CHAPTER-I

INTRODUCTION AND PRELIMINARIES

1.1 Background of composition multiplication, differentiation and weighted composition operators

Operator theory includes the study of operators arising in various branches of physics and mechanics as well as other parts of mathematics and indeed is sufficiently well developed to have a logic of its own.

Let \( D \) be the open unit disk in the complex plane \( \mathbb{C} \), \( \partial D \) its boundary and \( H(D) \) the class of all holomorphic functions on \( D \) and \( \varphi \) a non-constant analytic self-map (a so called Schur function) of \( D \). The composition operator \( C_\varphi \) induced by \( \varphi \) is defined by

\[
C_\varphi f = f \circ \varphi \quad \text{for} \quad f \in H(D).
\]

Composition operators first appeared implicitly in works dealing with theoretical mechanics published in the nineteen thirties by B. O. Koopman [Ko 31]. In physics and especially the area of dynamical systems, the composition operator is usually referred to as the Koopman operator. However they began being called composition operators and being studied explicitly in the late sixties [No 68]. Composition operators occur naturally in variety of problems, for example they arise in the study of commutants of multiplication operators and more general operators and play a vital role in the theory of dynamical systems, partial differential equations and semigroups.

For \( \psi \in H(D) \), the multiplication operator \( M_\psi \) is defined on \( H(D) \) by

\[
M_\psi f(z) = \psi(z)f(z), \quad z \in D.
\]

The product of these two operators \( W_{\varphi,\psi} = M_\psi C_\varphi \), so called weighted composition operators. Composition and weighted composition operators have gained increasing attention during the last three decades, mainly due to the fact that they provide, just
as, for example, Hankel and Toeplitz operators, ways and means to link classical function theory to functional analysis and operator theory. In fact, some of the well known conjectures can be linked to composition operators. Nordgren, Rosenthal and Wintrobe [NRW 87] have shown that the invariant subspace problem can be solved by classifying certain minimal invariant subspaces of certain composition operators on $H^2$, whereas, Louis de Branges used composition operators to prove the Bieberbach conjecture. The general idea in this endeavour has been to relate the function-theoretic properties of $\varphi$ to the behaviour of $C_\varphi$. Typically one restricts $C_\varphi$ to a given Banach space of analytic or harmonic functions and seeks to characterize its operator-theoretic properties such as boundedness, compactness, order boundedness, adjoint, spectra, etc.

Composition operators have been studied extensively on the Hardy spaces of the unit disk. We first recall, Hardy spaces and brief history of the composition operators on these spaces. In complex analysis, the Hardy spaces (or Hardy classes) $H^p$ are certain spaces of holomorphic functions on the unit disk or the upper half plane. They were introduced by Frigyes Riesz [Ri 23], who named them after G. H. Hardy, because of the paper [Ha 15]. In real analysis, Hardy spaces are certain spaces of distributions on the real line, which are (in the sense of distributions) boundary values of the holomorphic functions of the complex Hardy spaces are related to the $L^p$ spaces of functional analysis.

Recall that, for $0 < p < \infty$, the Hardy space $H^p$ is the space of all analytic functions $f$ on the open unit disk $\mathbb{D}$ for which

$$||f||_{H^p} = \lim_{r \to 1} \left\{ \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \right\}^{1/p} < \infty.$$  

For $1 \leq p < \infty$, $H^p$ becomes a Banach space under the norm $|| \cdot ||_{H^p}$, whereas for $p = \infty$, $H^\infty$ is a Banach algebra under the supremum norm

$$||f||_{\infty} = \sup_{z \in \mathbb{D}} |f(z)|.$$

For every $f \in H^p$ ($1 \leq p < \infty$), the radial limit

$$f^*(e^{i\theta}) = \lim_{r \to 1} f(re^{i\theta})$$
exists almost everywhere and \( f^* \in L^p \), the Banach space of all complex-valued measurable functions \( f \) on \( \partial \mathbb{D} \) such that
\[
\|f\|_{L^p} = \left( \frac{1}{2\pi} \int_0^{2\pi} |f(e^{i\theta})|^p \, d\theta \right)^{1/p} < \infty.
\]
Furthermore, for \( p = 2 \), \( H^2 \) is a Hilbert space under the inner product
\[
\langle f, g \rangle = \frac{1}{2\pi} \int_0^{2\pi} f^*(e^{i\theta}) g^*(e^{i\theta}) \, d\theta = \sum_{n=0}^{\infty} a_n \overline{b_n},
\]
where \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) and \( g(z) = \sum_{n=0}^{\infty} b_n z^n \) are Taylor's series expansions in the unit disk of \( f \) and \( g \), respectively. It also has a reproducing kernel
\[
k_z(w) = \frac{1}{1 - \overline{z} w}; \quad z, w \in \mathbb{D},
\]
called the Szegő (Riesz) kernel.

For detailed study of Hardy spaces, see [Ho 62], [Du 70], [Ga 81] and [Ru 87].

The study of holomorphic composition operators, from an operator-theoretic viewpoint, does not date back as far as the study of Toeplitz operators or the study of Hankel operators. A systematic study of composition operators began in the late sixties and it has subsequently evolved into a massive amount of research literature. One might also be interested in describing the structure of the whole set of composition operators acting on a given function space. As general references in the field, we mention the monographs [CoM 95] and [Sh 93a]. In operator theory, the normal operators are the only class of operators on infinite dimensional spaces that can be said to be well understood at this time, although great strides have been made in the past decade in the understanding of subnormal operators and operators whose spectrum is a spectral set. Composition operators generally do not fit into these classes. Although composition operators have been studied on a variety of spaces, the majority of the literature concerns spaces whose functions are analytic on some set and in which the vector and norm structures are closely connected to the analytic structure. Moreover, such spaces have played an important role in the development of analysis and composition operators on these spaces can be studied.
without additional special hypotheses. People used results from classical function theory such as Carleson type inequalities, Julia-Caratheodory Theorem, Denjoy-Wolff Theorem, Nevanlinna Counting function and Schroeder functional equation: $\sigma \circ \varphi = \lambda \sigma$ to characterize operator theoretic properties of these operators. The class of $\mathcal{H}^p$ spaces and especially the Hilbert space $\mathcal{H}^2$, is the most classical setting for studying composition operators. It is a consequence of Littlewood’s subordination principle (see e.g. [CoM 95] and [Du 70]) that every composition operator $C_\varphi$ restricts to a bounded operator on $\mathcal{H}^p$.

A classical result, the subordination principle of J. E. Littlewood [Lit 25], guarantees that $C_\varphi$ acts boundedly on Hardy spaces $\mathcal{H}^p$ and on Bergman spaces $\mathcal{A}^p$. J. V. Ryff [Ry 66], was the first mathematician who initiated the study of composition operators on Hardy spaces although in a different terminology. He actually exploited the idea of Littlewood [Lit 25], who employed the term subordinate to deal with such type of operators and established that if $f$ is subordinate to $F$, then

$$\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \leq \frac{1}{2\pi} \int_0^{2\pi} |F(re^{i\theta})|^p d\theta, \quad 0 < r < 1.$$ 

If $f$ and $F$ are analytic in $\mathbb{D}$ and if there exists a function $\varphi$ satisfying Schwarz’s Lemma such that $f(z) = F(\varphi(z))$, then $f$ is subordinate to $F$. Ryff in his paper [Ry 66] proved that if $f \in \mathcal{H}^p$ and $\varphi$ is analytic self-map of $\mathbb{D}$, then $f \circ \varphi \in \mathcal{H}^p$ and

$$\|f \circ \varphi\|_{\mathcal{H}^p} \leq \left(\frac{1 + |\varphi(0)|}{1 - |\varphi(0)|}\right)^{1/p} \|f\|_{\mathcal{H}^p}.$$

