpha^2}{(1+\pi^2 \alpha^2/3)}(1+3\alpha) \right.
+ \frac{8\pi^4 \alpha^4}{(1+\pi^2 \alpha^2/3)^2} - \frac{16\pi^4 \alpha^4}{3(1+\pi^2 \alpha^2/3)^3} \left( 1 - \frac{\pi^2 \alpha^2}{3} \right) \left] \right. \quad \text{(III.38e)}
\]

\[
\frac{\dddot{E}(\rho_{nm}, a_{nm})}{A} = \frac{W_0 \alpha^2}{(1+\pi^2 \alpha^2/3)} \left[ -3 + 18\alpha - \pi^2 \alpha^2 - \frac{8\pi^2 \alpha^3}{(1+\pi^2 \alpha^2/3)} \right.
- \frac{W_0 \alpha^2}{(1+\pi^2 \alpha^2/3)} \left[ 2 - \frac{4\pi^2 \alpha^2}{3(1+\pi^2 \alpha^2/3)} - \frac{8\pi^2 \alpha^2}{9(1+\pi^2 \alpha^2/3)^2} (\pi^2 \alpha^2 - 3) \right] \quad \text{(III.38f)}
\]

for the second order energy given in (III.20). In these expressions \( \alpha \) is the small ratio between the surface thickness \( a \) and the radius \( R \) (evaluated at \( \rho_0 = \rho_{nm} \) and \( a = a_{nm} \)) and the expression for \( R \) in (III.7) can be used to write it as
\begin{align*}
\alpha &= \frac{\frac{a_{\text{nm}}}{A^{1/3}}}{\left(1 - \frac{2r_{\text{nm}}^2}{3r_{\text{nm}}^2}A^{-2/3} + \cdots\right)} \quad \ldots(\text{III.39})
\end{align*}

In obtaining the relations for $E\left(\rho_{\text{nm}}, a_{\text{nm}}\right)$ in (III.37c&38c), we have used the relation

\begin{align*}
\delta(\rho_{\text{o}}) \simeq W_{\text{o}} \left(\frac{\rho_{\text{o}} - \rho_{\text{nm}}}{\rho_{\text{nm}}}\right)^2 + \delta(\rho_{\text{nm}}) \quad \ldots(\text{III.40})
\end{align*}

in the spirit of the approximation introduced in (III.18).

The expansion of $\alpha$ in powers of $A^{-1/3}$ in (III.39) can be used to expand $E\left(\rho_{\text{nm}}, a_{\text{nm}}\right)$ and the various derivatives $\dot{E}, \ddot{E}$ etc. to any desired order in $A^{-1/3}$ and the importance of the lower order terms in the variation of $\varepsilon$ and $\lambda$ can be studied. The expression of $E\left(\rho_{\text{nm}}, a_{\text{nm}}\right)$ allows one to obtain the surface, curvature and gaussian curvature contributions to the energy in the form,

\begin{align*}
\gamma_{s} &= \frac{5}{2} W_{\text{o}} \left(\frac{a_{\text{nm}}}{r_{\text{nm}}}\right), \quad \ldots(\text{III.41a})
\gamma_{c} &= \frac{25}{6} W_{\text{o}} \left(\frac{a_{\text{nm}}}{r_{\text{nm}}}\right)^2, \quad \ldots(\text{III.41b})
\gamma_{g} &= -\frac{(5n^2+6)}{6} W_{\text{o}} \left(\frac{a_{\text{nm}}}{r_{\text{nm}}}\right)^3, \quad \ldots(\text{III.41c})
\end{align*}

for the fourth order approximation in (III.19) and

\begin{align*}
\gamma_{s} &= 3W_{\text{o}} \left(\frac{a_{\text{nm}}}{r_{\text{nm}}}\right), \quad \ldots(\text{III.42a})
\end{align*}
\[
\mathcal{S}_c = 6W_0 \left( \frac{a_{nm}}{r_{nm}} \right)^2, \quad \text{...(III.42b)}
\]
and
\[
\mathcal{S}_g = -\pi^2 W_0 \left( \frac{a_{nm}}{r_{nm}} \right)^3, \quad \text{...(III.42c)}
\]
for the second order approximation in (III.20). It may be noted from the above relations that \(\mathcal{S}_c\) and \(\mathcal{S}_g\) in both the cases can be expressed in terms of \(\mathcal{S}_s\) and \(W_0\) as
\[
\mathcal{S}_c = \frac{2}{3} \mathcal{S}_s^2/W_0, \quad \text{...(III.43a)}
\]
\[
\mathcal{S}_g = -\frac{4(5n^2+6)}{375} \mathcal{S}_s^3/W_0^2, \quad \text{...(III.43b)}
\]
for the fourth order case and
\[
\mathcal{S}_c = \frac{2}{3} \mathcal{S}_s^2/W_0, \quad \text{...(III.44a)}
\]
\[
\mathcal{S}_g = -\frac{\pi^2}{27} \mathcal{S}_s^3/W_0^2, \quad \text{...(III.44b)}
\]
for the second order case.

