CHAPTER 6 TIME JITTER CAUSED BY THE SAMPLE-RATE CONVERTER
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In a proposed sample rate converter, the output samples are placed on the output time grid with a variable time delay, which results in time jitter. Due to this timing jitter, the output samples will not have the correct amplitude [23]. In this chapter, the time jitter signal in the conversion process will be investigated. Also an estimate of the output amplitude deviations will be given using a stochastic approach of the jitter process.

6.1 Construction of the time jitter signal

Each output sample has an individual delay, with respect to the previous input sample, when it is placed in the output sequence. This time difference is called timing jitter. In figure 6.1 a plot is given which shows a part of an input sequence, a conversion control signal (output of sigma-delta converter), the converted output sequence and the individual time delays of the output samples (time jitter signal).

Let $T_{\text{sin}}$ and $T_{\text{out}}$ be the sample period of the input sequence and the output sequence respectively. Assume that the PLL supplies a certain DC to the sigma-delta modulator, such that its output pulses are as shown in figure 6.1. The output samples of the sample-rate converter can then easily be constructed by applying the conversion control algorithm given in table 6.1. It should be noticed that the output time grid is determined by the output pulses of the sigma-delta modulator.
6.2 Individual time delays of the output samples

The time delay of each output sample with respect to the previous input sample can be determined by looking at the operation performed by the variable hold function (transfer a new input sample to the output sequence or repeat the previous output sample).

If the control action is “repeat”, then the delay of the corresponding output sample is, with respect to the previous output sample, incremented with $T_{S,\text{out}}$. This implies that the difference in individual delays between output samples 1 and 2 is equal to $T_2 - T_1 = T_{S,\text{out}}$ (see figure 6.1). Equally, for the remaining ‘repeat’ actions in figure 6.1: $T_3 - T_2 = T_{S,\text{out}}$, $T_5 - T_4 = T_{S,\text{out}}$, $T_6 - T_5 = T_{S,\text{out}}$, $T_9 - T_8 = T_{S,\text{out}}$, and $T_{10} - T_9 = T_{S,\text{out}}$.
If the control action is “take over”, then the increment in individual delay of the corresponding output sample, with respect to the previous output sample is also $T_{s,\text{out}}$, but since the ‘take over’ action places a new input sample on the output time grid, the individual time delay must be decremented with $T_{s,\text{in}}$. In conclusion, the increment in individual time delay with respect to the previous output sample is $T_{s,\text{out}} - T_{s,\text{in}}$ (since $T_{s,\text{in}}$ is larger than $T_{s,\text{out}}$, the individual time delay is decreased by $T_{s,\text{in}} - T_{s,\text{out}}$). For the situation as depicted by figure 6.1, the difference in individual time delays between output samples 3 and 4 is $T_4 - T_3 = T_{s,\text{out}} - T_{s,\text{in}}$. Equally, for the remaining ‘take over’ actions in figure 6.1: $T_7 - T_6 = T_{s,\text{out}} - T_{s,\text{in}}$ and $T_8 - T_7 = T_{s,\text{out}} - T_{s,\text{in}}$.

In one second, the sample rate converter executes $F_{s,\text{in}}$ ‘take over’ actions and $F_{s,\text{out}} - F_{s,\text{in}}$ ‘repeat’ actions. With this information, table 5.3 can be extended to obtain table 6.1:

Table 6.1 Time Error Increments Caused by the Conversion Process

<table>
<thead>
<tr>
<th>Sigma-delta output</th>
<th>Control Action</th>
<th>Time error increment</th>
<th>Number of executions per second</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td>Take over a new input sample</td>
<td>$T_{s,\text{out}} - T_{s,\text{in}}$</td>
<td>$F_{s,\text{in}}$</td>
</tr>
<tr>
<td>1</td>
<td>Repeat the previous output sample</td>
<td>$T_{s,\text{out}}$</td>
<td>$F_{s,\text{out}} - F_{s,\text{in}}$</td>
</tr>
</tbody>
</table>