In 1968, Nordgen initiated the study of spectra of composition operators on $\mathcal{H}^p$ [No 68]. His work was carried on by Caughran and Schwartz [CaSc 70], Shapiro and Taylor [ShT 73], Kamowitz [Ka 75], C. C. Cowen [Co 83] and [Co 88b], B. D. MacCluer [Ma 84] and [Ma 84a] and Shapiro [Sh 87]. The study of compact composition operators on $\mathcal{H}^p$, was first appeared in H. J. Schwartz [Sc 69] thesis in the late sixties. He calculated several upper and lower estimates for the norm of $C_\varphi$. By using orthonormal vectors $e_n$, defined by $e_n(z) = z^n$ for $n \in \mathbb{N}$, he was able to give a characterization for a bounded linear operator on $\mathcal{H}^p$ to be a composition operator. In fact, he proved that a non-zero operator $A$ on $\mathcal{H}^p$ is a composition operator if and only if $Ae_n = (Ae_1)^n$, $n = 0, 1, 2, \ldots$. He also observed that $C_\varphi$ is invertible if and only if $\varphi$ is a conformal automorphism.
of the unit disk. Normal and compact composition operators have been characterized. He proved that, if $C_\varphi$ is compact, then $|\varphi^*| < 1$ a.e. on the unit circle, where $\varphi^*$ is the radial limit. In other words, $C_\varphi$ is not compact whenever the set $\{|\varphi^*| = 1\}$ has positive measure. Schwartz also showed this condition is not sufficient, by showing the composition operator induced by

$$\varphi(z) = \frac{1+z}{2}$$

is not compact, even though $\varphi$ maps only a single point of the unit circle onto the unit circle, $\varphi(1) = 1$. Shapiro and Taylor began studying compact composition operators in [ShT 73], where among other results, the authors proved that if $C_\varphi$ is compact on one of the $H^p$ spaces, then it is compact on every $H^p$. They also obtained a necessary condition for $C_\varphi$ to be compact on $H^p$, namely, that $\varphi$ not possess an angular derivative in the sense of Carathéodory at any point of $\partial\mathbb{D}$. B. D. MacCluer and Shapiro [MaS 86], showed that the non-existence of the angular derivative is not sufficient for the compactness of $C_\varphi$ on the $H^p$ spaces, although the non-existence of the angular derivative is both necessary and sufficient in the Bergman spaces $A^p$. They also showed that, if $\varphi$ is univalent [MaS 86] then the angular derivative condition does imply $C_\varphi$ acts compactly on the $H^p$ spaces. They also provide several necessary and sufficient conditions for the compactness of $C_\varphi$ on $H^p$. They noted that the condition

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{1}{1 - |\varphi^*(e^{i\theta})|} d\theta < \infty,$$

established by Schwartz [Sc 69] as a sufficient condition for compactness of $C_\varphi$, turns out to be a necessary and sufficient condition for $C_\varphi$ to be a Hilbert-Schmidt composition operator on $H^2$. The connection with Hilbert-Schmidt operators and the fact that $C_\varphi$ is Hilbert-Schmidt whenever $\varphi(\mathbb{D})$ lie in an inscribed polygon comes from Shapiro and Taylor [ShT 73], where it is shown that such operators actually lie in every Schatten $p$-class. Examples of compact composition operators that were not Hilbert-Schmidt were also introduced in [ShT 73]. Cowen [Co 83] and [Co 88b] explored various properties of composition operators on Hardy spaces. He was able to compute essential norm estimate and essential spectra of certain nice composition operators on $H^p$. MacCluer brought Carleson measure conditions to study composition operators on $H^p(B_N)$ [Ma 85b].
Shapiro [Sh 87] made use of the Nevanlinna counting function of \( \varphi \). The roots of the counting function lie in Rolf Nevanlinnas renowned theory of value distribution for entire and meromorphic functions (see [Ne 70]). The Nevanlinna counting function, is a device from Nevanlinna theory that gives a biased measure of the number of times \( \varphi \) assumes a given value \( w \). The Nevanlinna counting function of \( \varphi \) is defined by

\[
N_{\varphi}(w) = \sum_{z \in \varphi^{-1}(w)} \log \frac{1}{|z|} \quad \text{for} \quad w \in \mathbb{D} \setminus \{\varphi(0)\}.
\]

In the sum on the right side, the points \( z \) in \( \varphi^{-1}(w) \) are counted with multiplicities and the sum is interpreted to be 0 if \( w \) is not in \( \varphi(\mathbb{D}) \). The convergence of the series, in case \( \varphi^{-1}(w) \) is infinite, follows by the Blaschke condition (assuming \( \varphi \) a non constant function). The number \( N_{\varphi}(w) \) should be viewed as a measure of the affinity that \( \varphi \) has for the value \( w \). It weights each pre-image \( z \) by the product of its multiplicity and logarithmic distance \(-\log |z| \) from the boundary. So pre-images that are located deep inside \( \mathbb{D} \) count more than those near \( \partial \mathbb{D} \). Since we are mainly interested in values close to the boundary (whose pre-images also lie near \( \partial \mathbb{D} \) by the Schwarz lemma), it is reasonable to think of \(-\log |z| \) as approximately the Euclidean distance \( 1 - |z| \).

Before we state Shapiro's characterization we recall that, the essential norm \( ||T||_e \) of a bounded linear operator \( T \) on a Banach space \( X \) is given by

\[
||T||_e = \inf \{ ||T + K||_X : K \text{ is compact on } X \},
\]

that is, its distance in the operator norm from the space of compact operators on \( X \). The essential norm provides a measure of non-compactness of \( T \). Clearly, \( T \) is compact if and only if \( ||T||_e = 0 \). For some results in the topic see, e.g. [CoM 95], [CuZ 04], [GLS 11], [Rod 00], [Sh 87], [StS 11b], [Stev 09d], [Stev 10g], [SSB 11a], [Ue 08] and the related references therein.

Shapiro's necessary and sufficient condition is roughly, that \( \varphi \) not take too many values near \( \mathbb{D} \) too often and is quantititized by means of the Nevanlinna counting function. In fact, he [Sh 87] obtained a relation between the essential norm of \( C_{\varphi} \) on \( H^2 \) and a quantity involving the Nevanlinna counting function of the inducing map. More precisely,

\[
||C_{\varphi}||_e^2 = \limsup_{|w| \to 1} \frac{N_{\varphi}(w)}{\log \frac{1}{|w|}} = \limsup_{|\varphi(z)| \to 1} \frac{N_{\varphi}(\varphi(z))}{\log \frac{1}{|\varphi(z)|}}.
\]
In particular, he proved that $C_\varphi$ is compact on $H^2$ if and only if

$$N_\varphi(w) = o\left(\frac{1}{\log|w|}\right) \quad \text{as} \quad |w| \to 1,$$  \hfill (1.1.1)

thus providing a complete function theoretic characterization of compact composition operators in terms of inducing map’s Nevanlinna counting function $N_\varphi$. Shapiro’s characterization (1.1.1) has inspired much additional interesting work. To mention just two results, D. H. Luecking and K. Zhu [LuZ 92] have characterized the function $\varphi$ for which $C_\varphi$ belongs to one of the Schatten classes as an operator on $H^2$ or on $A^2$, and P. Poggi-Corradini [Co 97] has characterized those univalent $\varphi$ for which $C_\varphi$ is a Riesz operator (an operator with essential spectral radius 0) on $H^2$. Shapiro’s book [Sh 93a] is a delightful elementary account of this work and related issues.

$C_\varphi$ as an operator on the Nevanlinna class $N$ was first studied by Masri [Mas 85] in his thesis, where he obtained several necessary conditions and sufficient conditions on $\varphi$ for $C_\varphi$ to be compact. The compactness of $C_\varphi$ as an operator on the spaces $N$ and $N^p$ was also studied by Choa and Kim in [ChK 97] and [ChK 01a]. The theory of vector-valued functions is discussed in Helson [Hel 64]. For detailed account of vector-valued Hardy and Nevanlinna classes, one can refer to [He 69], [RoR 71] and [RoR 85a]. In 1984, Cowen [Co 84a] studied the connection between the Cesaro operator and certain semigroups of composition operators. Berkson and Porta [BeP 78] initiated a different line of investigation into semigroups of composition operators, emphasizing function theoretic properties of the infinitesimal generators of these semigroups and their work was carried forward by Siskakis [Si 87a, Si 87b] and Aleman [Al 90].