iv(b). Expansion of \(\varepsilon\) and \(\lambda\) in powers of \(A^{-1/3}\):

The variational parameters \(\varepsilon\) and \(\lambda\) are given in (III.31&32) respectively, which involves the various derivatives of the energy evaluated at the SINM values \(\rho_0 = \rho_{nm}\) and \(a = a_{nm}\). Using the expressions for these functions given in (III.37b-f) and (III.38b-f) for the fourth order and second order energy densities respectively, one can study the variations of central density and
surface thickness as a function of $\alpha$. Alternatively, with the help of the expression for $\alpha$ in (III.39) each functional in the expression of $\varepsilon$ and $\lambda$ can be expanded in powers of $A^{-1/3}$ analogous to the expansion of $E(\rho_{nm}, a_{nm})$ in (III.43&44). Finally the expressions for $\varepsilon$ and $\lambda$ in the powers of $A^{-1/3}$ become

$$
\varepsilon = \frac{1}{3} \left( \frac{s}{W_0} \right) A^{-1/3} \left[ 1 - \frac{26}{225} \left( \frac{s}{W_0} \right) A^{-1/3} + \ldots \right], \quad \text{...(III.45a)}
$$

$$
\lambda = -\frac{1}{9} \left( \frac{s}{W_0} \right) A^{-1/3} \left[ 1 - \frac{72\pi^2 - 344}{135} \left( \frac{s}{W_0} \right) A^{-1/3} + \ldots \right], \quad \text{...(III.45b)}
$$

for the fourth order energy density in (III.19) and

$$
\varepsilon = \frac{1}{3} \left( \frac{s}{W_0} \right) A^{-1/3} \left[ 1 - \frac{2}{9} \left( \frac{s}{W_0} \right) A^{-1/3} + \ldots \right], \quad \text{...(III.46a)}
$$

$$
\lambda = -\frac{1}{3} \left( \frac{s}{W_0} \right) A^{-1/3} \left[ 1 - \frac{2(\pi^2 - 6)}{9} \left( \frac{s}{W_0} \right) A^{-1/3} + \ldots \right], \quad \text{...(III.46b)}
$$

for the second order energy density in (III.20).

The expansions of $\varepsilon$ and $\lambda$ in powers of $A^{-1/3}$ can now be used to obtain the corresponding expansions for the total energy in (III.33) and the mean square radius in (III.9) in the form,

$$
E = \mathcal{S}(\rho_{nm}) A + \frac{s}{a} A^{2/3} \left[ 1 + \frac{5}{9} \left( \frac{s}{W_0} \right) A^{-1/3} - \frac{4(135\pi^2 - 403)}{10125} \left( \frac{s}{W_0} \right)^2 A^{-2/3} + \ldots \right],
$$

\text{...(III.47a)}
\[
<r^2> = \frac{3}{5} \frac{r^2}{m_n} A^{2/3} \left[ 1 - \frac{2}{9} \left( \frac{s}{w} \right)^{1/3} A^{1/3} + \frac{(180\pi^2 + 59)}{675} \left( \frac{s}{w} \right)^2 A^{-2/3} + \ldots \right],
\]

...(III.47b)

for the fourth order case and

\[
E = \xi (\xi_{nm}) A + \xi_s A^{2/3} \left[ 1 + \frac{5}{9} \left( \frac{s}{w} \right)^{1/3} - \frac{(3\pi^2 - 8)}{81} \left( \frac{s}{w} \right)^2 A^{-2/3} + \ldots \right],
\]

...(III.48a)

\[
<r^2> = \frac{3}{5} \frac{r^2}{m_n} A^{2/3} \left[ 1 - \frac{2}{9} \left( \frac{s}{w} \right)^{1/3} A^{1/3} + \frac{(5\pi^2 + 3)}{27} \left( \frac{s}{w} \right)^2 A^{-2/3} + \ldots \right].
\]

...(III.48b)

for the second order case. Note that the above relations for \( E \) and \( <r^2> \) include appropriate corrections arising out of the variation of the central density and the surface thickness with mass number \( A \). It is also interesting to note in (III.47&48) that the total energy and the mean square radius for the two different surface energy densities considered are identical up to terms of order \( A^{1/3} \) and they differ only for the lower order terms in the \( A^{-1/3} \) expansions.