From table 6.1, the average time error increment of an output sample with respect to its previous output sample $\Delta T_{\text{INC,av}}$ can be calculated:

$$\Delta T_{\text{INC,av}} = F_{s,\text{in}} \cdot (T_{s,\text{out}} - T_{s,\text{in}}) + \left( F_{s,\text{out}} - F_{s,\text{in}} \right) \cdot T_{s,\text{out}} \ldots$$

$$\ldots = F_{s,\text{in}} \cdot T_{s,\text{out}} - F_{s,\text{in}} T_{s,\text{in}} + F_{s,\text{out}} T_{s,\text{out}} - F_{s,\text{in}} T_{s,\text{out}}$$

By using the following relations
The average time delay increment of the output samples becomes:

\[ \Delta T_{INC,av} = F_{S,in} T_{S,out} - 1 + 1 - F_{S,in} T_{S, out} = 0 \]  

(6-4)

It can be concluded that the average time error increment is zero as required. It follows from figure (6.1) that the average time error is greater than zero. It is however only the fluctuation in the individual time delays which causes distortion in the output signal. Therefore, the time error signal is obtained by subtracting the average delay from the individual time delays. Consequently, the time error will be balanced around zero.

### 6.3 Amplitude deviations due to timing jitter

Thus far we have only considered the time jitter signal and the individual output sample delays introduced by the variable hold operation. The sample rate conversion can be viewed as resampling the input signal by a jittered clock. Each output sample has, when its placed on the output time grid, a (positive) time delay with respect to the actual input sample. As a result of this timing jitter, the amplitude of the output samples will deviate from their correct value. In this sub-paragraph we want to study the usefulness of a first-order approximation of this amplitude deviations.

#### 6.3.1 A first-order model

The continuous-time input signal, constructed from the input samples, to be resampled by the output clock, is denoted by \( x(t) \); it is a realization of the stochastic process \( x \). In the following considerations, \( x(t) \) is assumed to be real, bandlimited (bandwidth \( W \)), second order \( (E\{x^2(t)\} < \infty) \), wide-sense stationary, with mean zero, having a correlation function \( R_{xx}(\tau) \) and a power spectral density \( S_{xx}(f) \), related to each other by Wiener-Khinchine relations [24]:
\begin{equation}
S_{xx}(f) = \int_{-\infty}^{\infty} R_{xx}(\tau).e^{-j2\pi ft} d\tau
\end{equation}

\begin{equation}
R_{xx}(\tau) = \int_{-\infty}^{\infty} S_{xx}(f).e^{j2\pi ft} df
\end{equation}

Since the process $x$ is bandlimited to $W$, $S_{xx}(f) = 0$ for $|f| > W$. Therefore the process $\hat{x}$ and all its higher derivatives are well defined. See for instance the first derivative:

\begin{equation}
S_{\hat{x}\hat{x}}(f) = (2\pi f)^2 S_{xx}(f)
\end{equation}

And therefore:

\begin{equation}
R_{\hat{x}\hat{x}}(0) = \int_{-W}^{W} (2\pi f)^2 S_{xx}(f) df < \infty
\end{equation}

Likewise, it can be shown that all higher order derivatives exist. In the remainder of this sub-paragraph, we assume a fourth-order time jitter process, which is independent of the process $x$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{Fig_6.2.png}
\caption{Amplitude error of the output sample due to timing jitter}
\end{figure}
Figure (6.2) demonstrate an example of the amplitude deviation of an output sample due to timing jitter. The input samples have the correct amplitude. The input sample at time $t_1$ has an amplitude $A$, this sample is placed on the output grid at time $t_2$ ($t_2 - t_1 = \Delta t$), causing an amplitude error $\Delta A$. The correct amplitude of the output sample at time $t_2$ can be determined by calculating the Taylor series expansion of the signal $x(t)$ around the time moment $t_1$ (Figure 6.2):

$$x(t_2) = A + \Delta A = x(t_1) + \dot{x}(t_1) \cdot \Delta t + \frac{1}{2} \ddot{x}(t_1) \cdot \Delta t^2 + \frac{1}{6} x^{(3)}(t_1) \cdot \Delta t^3 + \frac{1}{24} x^{(4)}(t_1) \cdot \Delta t^4 \quad (6-9)$$

The amplitude error $\Delta A$ is for this particular case, given that $x(t_1) = A$:

$$\Delta A = x(t_2) - x(t_1) = \dot{x}(t_1) \cdot \Delta t + \frac{1}{2} \ddot{x}(t_1) \cdot \Delta t^2 \quad (6-10)$$

(the third- and higher-order terms of the Taylor series have been neglected).