Motivated by his work with Porta on semigroups of composition operators, Berkson proved that each composition operator induced by an inner function is isolated in the operator norm topology from every other composition operator [Berk 81]. Shapiro and Sundberg [ShSu 90], related isolation phenomena with extreme points of $H^\infty$, Hunziker, Jarchow and Mascioni [HJM 90] have also studied similar problems for the class of Hilbert-Schmidt composition operators in the topology induced by the Hilbert-Schmidt norm.
In a more fruitful direction, Banach space theorists have identified various subclasses of compact operators that generalize classes like the Hilbert-Schmidt operators on Hilbert spaces. Jarchow and Hunziker have done interesting work on the question of which composition operators on $H^p$ or between $H^p$ spaces belong to such classes, see [Hu 89], [HuJ 91], [Ja 92], [Ja 93a] and [Ja 94b].

Nordgren, Rosenthal and Wintrobe [NRW 87] studied the algebras generated by various classes of composition operators. Guyker [Gu 89] classified the reducing subspaces of certain composition operators. C. C. Cowen and MacCluer [CoM 95] have written a comprehensive treatment of holomorphic composition operators. Composition operators played a key role in L. de Branges’s renowned proof of the Bieberbach-Robertson-Milin conjectures [Br 85] and [Br 87a].

The theory of multiplication operators has extensively been studied during the last three decades or so on different spaces of functions. It is evident from the literature that multiplication operators are appearing in different areas of mathematical sciences like dynamical systems and theory of semigroups (e.g. see [Ax 85a], [BrS 91] and isometries (e.g. see [At 84], [Ax 82] and [HoJ 01])) besides their role in the theory of operator algebras and operator spaces (e.g. see [Si 86c]). Evard and Jafari [EvJ 95] and Siskakis ([Si 86c] and [Si 98d]) have employed these operators to make a study of weighted composition semigroups and dynamical systems on Hardy Spaces. De Leeuw et al. [LRW 75] and Nagasawa [Na 59] have described isometries of Hardy spaces $H^1(\mathbb{D})$ and $H^\infty(\mathbb{D})$ as a product of multiplication operators and composition operators. Isometries on $H^p$-spaces and Bergman spaces are very much related with multiplication operators and composition operators and for details on these isometries, we refer to Forelli [Fo 64], Cambern and Jarosz [CaJ 89], Kolaski [Kol 89], Mazur [Maz 89] and Lin [Li 90]. In [Arv 98], Arveson has recently obtained Toeplitz $C^*$-algebras and operator spaces associated with these multiplication operators on Hardy Spaces.

In recent years, many authors like Attele [At 84], Axler ([Ax 82], [Ax 85a], [Ax 86b] and [AxS 85]), Bercovici [Ber 87], Eschmeier [Es 90], Luecking [Lu 86], Vukotic [Vu 99] and Zhu [Zhu 01b] have made a study of multiplication operators on Bergman spaces,
whereas Campbell and Leach [CaL 84], Feldman [Fe 99], Lin [Li 90], Arveson [Arv 98]
and Ohno and Takagi [OhT 01] have obtained a study of these operators on Hardy
spaces. On Bloch spaces, these operators are studied by Arazy [Ar 82], Axler [Ax 86b]
and Brown and Shields [BrS 91]. Also, Axler and Shields [AxS 85] and Stegenga [Ste 80]
have explored multiplication operators on Dirichlet spaces. On BMOA, these operators
are studied by Ortega and Fabregá [OrF 96]. Further, on Nevanlinna classes of analytic
functions, these operators are studied by Jarchow et al. [JMWX 01] and Yanagihara [Ya
73]. Besides these well-known analytic function spaces, a study of these operators on
some other Banach spaces of analytic functions has also been pursued by Bonet et al.
([BDL 99], [BDL 99a], [BDLT 98] and [BDL 01b]), Contreras and Hernández-Díaz [CoH
00], Ohno and Takagi [OhT 01] and Shields and Williams ([ShW 71] and [ShW 78a]). In
[CoH 01a], Contreras and Hernández-Díaz have made a study of weighted composition
operators on Hardy spaces, whereas Mirzakarimi and Siddighi [MiS 97] have considered
these operators on Bergman and Dirichlet spaces. On Bloch and Bloch-type spaces,
these operators are studied by MacCluer and Zhao [MaZ 03a], Ohno [Oh 01], Ohno and
Zhao [OhZ 01] and Ohno et al. [OSZ 03]. In [OhT 01], Ohno and Takagi have obtained
some properties of these operators on the disk algebra and the Hardy space $H^\infty(\mathbb{D})$.
Also recently, Montes-Rodríguez [Rod 00] and Contreras and Hernández-Díaz [CoH 00]
have studied the behaviour of these operators on weighted Banach spaces of analytic
functions. The applications of these operators can be found in the theory of semigroups
and dynamical systems (see [At 84], [Si 98d] and [JTTY 97]).

The classes of multiplication operators and composition operators are playing impor-
tant role in the study of weighted composition operators. Weighted composition opera-
tors appeared in about the same period and a first result supporting their importance is
the fact that Hardy space isometries are necessarily weighted composition operators [Fo
64], (if $p \neq 2$). Studying weighted composition operators between spaces of holomor-
phic functions attracted attention recently by number of authors. Weighted composition
operators appear naturally in different contexts. For example, Singh and Sharma [SiS
79] related the boundedness of composition operators on Hardy space of the upper half-
plane with the boundedness of weighted composition operators on the Hardy space of
the open unit disk $\mathbb{D}$. Weighted composition operators have also played an important role
in the study of compact composition operators on Hardy spaces and Bergman spaces of unbounded domains (see for example [ShS 03] and [Mat 99] for more details). Isometries in many Banach spaces of analytic functions are just weighted composition operators, for example see [Fo 64] and [HoJ 01]. Recently, several authors have studied weighted composition operators on different spaces of analytic functions. For example, one can refer to [At 92a], [CoH 01a] and [CoH 04b] for study of these operators on Hardy spaces [Ka 79a] and [OhT 01] for disk algebra, [OSZ 03] and [OhZ 01] for Bloch-type spaces and [CuZ 04] and [Mat 99] for weighted Bergman spaces.

The study of the weighted composition operators on the Bloch space began with the work of Ohno and Zhao in [OhZ 01], where the operators from the Bloch space of $\mathbb{D}$ into itself were considered. Characterizations of the boundedness and the compactness of the weighted composition operators from the Bloch space to $H^{\infty}$ were given by Hosokawa, Izuchi and Ohno in [HIO 05] in the one-dimensional case and by Li and Stević in the case of the unit ball [LiS 08d]. The study of the weighted composition operators from $H^{\infty}$ to the Bloch space in several variables was carried out by Li and Stević in [LiS 07]. For the case of the unit ball, the study of these operators from $H^{\infty}$ into a larger class of spaces known as the $\alpha$-Bloch spaces was carried out by Li and Stević in [LiS 08d] and Zhang and Chen in [ZhC 09].