III.v. Results And Discussions:

It is evident from the discussions of the previous sections that the simple approximation to the surface energy density in (III.19&20) together with the leptodermous approximation, allows
one to express the total energy of a spherical nucleus (with $N=Z$ and no Coulomb interaction) and its various derivatives as simple functions of the parameters $W_0$, $\rho_{nm}$ and $a_{nm}$. While the parameter $W_0$ is mainly determined by the saturation properties of nuclear matter, the central density $\rho_{nm}$ and the surface thickness $a_{nm}$ are to be obtained by requiring that they give good energies and good mean square radii for finite nuclei (which excludes too small nuclei). A simple and accurate estimation of $W_0$ for the Skyrme-type interaction have been discussed in the section-v(d). However, as will be shown in the followings, the crucial importance of the fourth order gradient terms in the energy density and the variation of central density and surface thickness with the size of the nucleus can be studied in a general way, independent of a detail knowledge of these three parameters.

v(a). Importance of fourth order gradient terms in the energy density.

The different contributions coming from the surface region of the spherical nucleus for the two different energy densities are given in (III.41a-c&42a-c). The relations for the surface energy $\gamma_s$ for both the energy densities show that for the same surface energy and same $W_0$ and $\rho_{nm}$, the surface thickness parameter $a_{nm}$ for the fourth order energy density is exactly 20% higher than the corresponding second order value. This is a very important
conclusion from the point of view that energy densities containing only second order derivative terms always underestimate the surface thickness and lead to a density which falls-off quickly in the outer part of the surface, whenever one tries to reproduce the correct surface energy\cite{16,61}. An improvement of this behavior of the density in the outer part of the surface can only be achieved at the expense of the surface energy, which is then overestimated. It may further be noted that for the same surface energy $\xi_b$, the curvature contributions $\xi_c$ are same for both the energy densities while the gaussian curvature contributions are different.

\[ v(b) \text{. Variation of central density and surface thickness} \]

The variation of central density and surface thickness with the size of the nucleus are described by the quantities $\varepsilon$ and $\lambda$ defined in (III.31&32) respectively. The relations for the various derivatives of the energy density functionals given in (III.37b-f&38b-f) show that $\varepsilon$ and $\lambda$ have no explicit dependence on $W_0$ and they depend on the size of the nucleus through the ratio $\alpha$. Thus, $\varepsilon$ and $\lambda$ as functions of $\alpha$ describe in a general way, the variation of central density and surface thickness with the size of the nucleus. However, $\alpha$ depends implicitly on $W_0$ through $\rho_{nm}$ and $a_{nm}$. It may be noted that the dependence of $\varepsilon$ and $\lambda$ on $\alpha$ is not a simple one unless the analytical forms of these quantities
are simplified. One may attempt to simplify this by expanding $\varepsilon$ and $\lambda$ in powers of $\alpha$. But, as we shall see, this is not allowed because $\alpha$ in itself is not a very small quantity for finite nuclei. The expansions of $\varepsilon$ and $\lambda$ in powers of $A^{-1/3}$ are given in (III.45&46). These relations can be written in powers of $\alpha$ as,

$$\varepsilon = \frac{5}{6} \alpha \left[ 1 - \frac{13}{45} \alpha + \cdots \right], \quad \text{...}(III.49a)$$

$$\lambda = -\frac{5}{18} \alpha \left[ 1 - \frac{(36\alpha^2-172)}{27} \alpha + \cdots \right], \quad \text{...}(III.49b)$$

for the fourth order case and

$$\varepsilon = \alpha \left[ 1 - \frac{2}{3} \alpha + \cdots \right], \quad \text{...}(III.50a)$$

$$\lambda = -\alpha \left[ 1 - \frac{2(\pi^2-6)}{3} \alpha + \cdots \right], \quad \text{...}(III.50b)$$

for the second order case. These relations can be used to obtain similar expansions for $E$ in (III.33)

$$\frac{E}{A} = \mathcal{E}(\rho_{nm}) + W_0 \left[ \frac{5 \alpha}{2} + \frac{125 \alpha^2}{36} - \frac{(270\alpha^2-403)\alpha}{162} + \cdots \right], \quad \text{...}(III.51a)$$

$$\frac{E}{A} = \mathcal{E}(\rho_{nm}) + W_0 \left[ 3\alpha + 5\alpha^2 - \frac{(6\pi^2-8)\alpha}{3} + \cdots \right], \quad \text{...}(III.51b)$$

for the fourth order and second order cases, respectively.