We are interested in a first order approximation of the amplitude deviations caused by timing jitter $\Delta t(\hat{t})$, because a first order model will simplify the calculations. We therefore truncate the generalized Taylor series expansion of $x(t)$ after the first order term to obtain:

$$x(\hat{t}) = x(t + \Delta t(\hat{t})) = x(t) + \dot{x}(t) \cdot \Delta t(\hat{t}) \quad (6-11)$$

Where $t$ denotes the nominal input sample moment and $\hat{t}$ denotes the actual output sample moment. The time error (jitter) is always positive in this sample-rate conversion process. When $x(t)$ is increasing, its first derivative will be positive and the actual amplitude error is negative (figure 6.2). When $x(t)$ is decreasing, its first derivative will be negative and the actual amplitude error is positive. In conclusion the first order approximation of the amplitude error of the output samples equals $-\dot{x}(t) \cdot \Delta t(\hat{t})$. 
6.3.2 Validity of a first-order approximation

The approximation by a first-order model is only valid when the contribution of the second order term in the amplitude deviations is much less than the first-order term. In order to check this restriction, we calculate the mean square error of the first-order approximation, neglecting the third and higher order terms of the Taylor series expansion of $x(t)$:

$$
E \left\{ \frac{1}{2} \ddot{x}(t). [\Delta t(\ddot{t})]^2 \right\} = E \left\{ \left[ (x(t + \Delta t(\ddot{t})) - x(t) + x(t). \ddot{t}(\ddot{t})) \right]^2 \right\} \tag{6-12}
$$

This expectation is with respect to the stochastic process $x$ and the process $\Delta t$. We assume that these processes are independent of each other. The expectation of a product of independent stochastic variables is equal to the product of the expectations of these variables. Firstly, we will average the outcome of (6-12) with respect to $x$ and then we will average the result with respect to $\Delta t$. For any fixed $\Delta t(\ddot{t}) = J$, we obtain for the right-hand side of (6-12):

$$
E_x \left\{ \left| x(t + J) - x(t) + J. \dot{x}(t) \right|^2 \right\} = E_x \{ [x(t + J)]^2 \} - 2.E_x \{ x(t). x(t + J) \} + E_x \{ [x(t)]^2 \} + \cdots
$$

$$
... + 2.J. E_x \{ x(t + J). x(t) \} - 2.J. E_x \{ x(t). \dot{x}(t) \} + J^2. E_x \{ [\dot{x}(t)]^2 \} \tag{6-13}
$$

For stationary bandlimited processes, the following equations will hold [23]:

$$
R_{xx}(J) = \check{R}_{xx}(J) \tag{6-14}
$$

$$
R_{xx}(J) = -\check{R}_{xx}(J) \tag{6-15}
$$

$$
R_{xx}(J) = -\ddot{R}(J) \tag{6-16}
$$

We assume the process $x$ to be real, so the correlation function is even. This implies that all its odd derivatives vanish at zero. We find for the different terms of the right-hand side of (6-13),
keeping in mind that \( x(t) \) is stationary, so that its statistical properties are not influenced by a time shift \( J \):

\[
E_x[[x(t)]^2] = E_x[x(t)x(t+\tau)]|_{\tau=0} = R_{xx}(0) \quad (6-17)
\]

\[
E_x[[x(t+J)]^2] = E_x[x(t+J)x(t+J+\tau)]|_{\tau=0} = R_{xx}(0) \quad (6-18)
\]

Using (6-14) to (6-16), we furthermore obtain:

\[
E_x[[\dot{x}(t)]^2] = E_x[\dot{x}(t)\dot{x}(t+\tau)]|_{\tau=0} = -\ddot{R}_{xx}(0) \quad (6-19)
\]

\[
E_x[x(t)x(t+J)] = E_x[x(t)x(t+\tau)]|_{\tau=J} = R_{xx}(J) \quad (6-20)
\]

\[
E_x[x(t+J)\dot{x}(t)] = R_{xx}(J) = -R_{xx}(J) \quad (6-21)
\]

\[
E_x[x(t)\dot{x}(t)] = R_{xx}(\tau)|_{\tau=0} = R_{xx}(0) = -R_{xx}(0) = 0 \quad (6-22)
\]