The $n$th order differentiation operator $D^n$ is defined by

$$D^n f = f^{(n)}, \quad f \in H(\mathbb{D}).$$

Hibschweiler and Portnoy [HiP 05] considered the behavior of the differentiation on the range of the composition operator on the Hardy or weighted Bergman spaces. The products of composition and differentiation operators are defined by

$$DC_\varphi f(z) = (C_\varphi f)'(z) = f'(\varphi(z))\varphi'(z)$$

and

$$C_\varphi D f(z) = (C_\varphi f)'(z) = f'(\varphi(z))$$

for $z \in \mathbb{D}$ and $f \in H(\mathbb{D})$. Generally $D$ is unbounded on analytic function spaces. Hibschweiler and Portnoy [HiP 05] characterized the boundedness and the compactness of
The products of composition, multiplication and differentiation operators can be defined in the following six ways as

\[(M_\psi C_\varphi D f)(z) = \psi(z) f(\varphi(z)),\]
\[(M_\psi DC_\varphi f)(z) = \psi(z) \varphi'(z) f'(\varphi(z)),\]
\[(C_\varphi M_\psi D f)(z) = \psi(\varphi(z)) f'(\varphi(z)),\]
\[(DM_\psi C_\varphi f)(z) = \psi'(z) f(\varphi(z)) + \psi(z) \varphi'(z) f'(\varphi(z)),\]
\[(C_\varphi DM_\psi f)(z) = \psi'(\varphi(z)) f(\varphi(z)) + \psi(\varphi(z)) f'(\varphi(z)),\]
\[(DC_\varphi M_\psi f)(z) = \psi'(\varphi(z)) f(\varphi(z)) \varphi'(z) + \psi(\varphi(z)) f'(\varphi(z)) \varphi'(z),\]

for \(z \in \mathbb{D}\) and \(f \in H(\mathbb{D})\).

Note that the operator \(M_\psi C_\varphi D\) induces many known operators. If \(\psi(z) = 1\), then \(M_\psi C_\varphi D = C_\varphi D\), while when \(\psi(z) = \varphi'(z)\), then we get the operator \(DC_\varphi\). These two operators have been studied in ([LiS 07a], [LiS 08b], [LiS 09g], [LiS 10h], [Oh 06a], [Sha 11b], [Stev 09e] and [Stev 09f]). If we put \(\varphi(z) = z\), then \(M_\psi C_\varphi D = M_\psi D\), that is product of differentiation operator and multiplication operator. Also note that \(M_\psi DC_\varphi = M_\psi \varphi' C_\varphi D\) and \(C_\varphi M_\psi D = M_\psi \varphi C_\varphi D\). Thus the corresponding characterizations of boundedness and compactness of \(M_\psi DC_\varphi\) and \(C_\varphi M_\psi D\) can be obtained by replacing \(\psi\), respectively by \(\psi \varphi'\) and \(\psi \circ \varphi\) in the results stated for \(M_\psi C_\varphi D\). In order to treat these operators between spaces of holomorphic functions in a unified manner, we consider the operator \(T_{\psi_1,\psi_2,\varphi}\) defined as

\[T_{\psi_1,\psi_2,\varphi} f(z) = \psi_1(z) f(\varphi(z)) + \psi_2(z) f'(\varphi(z)), \quad f \in H(\mathbb{D}),\]

where \(\psi_1, \psi_2\) and \(\varphi\) are holomorphic maps on \(\mathbb{D}\) such that \(\varphi(\mathbb{D}) \subset \mathbb{D}\). It is clear that all possible products of composition, multiplication and differentiation operators can be obtained from the operator \(T_{\psi_1,\psi_2,\varphi}\) by fixing \(\psi_1\) and \(\psi_2\). More specifically we have

\[M_\psi C_\varphi D = T_{0,\psi_1,\varphi}, \quad M_\psi DC_\varphi = T_{0,\psi_\varphi',\varphi}, \quad C_\varphi M_\psi D = T_{0,\psi \circ \varphi',\varphi},\]
\[ \text{DM}_\psi C_\varphi = T_{\psi', \psi \varphi', \varphi}, \quad C_\varphi \text{DM}_\psi = T_{\psi', \psi \varphi, \varphi}, \quad \text{DC}_\varphi M_\psi = T_{(\psi' \circ \varphi), \psi' \varphi, \varphi}. \]

1.2. Basic notations and Geometric Function Theory of some spaces of holomorphic functions.

Throughout this thesis, letters \( \mathbb{N} \), \( \mathbb{Z} \), \( \mathbb{R} \) and \( \mathbb{C} \) stand for the set of natural numbers, the set of integers, the real numbers and the set of complex numbers, respectively. The constants are denoted by \( C \), they are positive and not necessarily the same at each occurrence.

The notation
\[ A \asymp B \]
means that there is a positive constant \( C \) such that
\[ B/C \leq A \leq CB. \]

The notation
\[ a \lesssim b \]
means that \( a \leq C b \).

1.2.1. Automorphisms of the disk:

Most general conformal automorphism of the unit disk \( \mathbb{D} \) onto itself is a Möbius map of the form
\[ z \mapsto \frac{wz + \bar{c}}{cz + \bar{w}}, \quad w, c \in \mathbb{C}, \quad |w|^2 - |c|^2 = 1, \]
or of the equivalent form
\[ z \mapsto e^{i \theta} \beta_z(w), \quad \theta \in \mathbb{R}, \quad w \in \mathbb{D}, \quad \text{where} \quad \beta_z(w) = \frac{z - w}{1 - \overline{z}w}. \]

It is well known that these maps form a group \( \text{Aut}(\mathbb{D}) \) under composition and that \( \text{Aut}(\mathbb{D}) \) acts transitively on \( \mathbb{D} \) (that is, for all \( z \) and \( w \) in \( \mathbb{D} \) there is some \( g \) in \( \text{Aut}(\mathbb{D}) \)
such that \( g(z) = w \). Also \( \text{Aut}(\mathbb{D}, 0) \), the subgroup of conformal automorphisms that fix the origin, is the set of rotations of the complex plane about the origin. By direct verification,

\[
|\beta_{\lambda}(z) - \beta_{\lambda}(w)| = \frac{(1 - |\lambda|^2)|z - w|}{|1 - \lambda z||1 - \lambda w|}
\]

and

\[
|\beta'_{\lambda}(w)| = \frac{1 - |\lambda|^2}{|1 - \lambda w|^2} = \frac{1 - |\beta_{\lambda}(w)|^2}{1 - |w|^2}.
\]

It is well known that inverse of \( \beta_z \) under composition is \( \beta_z \), that is, \((\beta_z \circ \beta_z)(w) = w\) for \( w \in \mathbb{D} \).

The hyperbolic plane is the unit disk \( \mathbb{D} \) with the hyperbolic metric

\[
\lambda_{\mathbb{D}}(z)|dz| = \frac{2|dz|}{1 - |z|^2}.
\]

This metric induces a hyperbolic distance \( d_{\mathbb{D}}(z, w) \) between two points \( z \) and \( w \) in \( \mathbb{D} \) in the following way. We join \( z \) to \( w \) by a smooth curve \( \gamma \) in \( \mathbb{D} \) and define the hyperbolic length \( l_{\mathbb{D}}(\gamma) \) of \( \gamma \) by

\[
l_{\mathbb{D}}(\gamma) = \int_{\gamma} \lambda_{\mathbb{D}}(z)|dz|.
\]

Finally, we set

\[
d_{\mathbb{D}}(z, w) = \inf_{\gamma} l_{\mathbb{D}}(\gamma)
\]

where the infimum is taken over all smooth curves \( \gamma \) joining \( z \) to \( w \) in \( \mathbb{D} \). It is immediate from the construction of \( d_{\mathbb{D}} \), that it satisfies the requirements for a distance on \( \mathbb{D} \), namely

(a) \( d_{\mathbb{D}}(z, w) \geq 0 \), with equality if and only if \( z = w \).

(b) \( d_{\mathbb{D}}(z, w) = d_{\mathbb{D}}(w, z) \).

(c) for all \( u, v, w \in \mathbb{D} \), \( d_{\mathbb{D}}(u, w) \leq d_{\mathbb{D}}(u, v) + d_{\mathbb{D}}(v, w) \).

The hyperbolic area of a Borel measurable subset of \( \mathbb{D} \) is

\[
a_{\mathbb{D}}(E) = \int_E \lambda_{\mathbb{D}}^2(z)dxdy.
\]
A holomorphic self-map $f$ of $\mathbb{D}$ is an isometry of the metric $\lambda_\mathbb{D}(z)|dz|$ if for all $z$ in $\mathbb{D}$,

$$\lambda_\mathbb{D}(f(z))|f'(z)| = \lambda_\mathbb{D}(z)$$

and it is an isometry of the distance $d_\mathbb{D}$ if, for all $z$ and $w$ in $\mathbb{D}$,

$$d_\mathbb{D}(f(z), f(w)) = d_\mathbb{D}(z, w).$$

In fact, the two classes of isometries coincide and each isometry is a Möbius transformation of $\mathbb{D}$ onto itself.