The variation of $\varepsilon$ is shown as a function of $\alpha$ in the Figure-2 and 3 for the two different energy densities. In both the figures, Curve-1a represents the complete version of $\varepsilon$ without
any further expansion of the derivatives as given in (III.37a-f&38a-f) in powers of $\alpha$. Curve-IIa represents the variation when energy per particle $E/A$, and each of its derivative is expanded in powers of $\alpha$ and terms up to $\alpha^9$ (equivalent to retain terms up to $A^0$) are retained. Curve-IIIa represents the variation when $\varepsilon$ is given by (III.49a&50a). It is evident from a comparison of these curves in each of the Figure-2 and 3 that the complete version of $\varepsilon$ given in (III.31) with the derivatives defined in (III.37&38) is essential for an accurate description of the variation of the central density with the size of the nucleus. A comparison of the Curves-Ia and Ila also shows that the lower order terms in the energy in the expansion in powers of $\alpha$ are quite important for an accurate description of the variation of the central density with the size of the nucleus at least in the region $\alpha \geq 0.1$ (corresponding to the mass region $A \leq 100$), although these lower order terms are not so important as far as the total energy is concerned. From the Curves-IIIa in Figures-2&3, it is evident that it cannot describe the general behavior of $\varepsilon$ either quantitatively or qualitatively in the finite nuclei region and it is applicable only in the asymptotic region of superheavy nuclei extending to SINM. Hence the leptodermous expansion of $\varepsilon$, as given in (III.49a&50a), is not suitable for the study of the variation of the central density with the size of the nucleus.
Fig. 2 The variation of $\varepsilon$ shown as a function of $\alpha$ for the fourth order energy density. See text for explanation of curves Ia, IIa & IIIa.
Fig. 3 The variation of $\varepsilon$ shown as a function of $x$ for the second order energy density. See text for explanation of curves Ia, IIa & IIIa.
It is important to note that the simple approximation to the surface energy density, introduced in section-iii of this chapter, has the advantage of describing the variation of $\varepsilon$ in terms of the variation of $\alpha$, in a way independent of the nature of the interaction used to obtain the energy per particle in nuclear matter $\mathcal{E}(\rho)$ as a function of the density $\rho$. However, there is an implicit dependence through $\rho_{nm}$ and $a_{nm}$ and because of this, the complete range of $\alpha$ for the finite nuclei region will be different for different two body interactions, shifting towards higher and higher values of $\alpha$ for interactions having lower and lower values of $W_o$. In view of this, the bending of the Curve-Ia exhibited by the fourth order energy density will occur only in cases of those interactions which have lower values of $W_o$, i.e., lower values of incompressibility $K$. This is in conformity with the results of Brack et al.[16] and Treiner & Krivine[49,50]. However, in contrast to the fourth order case, the second order case does not show up such bending for Curve-Ia which is due to the restricted nature of the gradient term in the later case because of the imposition of an F-1 density distribution in SINM.

The variation of $\lambda$, as a function of $\alpha$, is shown in Figures-4 and 5 for the fourth and second order energy densities, respectively, using the same three approximations as in case of $\varepsilon$ mentioned above. It is evident from a comparison of these three
Curves-Ib, IIb and IIIb in each of the two Figures-4 and 5 that the complete version of λ defined through (III.32,37&38) is essential for an accurate description of the variation of the surface thickness with the size of the nucleus. A comparison of the Curves-Ib and IIb also show that the lower order terms in the energy in an expansion in powers of α are quite important for an accurate description of λ in the region α≥0.075. It is important to note here that similar conclusions were also drawn in connection with the variation of ε as a function of α in Figures-2 and 3.