We can expand the correlation function of (6-20) further by applying Taylor series expansion around \( v=0 \):

\[
R_{xx}(J) = R_{xx}(0) + \ddot{R}_{xx}(0).J + \frac{1}{2}. \dddot{R}_{xx}(0).J^2 + \frac{1}{6}. \dddot{R}_{xx}(0).J^3 + \frac{1}{24}. \dddot{R}_{xx}^{(4)}(0).J^4 (|u| \leq |J|) \quad (6-23)
\]

Because the odd derivatives of the correlation function vanish at zero, (6-23) becomes:

\[
R_{xx}(J) = R_{xx}(0) + \frac{1}{2}. \dddot{R}_{xx}(0).J^2 + \frac{1}{24}. \dddot{R}_{xx}^{(4)}(0).J^4 \quad (6-24)
\]

Expansion of the correlation function of (6-21) gives:

\[
R_{xx}(J) = R_{xx}(0) + R_{xx}(0).J + \frac{1}{2}. R_{xx}(0).J^2 + \frac{1}{6}. R_{xx}^{(3)}(0).J^2 + \frac{1}{24}. R_{xx}^{(4)}(s).J^4 (|s| \leq |J|)
\]
\begin{equation}
R_{xx}(0) - \dot{R}_{xx}(0).J - \frac{1}{2}.R^{(3)}_{xx}(0).J^2 - \frac{1}{6}.R^{(4)}_{xx}(0).J^3 - \frac{1}{24}.R^{(5)}_{xx}(s).J^4
\end{equation}

Because the odd derivatives of the correlation function are zero for \( \tau = 0 \).

When we substitute these results in (6-13), we obtain:

\begin{equation}
E_x[|x(t + J) - x(t) + J.\dot{x}(t)|^2] = - \frac{1}{12}.R^{(4)}_{xx}(s).J^4 - \frac{2}{3}.R^{(4)}_{xx}(u).J^4 \leq \frac{1}{4}.R^{(4)}_{xx}(0).J^4
\end{equation}

When we finally average this result with respect to the jitter process \( \Delta t \), we obtain:

\begin{equation}
E_{x,\Delta t}[|x(t + \Delta t) - x(t) + \Delta t(t).\dot{x}(t)|^2] \leq \frac{1}{4}.R^{(4)}_{xx}(0).E[|\Delta t(f)|^4]
\end{equation}

It follows that the mean-square error resulting from the first-order approximation is of the order of \( |\Delta t(f)|^4 \).

We assume that process \( x \) is white, stationary with a spectral density given by:

\begin{equation}
S_{xx}(f) = \begin{cases} 
\frac{\sigma_x^2}{2W} & |f| < W \\
0 & |f| \geq W 
\end{cases}
\end{equation}

From the spectral representation of a stationary process, the following relations can be derived [23]:

\begin{equation}
-\dot{R}_{xx}(0) = \int_{-W}^{+W} (2\pi f)^2.S_{xx}(f)df = \sigma_x^2 = \frac{1}{3}.\sigma_x^2.(2\pi W)^2
\end{equation}

\begin{equation}
R^{(4)}_{xx}(0) = \int_{-W}^{+W} (2\pi f)^4.S_{xx}(f)df = \sigma_x^2 = \frac{1}{5}.\sigma_x^2.(2\pi W)^4
\end{equation}
The second moment of the amplitude error is, when using the first-order approximation of (6-11), estimated as:

\[
E \left[ |x(t) - x(t + \Delta t(\hat{t}))|^2 \right] = E \left[ |\dot{x}(t).\Delta t(\hat{t})|^2 \right] = \sigma_x^2. E \left[ |\Delta t(\hat{t})|^2 \right] = \frac{4}{3}. \pi^2. W^2. \sigma_x^2. E \left[ |\Delta t(\hat{t})|^2 \right]
\]

(6-31)

The product of second order moments in (6-31) reflects the approximated noise power generated by timing errors. The error of this approximation is given by (6-27):

\[
E \left\{ \frac{1}{2}.\dot{x}(t).|\Delta t(\hat{t})|^2 \right\} \leq \frac{1}{4}. R_{xx}^{(4)}(0). E\{\Delta t(\hat{t})^4\} = \frac{4}{3}. \pi^4. W^4. \sigma_x^2. E\{\Delta t(\hat{t})^4\}
\]

(6-32)

Only when the error given by (6-32) is much smaller than the estimate of the first-order amplitude deviations given by (6-31), the first-order model will be accurate enough to predict noise power figures introduced by the sample-rate conversion process.