**Theorem 1.2.2.** [BeM 05] For any holomorphic self-map $f$ of $\mathbb{D}$ the following are equivalent:

1. $f$ is a conformal automorphism of $\mathbb{D}$.
2. $f$ is an isometry of the metric $\lambda_\mathbb{D}$.
3. $f$ is an isometry of the distance $d_\mathbb{D}$.

1.2.3. Linear fractional self-maps of $\mathbb{D}$:

A holomorphic map $\eta$ of $\mathbb{D}$ is a linear fractional map if it has the form

$$\eta(z) = \frac{az + b}{cz + d}$$

for complex numbers $a, b, c, d$ such that $ad - bc \neq 0$. It can be easily shown that linear fractional map $\eta$ defined above is the self-map of $\mathbb{D}$ if and only if

$$|b\overline{d} - a\overline{c}| + |ad - bc| \leq |d|^2 - |c|^2. \quad (1.2.1)$$

The following important lemma due to Cowen [Co 88b] played an important role in the computation of adjoints of linear fractional map $C_\eta$ in $\mathcal{H}^2$, $\mathcal{A}_\alpha^2$ and $\mathcal{D}$.

**Lemma 1.2.4.** Let $\eta(z) = \frac{az + b}{cz + d}$ be a linear fractional map. Then $\eta$ is a self-map of the disk if and only if the linear fractional transformation

$$\eta^*(z) = \frac{1}{\eta^{-1}(1/z)} = \frac{\overline{az - c}}{-\overline{bz + d}} \quad (1.2.2)$$
is also a self-map of the disk.

A proof of above lemma can be found in [MaV 06].

1.2.5. The Pseudo-hyperbolic Metric:

For \( z, w \) in \( \mathbb{D} \), the pseudo-hyperbolic distance is given by

\[
\rho(z, w) = |\beta_z(w)| = \left| \frac{z - w}{1 - \overline{z}w} \right|, \quad z, w \in \mathbb{D}.
\]

Most important property of the pseudo-hyperbolic distance is that, it is Möbius invariant, that is,

\[
\rho(\beta(z), \beta(w)) = \rho(z, w)
\]

for all \( \beta \in \text{Aut}(\mathbb{D}) \), the Möbius group of \( \mathbb{D} \) and all \( z, w \in \mathbb{D} \). For any \( z \in \mathbb{D} \) and \( r \in (0, 1) \) let

\[
B(z, r) = \{ w \in \mathbb{D} : \rho(z, w) < r \}
\]

be the pseudo-hyperbolic disk with center \( z \) and radius \( r \). \( B(z, r) \) is the image of the Euclidean disk \( |\zeta| < r \) under the Möbius transformation \( w = \beta_z(\zeta) \). It follows that \( B \) is actually a Euclidean disk contained in \( \mathbb{D} \). The Euclidean center and Euclidean radius of \( B(z, r) \) are

\[
\frac{1 - r^2}{1 - r^2|z|^2} z \quad \text{and} \quad \frac{1 - |z|^2}{1 - r^2|z|^2} r
\]

respectively. See Garnett [Ga 81].

1.2.6. The Bergman Metric:

The Bergman metric on \( \mathbb{D} \) also called the hyperbolic metric or the Poincaré metric is given by

\[
\beta(z, w) = \frac{1}{2} \log \left\{ \frac{|1 - \overline{z}w| + |z - w|}{|1 - z\overline{w}| - |z - w|} \right\}.
\]

Throughout this thesis, we fix some positive radius \( 0 < r < \infty \) and consider disks \( D(z, r) \) in the Bergman metric. The set

\[
D(z, r) = \{ w \in \mathbb{D} : \beta(z, w) < r \}, \quad z \in \mathbb{D},
\]
is called hyperbolic disk or Bergman disk of radius \( r \) about \( z \). It is well known that
\( D(z, r) \) is a Euclidean disk whose Euclidean center and Euclidean radius are
\[
\frac{(1 - s^2)z}{1 - s^2|z|^2} \quad \text{and} \quad \frac{(1 - |z|^2)s}{1 - s^2|z|^2},
\]
where \( s = \tanh r \in (0, 1) \) respectively.

The next known lemmas list some additional properties of the Bergman disks. Their proofs can be found, for example, in [HKZ 00] and [Zhu 90].

**Lemma 1.2.7.** Let \( r, s \) and \( R \) be positive numbers. Then there exists a positive constant \( C \) such that for all \( z \) and \( w \) in \( \mathbb{D} \), we have

1. \( (1 - |z|^2) \approx 1 - |w|^2 \approx |1 - zw| \) when \( \beta(z, w) < r \).
2. \( m(D(w, s)) \approx m(D(z, r)) \) when \( \beta(z, w) < R \).

**Lemma 1.2.8.** Let \( r \in (0, 1] \) be fixed. Then there is a positive integer \( N \) and a sequence \( (a_n)_{n \in \mathbb{N}} \) in \( \mathbb{D} \) such that :

1. \( \mathbb{D} \) is covered by \( \{D(a_n, r)\}_{n \in \mathbb{N}} \).
2. Every point in \( \mathbb{D} \) belongs to at most \( N \) sets in \( \{D(a_n, 2r)\}_{n \in \mathbb{N}} \).
3. If \( n \neq m \), then \( \beta(a_n, a_m) \geq \frac{r}{2} \).

**1.2.9. Submean Value Property:** Let \( f \) be a holomorphic map of \( \mathbb{D} \). Then there is a constant \( C > 0 \) such that
\[
|f(z)|^p \leq \frac{C}{|D(z, r)|} \int_{D(z, r)} |f(w)|^p \, dm(w)
\]
for all \( z \in \mathbb{D} \), \( p > 0 \) and \( r \in (0, 1) \).

**1.2.10. Some special sets:**
We define some special subsets (see [CoM 95]) of $\mathbb{D}$ which will be used throughout this thesis along with the pseudo-hyperbolic disks and Bergman disks.

For $\zeta \in \partial \mathbb{D}$ and $0 < \delta < 2$, let

$$S(\zeta, \delta) = \{ z \in \mathbb{D} : |z - \zeta| < \delta \}.$$

That is, the unit disk intersected with the disk of radius $\delta$ centered at $\zeta$. The sets $S(\zeta, \delta)$ can be replaced by equivalent choices, which in specific applications, may be easier to use. In particular, associated with each $S(\zeta, \delta)$ with center $\zeta$ on $\partial \mathbb{D}$, there is a disk

$$\Delta(a, \eta) = \{ z : |z - a| < \eta(1 - |a|) \}$$

whose center $a$ lies on radius from 0 to $\zeta$. If $z \in \Delta(a, \eta)$ and $\frac{2\eta}{1 + \eta^2} \leq r < 1$, then

$$\Delta(a, \eta) \subset D(a, r).$$

Let $I \subset \partial \mathbb{D}$ be an arc of length $|I| > 0$. The Carleson box $S(I)$ based on the arc $I$ is defined as

$$S(I) = \{ z : 1 - |I| \leq |z| < 1, \ z/|z| \in I \}.$$

One sometimes uses Carleson windows defined for $\zeta \in \partial \mathbb{D}$ and $0 < \delta < 1$:

$$W(\zeta, \delta) = \{ z \in \mathbb{D} : 1 - \delta < |z| < 1 \text{ and } \frac{z}{|z|} \in S(\zeta, \delta) \}.$$

Also for each $D(z, r)$, there is a $\zeta \in \partial \mathbb{D}$ so that $D(z, r) \subset S(\zeta, \delta)$ for $\delta \approx 1 - |z|$ and for fixed $r$, $0 < r < 1$

$$(\alpha + 1) \int_{D(z, r)} (1 - |w|^2)^\alpha d\mu(w) \approx (1 - |z|^2)^{\alpha + 2}.$$