An inspection into (III.45&46), where ε and λ are expanded in powers of \((\frac{\varepsilon}{W_0})^{1/3}\) shows that for the same surface energy and same \(W_0\), the leading term in the expansions of ε is the same for both the energy densities, whereas it is different for the quantity λ. As a result, the variation of surface thickness over the whole region is much more wider in the second order case than in the fourth order one. Moreover, the Curves-Ib, IIb and IIIb in Figure-4 show that for all the three approximations considered, the variation of λ with α is small for the fourth order energy density. The fact that the surface thickness of finite nuclei is almost independent of the size of the nucleus favors an energy density containing gradient terms up to fourth order.
Fig. 4 The variation of $\lambda$ shown as a function of $\alpha$ for the fourth order energy density. See text for explanation of the curves $I_b$, $II_b$, and $III_b$. 
Fig. 5 The variation of $\lambda$ shown as a function of $\alpha$ for the second order energy density. See text for explanation of curves Ib, IIb & IIIb.
The above considerations for the variation of $\varepsilon$ and $\lambda$ as a function of $\alpha$ is based on the Taylor series expansion of the total energy about $\rho_0 = \rho_{nm}$ and $a = a_{nm}$ given in (III.29) and retaining terms upto quadratic in $\varepsilon$ and $\lambda$. This is quite reasonable since $\varepsilon$ and $\lambda$ are rather small over the entire range considered here. The inclusion of higher order terms in the expansion of the energy will, therefore, have very little effect on $\varepsilon$ and $\lambda$. It is worth mentioning here that in the present formalism, we have ignored the effect of Coulomb energy and asymmetry in neutron and proton distributions which may considerably modify the values of $\varepsilon$ and $\lambda$.

$V(c)$. Total energy and r.m.s. radius of a spherical nucleus:

The total energy of a spherical nucleus (without Coulomb energy) approximated by Taylor series expansion upto quadratic terms in $\varepsilon$ and $\lambda$ is given in (III.33). From it the energy per particle can be written in the form

$$\frac{E}{A} = \psi(\rho_{nm}) + W h(\alpha) + W f(\alpha), \quad \ldots(III.52)$$

where $h(\alpha)$ and $f(\alpha)$ are functions of $\alpha$ only and are defined through (III.33,37&38). The main contribution coming from the surface region is contained in $h(\alpha)$, whereas $f(\alpha)$ accounts for the changes brought by the variation of $\varepsilon$ and $\lambda$. The variation of $h(\alpha)$ and $f(\alpha)$ as functions of $\alpha$ are shown in Figures-6 and 7, respectively, for the two different energy densities.
emphasize that the behavior of $f(\alpha)$ shown in Figure-7 cannot be described accurately in terms of a further expansion in powers of $\alpha$. However, since the magnitude of $f(\alpha)$ is small compared to $h(\alpha)$, the expansion for $E/A$ up to $\alpha^3$-terms, as given by (III.51a&b), is an accurate approximation for $E/A$ given in (III.52) at least for not too light nuclei. A comparison of $E(\rho_{nm}^{} , a_{nm}^{} )$ in (III.37a&38a) with $E$ in (III.51a&b), respectively, shows that when the variations in $\rho_o^{}$ and $a$ are taken into consideration, the terms of order $\alpha^2$ and higher are modified. Relations in (III.47&48) show that this modification is identical for the curvature contribution for the two different energy densities, whereas the modification in lower order terms brought by the variation of $\varepsilon$ and $\lambda$ are different.

We further note that (III.47&48) clearly exhibit the dependence of the total energy and the mean square radius on the three parameters $W_o^{}, r_{nm}^{}$ and $a_{nm}^{}$. It is apparent from these relations that two-body effective interactions with higher values of incompressibility $K$ (i.e., higher values of $W_o^{}$) will be associated with higher values of $r_{nm}^{}$ and lower values of $a_{nm}^{}$. On the other hand, interactions with lower values of $K$ will be associated with lower values of $r_{nm}^{}$ and higher values of $a_{nm}^{}$. These results are in conformity with the work of Sharma and
Fig. 6 The variation of \( h(\alpha) \) in (III.52) shown as a function of \( \alpha \). The two curves Ia and Ib correspond, respectively, to the fourth and second order energy densities.
Fig. 7 The variation of $f(\alpha)$ in (III-52) shown as a function of $\alpha$. The two curves Ia and Ib correspond, respectively, to the fourth and second order energy densities.
Nagarajan[62,63], where it has been shown that within the
Hartree-Fock approximation it is possible to construct a number of
Skyrme-type interactions, each one reproducing the correct energies
and the mean square radii of finite nuclei. These different sets
of Skyrme-type interactions differ primarily in the values of
incompressibility $K$ and the central density $\rho_{nm}$ in nuclear matter.
The interactions with higher values of $K$ are associated with lower
$\rho_{nm}$ and conversely.