6.3.3 Distribution Function Estimation of the Timing Jitter

The output sequence of a first order sigma-delta modulator is fixed for a given DC input, that is, when the DC level at the input is for instance 0.5 Volts, then the output sequence is a fixed pattern, only consisting of repetitions of the sequence “111-1”.

The output sequence of a higher-order sigma-delta modulator is not fixed, it also contains patterns differing from the above mentioned. For third and higher-order sigma-delta modulators, the diversity of the patterns is such that an exact analysis is not available because of its complexity. In order to simplify the analysis, we will have to make a few assumptions. It is observed that a repetition sample causes an increment in time delay of \(T_{s,\text{out}}\) seconds. When an output sample is for instance repeated \(n\) times, the \(n\)-th repetition sample has an increment in time delay of \(n.T_{s,\text{out}}\) seconds with respect to the first repetition sample. When the output sampling frequency is kept constant, it is to be expected that the maximum possible value of the
time delay is larger when the input sampling frequency is smaller, because a small input sampling frequency implies much repetitions (more “1” pulses).

We now make the assumption that the average delay of the copied input samples is zero, so the output samples obtained by copying the input samples have on the average the correct timing moment. As a consequence, its only the repetition output samples that have on the average a contribution to the absolute value of the time delay.

In one second \( F_{s,\text{in}} \) input samples are copied to the output. The remaining \( F_{s,\text{out}} - F_{s,\text{in}} \) output samples are repetition samples. The average number of repetitions of each input sample equals:

\[
R_{av} = \frac{F_{s,\text{out}} - F_{s,\text{in}}}{F_{s,\text{in}}} = \frac{F_{s,\text{out}}}{F_{s,\text{in}}} - 1 \quad (6-33)
\]

Since a repetition output sample introduces an increment in time delay of \( T_{s,\text{out}} \) this corresponds to an average time delay of:

\[
\Delta T_{av} = \left( \frac{F_{s,\text{out}}}{F_{s,\text{in}}} - 1 \right) T_{s,\text{out}} = \frac{1}{F_{s,\text{in}}} - \frac{1}{F_{s,\text{out}}} = T_{s,\text{in}} - T_{s,\text{out}} \quad (6-34)
\]

Equation (6-34) shows that at lower input sampling frequencies the average time delay of the output samples is larger, which satisfies the expectations.

We want to determine the second moment (variance) and the fourth moment of the time delay, so we are confronted with the diversity in the output patterns of the sigma-delta modulator. We expect that when the order of the sigma-delta modulator is higher, the variance of the time error is larger. This is because a higher-order sigma-delta modulator produces a larger variety in output patterns. The most evident thing to do is to estimate the distribution function by means of simulation results.
Numerical Analysis by simulation of the system

A third-order sample-rate converter has been simulated using 131072 simulation points. The output sampling frequency is fixed at 128Fs, out while the input sampling frequency equals 19.23Fs (Fs=44.1kHz). It is calculated that the DC level at the input of the sigma-delta converter must be +0.69953125 Volts, which is about the upper bound of the usable input range. From (6-2) it follows that in terms of time delay, this is the worst case situation.

In figure 6.3 a plot of the corresponding converted sinewave is given. From this plot it follows that the repetition sequences of the input samples are subject to fluctuation. These fluctuations are caused by the diversity in the output patterns of the sigma-delta modulator.