1.2.11. Carleson Measures:

Let $0 < \beta < \infty$. Recall that, a positive Borel measure $\mu$ on $\mathbb{D}$ is called a $\beta$-Carleson measure if

$$\sup_{I \subset \partial \mathbb{D}} \frac{\mu(S(I))}{|I|^\beta} < \infty. \quad (1.2.3)$$

Measure $\mu$ is a vanishing $\beta$-Carleson measure if

$$\sup_{|I| \to 0} \frac{\mu(S(I))}{|I|^\beta} = 0.$$
It is also well known that a positive Borel measure \( \mu \) on \( \mathbb{D} \) is called a \( \beta \)-Carleson measure if
\[
\|\mu\|_\beta := \sup_{z \in \mathbb{D}} \frac{\mu(D(z, r))}{(1 - |z|^2)^\beta} < \infty,
\]
while it is a vanishing \( \beta \)-Carleson measure if
\[
\lim_{|z| \to 1} \frac{\mu(D(z, r))}{(1 - |z|^2)^\beta} = 0. \tag{1.2.4}
\]

1.2.12. Littlewood’s Subordinate Theorem: If \( \varphi \) is a holomorphic self-map of \( \mathbb{D} \) such that \( \varphi(0) = 0 \) and \( G \) is a subharmonic function in \( \mathbb{D} \), then for \( 0 < r < 1 \),
\[
\int_0^{2\pi} G(\varphi(r e^{i\theta})) d\theta \leq \int_0^{2\pi} G(re^{i\theta}) d\theta.
\]

1.2.13. Schwarz-Pick Lemma: If \( \varphi \) is a holomorphic self-map of \( \mathbb{D} \), then
\[
\left| \frac{\varphi(w) - \varphi(z)}{1 - \varphi(w)\varphi(z)} \right| \leq \left| \frac{w - z}{1 - wz} \right|
\]
and if equality holds for any \( z \neq w \), then \( \varphi \) is automorphism of the disk.

1.2.14. Julia’s Lemma: \( \varphi \) is a holomorphic self-map of \( \mathbb{D} \). Suppose \( \zeta \) is in the unit circle and
\[
d_{\varphi}(\zeta) = \liminf_{z \to \zeta} \frac{1 - |\varphi(z)|}{1 - |z|}
\]
is finite where the lower limit is taken as \( z \) approaches \( \zeta \) unrestrictedly in \( \mathbb{D} \). Suppose \( \{a_n\} \) be a sequence along which this lower limit is achieved and for which \( \varphi(a_n) \) converges to \( \eta \in \partial \mathbb{D} \). Then \( |\eta| = 1 \) and for every \( z \in \mathbb{D} \)
\[
\frac{|\eta - \varphi(z)|^2}{1 - |\varphi(z)|^2} \leq d_{\varphi}(\zeta) \frac{|\zeta - z|^2}{1 - |z|^2}.
\]
Moreover, if equality holds for some \( z \in \mathbb{D} \), then \( \varphi \) is an automorphism of \( \mathbb{D} \).

The geometric interpretation of this result is that, \( \varphi \) maps each disk \( E(\zeta, k) = \{z \in \mathbb{D} : |z - \zeta| < k\} \) into the corresponding disk \( E(\zeta, kd_{\varphi}(\zeta)) \).
1.2.15. **Sector:** A sector (in \( \mathbb{D} \)) at a point \( \zeta \in \partial \mathbb{D} \) is the region between two straight lines in \( \mathbb{D} \) that meets at \( \zeta \) and are symmetric about the radius to \( \zeta \).

1.2.16. **Angular limit:** If \( f \) is a holomorphic function defined on \( \mathbb{D} \) and \( \zeta \in \partial \mathbb{D} \), then by

\[
\text{ang lim}_{z \to \zeta} f(z) = L,
\]

means that \( f(z) \to L \) as \( z \to \zeta \) through any sector at \( \zeta \). When this happens, we say that \( L \) is the angular limit of \( f \) at \( \zeta \).

1.2.17. **Angular derivative** ([Sh 93a], p. 56): A holomorphic self-map \( \varphi \) of \( \mathbb{D} \) has a finite angular derivative at a point \( \zeta \in \partial \mathbb{D} \) if there is a point \( w \in \partial \mathbb{D} \) such that the difference quotient

\[
\frac{\varphi(z) - w}{z - \zeta}
\]

has a finite limit, as \( z \) tends non-tangentially to \( \zeta \).

Note that existence of angular derivative at \( \zeta \) guarantees several things:

1. \( \varphi \) itself has a non-tangential limit of modulus 1 at \( \zeta \),
2. \( \varphi' \) also has a non-tangential limit at \( \zeta \) and
3. \( \varphi \) is in a certain sense 'conformal' at \( \zeta \).

The connection between composition operators and angular derivative is made by the following classical theorem.

1.2.18. **Julia-Caratheodory Theorem** ([Sh 93a], p. 57): Let \( \varphi \) be a holomorphic self-map and \( \zeta \in \partial \mathbb{D} \), the following statements are equivalent:

1. \( \varphi \) has finite angular derivative \( \varphi' \) at \( \zeta \).

2. Both \( \varphi \) and \( \varphi' \) have finite non-tangential limits at \( \zeta \), with \( |\eta| = 1 \) for

\[
\eta = \lim_{r \to 1} \varphi(r\zeta).
\]
(iii) \( d_\varphi(\zeta) = \liminf_{z \to \zeta} \frac{1 - |\varphi(z)|}{1 - |z|} < \infty \), where the limit is taken as \( z \) approaches \( \zeta \) unrestrictedly in \( \mathbb{D} \).

Moreover, when these conditions hold, we have

\[
\lim_{r \to 1} \varphi'(r\zeta) = \varphi'(\zeta) = d(\zeta)\zeta\eta
\]

and \( d(\zeta) \) is the non-tangential limit.

(For more information on the Julia-Caratheodory Theorem and its connection with composition operators, see Section 3 of [Sh 93a] Chapter 4.)

1.2.19. Nevanlinna Counting Function ([Sh 93a], p. 190): For a holomorphic self-map \( \varphi \) of \( \mathbb{D} \), the Nevanlinna counting function \( N_\varphi \) provides a measure of how often and where \( \varphi \) takes values near \( \partial \mathbb{D} \) is defined by:

\[
N_\varphi(w) = \sum_{z \in \varphi^{-1}(w)} \log \frac{1}{|z|}, \quad w \in \mathbb{D} \setminus \{\varphi(0)\},
\]

where \( z \in \varphi^{-1}(w) \) is repeated according to the multiplicity of zeros of the function \( \varphi_1(z) := \varphi(z) - w \).

1.2.20. Nevanlinna Counting Function \( N_{\varphi, \lambda} \) induced by \( \varphi \) and real number \( \lambda > 0 \) ([Sh 93a], p. 190): For a holomorphic self-map \( \varphi \) of \( \mathbb{D} \), the generalized Nevanlinna counting function \( N_{\varphi, \lambda} \) is defined for \( \lambda > 0 \) by:

\[
N_{\varphi, \lambda}(w) = \sum_{z \in \varphi^{-1}(w)} \left( \log \frac{1}{|z|} \right)^\lambda, \quad w \in \mathbb{D} \setminus \{\varphi(0)\}.
\]

Function \( N_{\varphi, \lambda} \) is used in the study of composition operators on weighted Bergman spaces [Sh 87]. It is well known that these counting functions, while not subharmonic themselves, satisfy a sub-mean value property [Sh 87] and [CoM 95].