\v(d). Determination of $W_o$:

The function $\rho[\delta(\rho)-\delta(\rho_{nm})]$ in the non-gradient part of the
energy density in (III.16) has been approximated by the function
$W o \rho[(\rho-\rho_{nm})/\rho_{nm}]^2$. In order that this approximation is valid
within the entire range $0$ to $\rho_{nm}$ (which corresponds to the surface
region of finite nuclei), the average behavior of this function
and its first derivative must be in good agreement over this
range. This has been expressed by the following relations

\begin{align}
\int_0^{\rho_{nm}} \rho[\delta(\rho)-\delta(\rho_{nm})]d\rho &\approx \int_0^{\rho_{nm}} W o \rho \left(\frac{\rho-\rho_{nm}}{\rho_{nm}}\right)^2 d\rho, \quad \ldots (III.53) \\
&\text{and} \quad \int_0^{\rho_{nm}} \rho d\rho \left[\rho[\delta(\rho)-\delta(\rho_{nm})]\right]d\rho = \int_0^{\rho_{nm}} \frac{d}{d\rho} \left[W o \rho \left(\frac{\rho-\rho_{nm}}{\rho_{nm}}\right)^2\right] d\rho. \quad \ldots (III.54)
\end{align}

From these two conditions, the expression for $W_o$ is obtained to be

$$W_o = -6 \frac{\rho_{nm}}{2} \int_0^{\rho_{nm}} \rho^2 \frac{d\sigma}{d\rho} d\rho, \quad \ldots (III.55)$$
or, equivalently \( W_0 = -6 \int \frac{y^2}{y} \frac{dy}{dy}, \) ...(III.56)

where, \( y = \rho/\rho_{nm} \). The values of \( W_0 \) have been calculated for the eight sets of Skyrme-type interactions used by Sharma & Nagarajan[63] and these values along with the parameters of the interaction sets have been given in Table-9. It is evident from these values that it is nearly equal to \( K/18 \) for those interactions for which the higher derivatives \( \rho_{nm}^n d^n \varepsilon/\rho_{nm}^n \) for \( n \geq 3 \) have small values and the difference increases as the values of these higher order derivatives increases. With these values of \( W_0 \), the total energy and r.m.s radii of hypothetical spherical nuclei with \( N=Z \) have been calculated from (III.47a&b) for the fourth order case over the range \( A^{1/3} = 2.5 \) to 6.5 for the eight sets of interactions of Sharma & Nagarajan and their values have been given in Tables-10&11, respectively. From these tables it is evident that although all these sets give nearly equal energies, the interactions with higher values of incompressibility give higher values of r.m.s radii for heavy nuclei. On the other hand, the interactions with lower values of incompressibility give higher values of r.m.s radii for light nuclei. This substantiates the claim made in this connection in the last section.
TABLE-9: The values of $W_0$ obtained from (III.56) for the eight sets of Skyrme-type interactions of Sharma and Nagarajan along with the other nuclear matter characteristics $\rho_{nm}$, $K$ and $\alpha$. The value of $m^*/m$ for these sets have been taken to be 0.79.

<table>
<thead>
<tr>
<th>Force</th>
<th>$\rho_{nm}$ (fm$^{-3}$)</th>
<th>$K$ (MeV)</th>
<th>$\alpha$</th>
<th>$W_0$ (MeV)</th>
</tr>
</thead>
<tbody>
<tr>
<td>SKSM I</td>
<td>0.140</td>
<td>392.6</td>
<td>1.2</td>
<td>19.952</td>
</tr>
<tr>
<td>SKSM II</td>
<td>0.145</td>
<td>359.8</td>
<td>1.0</td>
<td>19.114</td>
</tr>
<tr>
<td>SKSM III</td>
<td>0.150</td>
<td>327.0</td>
<td>0.8</td>
<td>18.171</td>
</tr>
<tr>
<td>SKSM IV</td>
<td>0.155</td>
<td>304.9</td>
<td>2/3</td>
<td>17.497</td>
</tr>
<tr>
<td>SKSM V</td>
<td>0.160</td>
<td>248.0</td>
<td>1/3</td>
<td>15.442</td>
</tr>
<tr>
<td>SKSM VI</td>
<td>0.165</td>
<td>219.8</td>
<td>1/6</td>
<td>14.266</td>
</tr>
<tr>
<td>SKSM VII</td>
<td>0.170</td>
<td>200.3</td>
<td>1/20</td>
<td>13.371</td>
</tr>
<tr>
<td>SKSM*</td>
<td>0.153</td>
<td>313.7</td>
<td>0.72</td>
<td>17.773</td>
</tr>
</tbody>
</table>