![Fig. (6.3). An example of a converted sinewave. The input sampling frequency equals 19.23 F, while the output sampling frequency equals 128 F.](image)

The number of repetitions of each input sample, that is the length of the “+1”-sequences in the sigma-delta output, has been counted for the 131072 simulation points mentioned above. Table 6.2 shows the results of this count: the number of occurrences of all repetition sequences is shown and the corresponding probabilities have been calculated.
Table 6.2 Distribution function of time delays

<table>
<thead>
<tr>
<th>Number of Repetitions R</th>
<th>Number of Occurrences</th>
<th>Probability P(Ri=R) (1≤i≤#input_samples)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>0.000152354</td>
</tr>
<tr>
<td>3</td>
<td>575</td>
<td>0.029201158</td>
</tr>
<tr>
<td>4</td>
<td>3224</td>
<td>0.163729623</td>
</tr>
<tr>
<td>5</td>
<td>5487</td>
<td>0.278655223</td>
</tr>
<tr>
<td>6</td>
<td>5424</td>
<td>0.275455792</td>
</tr>
<tr>
<td>7</td>
<td>3292</td>
<td>0.167182977</td>
</tr>
<tr>
<td>8</td>
<td>1454</td>
<td>0.073840841</td>
</tr>
<tr>
<td>9</td>
<td>223</td>
<td>0.013249708</td>
</tr>
<tr>
<td>10</td>
<td>9</td>
<td>0.000457062</td>
</tr>
<tr>
<td>&gt;10</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Figure 6.4 shows the distribution function of the repetition sequences of the output signal of the sigma-delta modulator. Note that this distribution function resembles the well-known Gaussian distribution.

Fig. (6.4) Distribution function of the time delays obtained by a simulation
From table 6.2 the average time delay (first moment) can be calculated:

\[
E\{\Delta t\} = \left[ \sum_i R_i.P(R = R_i) \right] T_{s,\text{out}} = 5.66.1.77 \times 10^{-7} = 1.00 \mu \text{sec}
\]  

(6-35)

When the values of the sampling frequencies are filled in (6-34), it follows that this equation predicts exactly the same average delay as the one obtained by simulation results.

This average time delay will not give rise to amplitude deviations in the output signal. It’s the fluctuations in time delay that are responsible for the amplitude errors.

The variance (second moment) of the time jitter can be calculated as:

\[
E\left\{ (\Delta t - E[\Delta t])^2 \right\} = \sum_i \left[(NR_i - 5.66)T_{s,\text{out}}\right]^2.P(\text{delay} = NR_i) = ...
\]

... = 1.65.T_{s,\text{out}}^2 = 5.19 \times 10^{-14} \text{ sec}^2

(6-36)

The standard deviation is the square root of the variance and equals:

\[
\sigma_{\Delta t} = \sqrt{\sigma_{\Delta t}^2} = \sqrt{1.65.T_{s,\text{out}}^2} = 1.28.T_{s,\text{out}} = 0.228 \mu \text{sec}
\]  

(6-37)

For the fourth moment we find from the simulation results:

\[
E\left\{ (\Delta t - E[\Delta t])^4 \right\} = \sum_i \left[(NR_i - 5.66)T_{s,\text{out}}\right]^4.P(\text{delay} = NR_i) = 7.13.T_{s,\text{out}}^4 = 7.02 \times 10^{-27} \text{ sec}^4
\]  

(6-38)

Finally, when we substitute (6-36) and (6-38) in the first order approximation equations, the second moment of the amplitude error and the error of the first–order approximation become:

\[
E\left\{ x(t) - x(t + \Delta t(t))^2 \right\} = \frac{4}{3} \Pi^2 W^2 \sigma_x^2.E\{\Delta t(t)^2\} = \frac{4}{3} \Pi^2 W^2 .1.77 \times 10^{-14}.\sigma_x^2
\]  

(6-39)

\[
E\left\{ \frac{1}{2}x(t).[\Delta t(t)]^2 \right\} = \frac{4}{5} \Pi^4 W^4 \sigma_x^2.E\{\Delta t(t)^3\} = \frac{4}{5} \Pi^4 W^4 .7.02 \times 10^{-27}.\sigma_x^2
\]  

(6-40)
When we divide (6-40) by (6-39) we obtain the relative error of the first-order approximation. This relative error $\varepsilon_{\text{rel}}$ becomes, when we assume that the signal $x(t)$ has a bandwidth $W$ of 20kHz (which is true for audio signals):

$$
\varepsilon_{\text{rel}} = 8.12 \times 10^{-14} \Pi^2 W^2 = 3.2 \times 10^{-4} = 0.032\%
$$  \hspace{1cm} (6-41)

It appears that the worst case relative error caused by a first-order approximation is 0.032%. From this figure we may conclude that the first-order approximation of the amplitude error is accurate enough to predict amplitude deviations as well as noise power introduced by the sample-rate conversion process.