1.2.21. Submean Value Property: Let \( \tau \) be a holomorphic self-map of \( \mathbb{D} \) and let \( \lambda \geq 1 \). If \( \tau(0) \neq 0 \) and \( 0 < r < |	au(0)| \), then

\[
N_{\tau, \lambda}(0) \leq \frac{1}{r^2} \int_{r\mathbb{D}} N_{\tau, \lambda}(z)dm(z).
\]
1.2.22. Littlewood’s inequality ([Sh 93a], p. 187): If $\varphi$ is a holomorphic self-map of $\mathbb{D}$, then for $z \in \mathbb{D} \setminus \{\varphi(0)\}$, we have

$$N_\varphi(z) \leq \log \left| \frac{1 - \varphi(0)z}{z - \varphi(0)} \right|.$$ 

It implies that $N_\varphi(w) = O(-\log|w|)$ and in the special case $\varphi(0) = 0$ it reduces to $N_\varphi(w) \leq -\log|w|$, which is actually an improvement of the Schwarz lemma.

Some spaces of holomorphic functions

1.2.23. Functional Hilbert space:

Let $\Omega$ be a non-empty set. A Hilbert space $(H(\Omega), \langle \cdot, \cdot \rangle)$ of complex-valued functions on $\Omega$ is called a functional Hilbert space if the following conditions are satisfied:

1. $f(z) = g(z)$ for all $z \in \Omega \Rightarrow f = g,$
2. $f(z_1) = f(z_2)$ for all $f \in H(\Omega) \Rightarrow z_1 = z_2,$
3. for each $z \in \Omega$, the evaluation map $f \rightarrow f(z)$ is continuous.

1.2.24. Weighted Bergman space:

Let

$$dm(z) = \frac{1}{\pi} dx dy = \frac{1}{\pi} r dr d\theta$$

be the normalized area measure on $\mathbb{D}$. For each $\alpha \in (-1, \infty)$, we set

$$dm_\alpha(z) = (\alpha + 1)(1 - |z|^2)^\alpha dm(z), \; z \in \mathbb{D}.$$ 

Then $dm_\alpha$ is a probability measure on $\mathbb{D}$. Let $L^p_{\alpha}$ be the weighted Lebesgue space consisting of measurable functions $f$ on $\mathbb{D}$ such that

$$\int_{\mathbb{D}} |f(z)|^p dm_\alpha(z) < \infty.$$
Also denote by $\mathcal{A}^p_\alpha = L^p_\alpha \cap H(D)$, the weighted Bergman space with the norm defined by
\[
\|f\|_{\mathcal{A}^p_\alpha} = \left( \int_D |f(z)|^p \, dm_\alpha(z) \right)^{1/p} < \infty.
\]
Note that $\mathcal{A}^p_\alpha$ is a Banach space only if $1 \leq p < \infty$. When $0 < p < 1$, $\mathcal{A}^p_\alpha$ is an F-space with respect to the translation invariant metric defined by
\[
d^\alpha_p(f, g) = \|f - g\|_{\mathcal{A}^p_\alpha}.
\]

The standard Bergman space is obtained as a special case: $\mathcal{A}^2 = \mathcal{A}^2_0$. The reproducing kernel in $\mathcal{A}^2_\alpha$ for the point $z$ in the disk is given by
\[
K_{z,\alpha}(w) = \frac{1}{(1 - \bar{z}w)^{\alpha+2}}
\]
with a suitably chosen analytic branch of the power. The space $H^2$ can be considered as the limit case of $\mathcal{A}^2_\alpha$ as $\alpha \to -1^+$, in the sense that
\[
\lim_{\alpha \to -1^+} \|f\|_{\mathcal{A}^2_\alpha} = \|f\|_{H^2}.
\]
See [Zhu 04c] for a detailed proof.

It is well known that $f \in \mathcal{A}^p_\alpha$ if and only if $f'(z)(1 - |z|^2) \in L^p_\alpha$. Moreover the following asymptotic relation holds
\[
\|f\|_{\mathcal{A}^p_\alpha} \asymp |f(0)|^p + \int_D |f'(z)|^p (1 - |z|^2)^p \, dm_\alpha(z). \tag{1.2.5}
\]
Also, for every $z$ in $D$, the following (exact) point evaluation estimate holds [BBu 89]
\[
|f(z)| \leq \frac{\|f\|_{\mathcal{A}^p_\alpha}}{(1 - |z|^2)^{\frac{1+\alpha}{p}}}, \quad f \in \mathcal{A}^p_\alpha \tag{1.2.6}
\]
The following growth estimate of derivatives of functions in $\mathcal{A}^p_\alpha$ is useful to us
\[
|f'(z)| \leq \frac{\|f\|_{\mathcal{A}^p_\alpha}}{(1 - |z|^2)^{\frac{1+2+\alpha}{p}}}, \quad f \in \mathcal{A}^p_\alpha \tag{1.2.7}
\]
See [HKZ 00], [Zhu 90], [Lu 85c] and [Sm 96] for more details on weighted Bergman spaces.
1.2.25. Dirichlet space:

The Dirichlet space $\mathcal{D}$ is the set of all functions analytic on the unit disk for which the Dirichlet integral
\[
\int_{\mathbb{D}} |f'(z)|^2 \, dm(z)
\]
is finite.

The norm of the Dirichlet space $\mathcal{D}$ is given by
\[
||f||_{\mathcal{D}}^2 = |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 \, dm(z).
\]

Note that $f \in \mathcal{D}$ if $f' \in A^2$. Also $\mathcal{D}$ is a Hilbert space with inner product given by
\[
\langle f, g \rangle_{\mathcal{D}} = f(0)g(0) + \int_{\mathbb{D}} f'(z)\overline{g'(z)} \, dm(z) \quad (1.2.8)
\]

where $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ are Taylor’s series expansions of $f$ and $g$ in $\mathbb{D}$.

The reproducing kernel consistent with above inner product is
\[
R_z(w) = 1 + \log \frac{1}{1 - \overline{w} z}
\]
with the suitably chosen analytic branch of the logarithm so that the following property holds
\[
\overline{R_z(w)} = \langle R_z, R_w \rangle = \langle R_w, R_z \rangle = R_w(z).
\]

Moreover, standard computation involving the Taylor coefficient shows that
\[
||f||_{H^2} \leq ||f||_{\mathcal{D}}
\]
and so $\mathcal{D} \subset H^2 \subset A^2$, a property that will be useful in the rest of this thesis.

It is well known that, classical spaces of holomorphic functions such as the Hardy space, the weighted Bergman space and the Dirichlet space are functional Hilbert spaces with reproducing kernel functions, respectively, as $K_z$, $K_{z,\alpha}$ and $R_z$, defined, respectively, by
\[
K_z(w) = \frac{1}{1 - \overline{w} z}, \quad K_{z,\alpha}(w) = \frac{1}{(1 - \overline{w} z)^{\alpha+2}} \quad \text{and} \quad R_z(w) = 1 + \log \frac{1}{1 - \overline{w} z}.
\]
Moreover, 
\[ \|K_z\|_{H^2} = \frac{1}{\sqrt{1 - |z|^2}}, \quad \|K_{z,\alpha}\|_{A^2_\alpha} = \frac{1}{(1 - |z|^2)^{(\alpha+2)/2}} \]

and 
\[ \|R_z\|^2 = 1 + \log \frac{1}{1 - |z|^2}. \]

Also, it is not surprising that evaluation of any derivative at a point of the disk is a bounded linear functional. In fact, for each point \( z \) in the open unit disk \( \mathbb{D} \) and positive integer \( m \), evaluation of the \( m \)th derivative of functions, respectively, in \( H^2 \), \( A^2_\alpha \), or \( \mathbb{D} \) at \( z \) is a bounded linear functional and 
\[ f^{(m)}(z) = \langle f, K_{z}^{(m)} \rangle_{H^2}, \quad f^{(m)}(z) = \langle f, K_{z,\alpha}^{(m)} \rangle_{A^2_\alpha}, \quad f^{(m)}(z) = \langle f, R_{z}^{(m)} \rangle_{\mathbb{D}}. \]

1.2.26. Bloch-type and Little Bloch-type spaces:

Let \( \nu \) be a strictly positive continuous function on \( \mathbb{D} \). If \( \nu(z) = \nu(|z|) \) for every \( z \in \mathbb{D} \), we call it a radial weight. A radial weight \( \nu \) is called typical if it is non-increasing with respect to \( |z| \) and \( \nu(z) \to 0 \) as \( |z| \to 1 \). For a typical weight \( \nu \), the Bloch-type space \( B_\nu \) on \( \mathbb{D} \) is the space of all analytic functions \( f \) on \( \mathbb{D} \) such that 
\[ b(f) = \sup_{z \in \mathbb{D}} \nu(z)|f'(z)| < \infty. \]