TABLE-10: Total energy $E$ obtained from (III.47a) for the eight sets of Skyrme-type interactions of Sharma & Nagarajan for the fourth order surface energy density approximation.

<table>
<thead>
<tr>
<th>$\frac{1}{3}A$</th>
<th>SKSM-I</th>
<th>II</th>
<th>III</th>
<th>IV</th>
<th>V</th>
<th>VI</th>
<th>VII</th>
<th>SKSM*</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.5</td>
<td>-119.4</td>
<td>-119.2</td>
<td>-119.1</td>
<td>-119.0</td>
<td>-118.7</td>
<td>-118.6</td>
<td>-118.6</td>
<td>-119.0</td>
</tr>
<tr>
<td>3.0</td>
<td>-247.0</td>
<td>-246.7</td>
<td>-246.5</td>
<td>-246.5</td>
<td>-245.9</td>
<td>-245.8</td>
<td>-245.6</td>
<td>-246.4</td>
</tr>
<tr>
<td>3.5</td>
<td>-437.5</td>
<td>-437.2</td>
<td>-437.0</td>
<td>-436.9</td>
<td>-436.3</td>
<td>-436.0</td>
<td>-435.8</td>
<td>-436.8</td>
</tr>
<tr>
<td>4.0</td>
<td>-703.0</td>
<td>-702.6</td>
<td>-702.5</td>
<td>-702.4</td>
<td>-701.7</td>
<td>-701.4</td>
<td>-701.1</td>
<td>-702.3</td>
</tr>
<tr>
<td>4.5</td>
<td>-1055.4</td>
<td>-1055.1</td>
<td>-1054.9</td>
<td>-1054.9</td>
<td>-1054.2</td>
<td>-1053.9</td>
<td>-1053.6</td>
<td>-1054.7</td>
</tr>
<tr>
<td>5.0</td>
<td>-1506.8</td>
<td>-1506.4</td>
<td>-1506.4</td>
<td>-1506.4</td>
<td>-1505.8</td>
<td>-1505.6</td>
<td>-1505.3</td>
<td>-1506.1</td>
</tr>
<tr>
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<td>-2069.1</td>
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<td>-2068.6</td>
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<tr>
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<td>-2754.1</td>
<td>-2754.2</td>
<td>-2754.5</td>
<td>-2754.3</td>
<td>-2754.2</td>
<td>-2754.2</td>
<td>-2754.0</td>
</tr>
<tr>
<td>6.5</td>
<td>-3574.5</td>
<td>-3574.3</td>
<td>-3574.6</td>
<td>-3575.1</td>
<td>-3575.1</td>
<td>-3575.4</td>
<td>-3575.4</td>
<td>-3575.4</td>
</tr>
</tbody>
</table>
TABLE-11 : The r.m.s radius $<r^2>$ obtained from (III.47b) for the eight sets of Skyrme-type interactions of Sharma & Nagarajan for the fourth order surface energy density approximation.

<table>
<thead>
<tr>
<th>$A^{1/3}$</th>
<th>SKSM-I</th>
<th>II</th>
<th>III</th>
<th>IV</th>
<th>V</th>
<th>VI</th>
<th>VII</th>
<th>SKSM*</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.5</td>
<td>18.485</td>
<td>18.185</td>
<td>17.939</td>
<td>17.678</td>
<td>17.801</td>
<td>17.813</td>
<td>17.809</td>
<td>17.779</td>
</tr>
<tr>
<td>6.0</td>
<td>31.714</td>
<td>31.095</td>
<td>30.550</td>
<td>30.007</td>
<td>29.839</td>
<td>29.584</td>
<td>29.328</td>
<td>30.219</td>
</tr>
<tr>
<td>6.5</td>
<td>36.980</td>
<td>36.238</td>
<td>35.572</td>
<td>34.917</td>
<td>34.635</td>
<td>34.275</td>
<td>33.920</td>
<td>35.172</td>
</tr>
</tbody>
</table>