In order to get some insight in the absolute value of the noise power we have to determine the variance of the input signal $x(t)$.

**Stochastic properties of the input signal**

Assume that the input signal consists of a realization of the process $x$, given by:

$$
x(t) = A \sin (\omega t + \varphi)
$$  \hspace{1cm} (6-42)

This signal is periodical with mean zero. Normally we need all realizations of a process to determine the first, second and higher moments. There is yet a class of stochastic processes, characterized by the fact that all statistical properties of the process can be deduced from one realization. This is the class of the **ergodic** processes. The process $x$ is an example of an ergodic process. We can therefore determine all moments of the process $x$ using the realization given by (6-42). The time derivative of the input signal (6-42) equals:

$$
x(t) = A\omega \cos (\omega t + \varphi)
$$  \hspace{1cm} (6-43)

The process $\dot{x}$ is ergodic, and we can write:
\[\sigma^2_x = E\{\dot{x}^2(t)\} = \frac{1}{T} \int_{-T/2}^{T/2} \dot{x}^2(t) dt\]  \hspace{1cm} (6-44)

Using (6-43), we get:

\[
\sigma^2_x = \frac{A^2 \omega^2}{T} \int_{-T/2}^{T/2} \cos(\omega t + \varphi)^2 dt = \frac{A^2 \omega^2}{2T} \left( \int_{-T/2}^{T/2} dt + \int_{-T/2}^{T/2} \cos(2\omega t - 2\varphi) dt \right) = \frac{A^2 \omega^2}{2} + \\
\frac{A^2 \omega}{4T} \cdot \sin(2\omega t + 2\varphi) \bigg|_{-T/2}^{T/2} = \frac{A^2 \omega^2}{2} + \frac{A^2 \omega}{4T} \cdot [\sin(\omega T + 2\varphi). \sin(-\omega T + 2\varphi)]
\]

(6-45)

Because \(\omega = 2\pi f = \frac{2\pi}{T}\), we obtain:

\[
\sigma^2_x = \frac{A^2 \omega^2}{2} + \frac{A^2 \omega}{4T} \cdot [\sin(2\pi + 2\varphi). \sin(-2\pi + 2\varphi)] = \frac{A^2 \omega^2}{2}
\]

(6-46)

An estimate of the absolute value of the noise power can be obtained when we use the first-order approximation of (6-11). We assume that the amplitude of the input sinewave is 1[Volt] and that the frequency is 20 [kHz]. The noise power now becomes (second moment of the first-order amplitude error), using (6-36) and (6-46):

\[
E \left\{ |x(t) - x(t + \Delta t\hat{t})|^2 \right\} = E \left\{ |\dot{x}(t). \Delta t\hat{t}|^2 \right\} = \sigma^2_x. \sigma^2_{\Delta t} = \frac{1.65A^2 \omega^2 T^2_{in}}{2}
\]

\[= 4.09 \times 10^{-4} [Volt]^2\]  \hspace{1cm} (6-47)

This corresponds to a noise power of -33.9 [dB]. The effective amplitude of the input sinewave amounts \(\frac{1}{\sqrt{2}} [volt]\), the signal power is the square of this value and amounts 0.5 [volt^2], which corresponds to -3[dB]. The noise power is, when compared to the input signal power, 30.9[dB] down for the worst case situation.
6.4 Overview of this chapter

In this chapter, it is observed that the digital sample-rate converter manifests itself as a jitter generator. The output samples are placed on the output time grid with a variable time delay. The time jitter signal has been constructed by applying the conversion control algorithm.

The timing jitter causes amplitude deviations in the output signal. In this chapter, a first-order approximation of this amplitude error has been given based on a stochastic approach. It has shown that for the worst case situation, this first-order model is accurate enough to predict amplitude deviations as well as noise power introduced by the sample-rate converter (worst case relative error of 0.032%, worst case noise power 30.9 [dB] below signal power). The distribution function of the output repetition sequences of the sigma-delta converter resembles that of the Gaussian distribution.

In the next chapter, we will investigate the relation between the power spectral densities of the input signal, the output signal and the sigma-delta control signal. We will use the first-order model developed in this chapter.