III.vi. Summary And Conclusions:

In this chapter the semi-classical nuclear properties have been studied in the framework of the energy density formalism. Two different energy densities, one containing derivative terms upto second order only and other containing fourth order derivative terms too have been considered for the purpose. The surface energy densities in each of the two cases have been suitably approximated so that the EL equations in an SINM lead to a pure Fermi-shape ($F_1$) for the density. The restricted form of the surface energy density resulting from this approximation involves simply the central density $p_{nm}$ and the surface thickness $a_{nm}$ in SINM and a parameter $W_0$. 
mainly determined by the incompressibility $K$ in nuclear matter. The relations for the surface energy in SINM for the two different energy densities explicitly demonstrate the crucial importance of the inclusion of fourth order derivative terms in the energy density. It is observed that for the same surface energy and same values of the parameters $W_0$ and $\rho_{nm}$, the surface thickness parameter $a_{nm}$ for the fourth order energy density is exactly 20% higher than the corresponding value for the second order energy density. This is quite important from the point of view that energy densities containing only second order derivative terms always underestimate the surface thickness and lead to a density which falls-off quickly in the outer part of the surface whenever one tries to reproduce the correct surface energy.

The resulting surface energy density has been used in conjunction with a leptodermous expansion to obtain simple analytical relations for the total energy and the mean square radius of a finite nucleus, in terms of its central density $\rho_0$ and the surface thickness $a$. Under the assumption that the departure of $\varepsilon$ and $\lambda$ of the central density and the surface thickness of a finite nucleus from their respective SINM values are small, the contribution to the total energy per particle coming from the surface region and the departures $\varepsilon$ and $\lambda$, have been obtained as
simple functions of the small ratio $\alpha$ between the surface thickness and the half density radius. These departures $\varepsilon$ and $\lambda$ as functions of the small ratio $\alpha$ describe the variation of the central density and the surface thickness with the size of the nucleus in a general way quite independent of the parameters $W_0$, $\rho_{nm}$ and $\alpha_{nm}$. It is found that these variations cannot be reproduced by an expansion of $\varepsilon$ and $\lambda$ in powers of the ratio $\alpha$ and retaining the first few terms. It is further observed that the lower order terms in the total energy per particle in an expansion in powers of $\alpha$ are quite important for a correct description of the departures $\varepsilon$ and $\lambda$ at least in the region $\alpha \geq 0.1$, although these lower order terms are not so important so far as the total energy is concerned.

A comparison of the variations of the surface thickness with the size of the nucleus for the two different energy densities shows that the variation $\lambda$ is much wider for the second order energy density, whereas, it is rather small for the fourth order energy density, in conformity with the fact that the surface thickness is almost independent of the size of the nucleus. This fact demonstrates the inadequacy of a second order energy density and favors an energy density containing derivative terms upto fourth order. However, we must mention here that the variation of
\( \rho_0 \) and \( a \) with the size of the nucleus considered in this work are rather qualitative in nature because of the fact that the modifications in these variations due to Coulomb and asymmetry effects have not been taken into consideration.

Lastly, we have discussed the significance of the parameter \( W_0 \) in the approximation of the non-gradient part of the energy density in (III.18) and its determination. Its values for the eight sets of Skyrme-type interactions of Sharma and Nagarajan have been given in Table-9. It has been found that this parameter \( W_0 \) is closely related to the nuclear matter incompressibility \( K \) of the effective interaction and it is very close to the value \( K/18 \) for those interactions for which the higher derivatives, \( \rho^n(\partial^m \rho/\partial \rho^n)_{n>3} \) have small values and the difference increases as these higher derivatives have larger values. However, from the Tables-10&11, it is evident that although these effective interactions varies widely in their nuclear matter properties, especially in their nuclear matter incompressibility values, still then they reproduce the ground state properties of the finite nuclei, such as, total energy and r.m.s. radii well within a fair accuracy. Moreover, from the analysis of the data for the various modes of nuclear excitations made by Slomo and Youngblood[30], they have predicted the range of the nuclear matter
incompressibility from 200 Mev to 350 Mev. In the forthcoming chapter, an attempt has been made to ascertain the correct value of the nuclear matter incompressibility from a quite general consideration avoiding the direct fit to the observed ground state properties like binding energies and radii of finite nuclei.