The little Bloch-type space \( B_{\nu,0} \) consists of all \( f \in B_\nu \) such that 
\[ \lim_{|z| \to 1} \nu(z)|f'(z)| = 0. \]

\( B_\nu \) is a Banach space with the norm \( \|f\|_{B_\nu} = |f(0)| + b(f) \) and \( B_{\nu,0} \) is a closed subspace of \( B_\nu \). When \( \nu(z) = (1 - |z|^2) \), \( B_\nu \) reduces to the usual Bloch space, denoted by \( B \) and \( B_{\nu,0} \) reduces to the little Bloch space \( B_0 \). Likewise for a typical weight \( \nu \), the weighted space of controlled growth \( A^\infty_\nu \) on \( \mathbb{D} \) is the space of functions \( f \in H(\mathbb{D}) \) such that 
\[ \sup_{z \in \mathbb{D}} \nu(z)|f(z)| < \infty \] and the little weighted space of controlled growth \( A^\infty_{\nu,0} \) consists of all \( f \in A^\infty_\nu \) such that 
\[ \lim_{|z| \to 1} \nu(z)|f(z)| = 0. \]

1.2.27. Weighted Hardy space:

Let \( \sigma \in C^2[0,1) \) be a weight such that \( \sigma(z) = \sigma(|z|), \quad z \in \mathbb{D} \), we call it a weight function. By \( H_\sigma \), we denote the weighted Hardy space consisting of all \( f \in H(\mathbb{D}) \) such
that
\[ \|f\|_{H_\sigma}^2 = |f(0)|^2 + \int_\mathbb{D} |f'(z)|^2 \sigma(z) \, dm(z) < \infty. \]

A simple computation shows that a function \( f(z) = \sum_{n=0}^\infty a_n z^n \) belongs to \( H_\sigma \) if and only if
\[ \sum_{n=0}^\infty |a_n|^2 \sigma_n < \infty, \]
where \( \sigma_0 = 1 \) and
\[ \sigma_n = 2n^2 \int_0^1 r^{2n-1} \sigma(r) \, dr, \quad n \in \mathbb{N}. \]

The sequence \( (\sigma_n)_{n \in \mathbb{N}_0} \) is called the weight sequence of the weighted Hardy space \( H_\sigma \).

The properties of the weighted Hardy space with the weight sequence \( (\sigma_n)_{n \in \mathbb{N}_0} \), clearly depends upon \( \sigma_n \). Writing \( (\sigma_n) = ||z^n||^2 \), the orthogonality is easily seen to imply that the norm on \( H_\sigma \) is given by
\[ \left( \left\| \sum_{n=0}^\infty a_n z^n \right\| \right)^2 = \sum_{n=0}^\infty |a_n|^2 \sigma_n \]
and the inner product by
\[ \left\langle \sum_{n=0}^\infty a_n z^n, \sum_{n=0}^\infty c_n z^n \right\rangle = \sum_{n=0}^\infty a_n \overline{c_n} \sigma_n. \]

The classical Hardy space, the classical Bergman space and the classical Dirichlet space are weighted Hardy spaces with \( \sigma_n \equiv 1 \), \( \sigma_n = (n + 1)^{-1/2} \) and \( \sigma_n = (n + 1)^{1/2} \) respectively. The generating function for the weighted Hardy space \( H_\sigma \) is the function
\[ k(z) = \sum_{n=0}^\infty \frac{z^n}{\sigma_n}. \]

Let \( \mathcal{H}_\sigma \) be a weighted Hardy space with weight sequence \( (\sigma_n) \). Then for each \( \lambda \in \mathbb{D} \), the evaluation functional in \( \mathcal{H}_\sigma \) at \( \lambda \) is a bounded linear functional and for \( f \in \mathcal{H}_\sigma \), \( f(\lambda) = \langle f, K_\lambda \rangle \), where
\[ K_\lambda(z) = \sum_{k=0}^\infty \frac{(\lambda z)^k}{\sigma(k)} \quad \text{and} \quad ||K_\lambda||_{\mathcal{H}_\sigma}^2 = \sum_{k=0}^\infty \frac{|\lambda|^{2k}}{\sigma(k)}. \]

Moreover,
\[ |f(z)| \leq \|f\|_{\mathcal{H}_\sigma} \left( \sum_{k=0}^\infty \frac{r^{2k}}{\sigma(k)} \right)^{1/2} \quad (1.2.9) \]
\[ |f'(z)| \leq \|f\|_{H^p} \left( \sum_{k=0}^{\infty} \frac{k^2 + 2(k-1)}{\sigma(k)} \right)^{1/2} \] (1.2.10)

for \(|z| \leq r\) where \(\sigma(k) = \|z^k\|_{H^p}\), see Theorem 2.10 in [CoM 95]. For more about weighted Hardy spaces and some related topics, see [CoM 95], [KeL 12] and [ShSh 11]. Also for \(0 < p < \infty\) and \(0 < q < \infty\), define
\[ \|f_r\|_{p,q} = \left( \frac{1}{2\pi} \int_{0}^{2\pi} |f(re^{|i\theta|})|^p (1 - r^2)^q \, d\theta \right)^{1/p}. \]

The Hardy space \(H_{p,q}(\mathbb{D}) = H_{p,q}\) is the family of all holomorphic functions \(f\) in \(\mathbb{D}\) satisfying
\[ \|f\|_{H_{p,q}} = \sup_{0 < r < 1} \|f_r\|_{p,q} < \infty. \]

It is trivial that \(H_{p,0}\) is just the classical Hardy space \(H^p\) and \(H_{p,q}\) is a Banach space for \(p \geq 1\). We write \(\|f\|_{H^p}\) for \(\|f\|_{H_{p,0}}\). For more about \(H^p\) and \(H_{p,q}\) see [Du 70] and [Zhu 90].

### 1.3 Structure of work

The main purpose of this thesis is to carry out the study of generalized composition operators on spaces of holomorphic functions defined in the previous section using some properties of holomorphic functions. In fact, our aim is to answer some of the questions that operator theory demands viz the question of boundedness, compactness, essential norm, adjoint, approximation number, rate of increase of mean values etc. of the generalized composition operators.

The present thesis is divided into five chapters. The chapter I is introductory in nature and its aim is to keep rest of the thesis reasonably self-contained. In this chapter we give a brief history of the subject and the range of the topics taken up in this thesis. The chapter I consists of three sections. In the first section, we give the background of composition, multiplication and weighted composition operators. In the second section we present the basic notations, some special sets such as Carleson windows, Carleson sets, psuedo-hyperbolic disks, Bergman disks and some of their elementary properties and useful relations between them. This section also contain some results from geometric function theory such as Carleson measures, Nevanlinna counting function, subordinate
principle and spaces of holomorphic functions along with useful properties and growth estimates of functions in these spaces. The final section of this chapter gives a brief and chapter wise account of the results obtained in this thesis.

Chapter second, deals with the study of boundedness, compactness and essential norm of products of composition, multiplication and differentiation operators on weighted Bergman spaces. This chapter comprises of two sections. Section 2.1, gives the boundedness of generalized composition operators on weighted Bergman spaces. The boundedness is established by using Carleson-type measures and the behaviour of certain integral transforms involving the inducing symbols. In section 2.2 of this chapter, we estimate the essential norm of products of multiplication composition and differentiation operators on the weighted Bergman spaces and give its numerous applications.

Chapter third is devoted to study of approximation numbers and adjoints of products of composition, multiplication and differentiation operators on some functional Hilbert spaces. The chapter consists of two sections. We find upper and lower bounds for approximation numbers of compact composition operators on the weighted Hardy spaces $H_\sigma$ under some conditions on the weight function $\sigma$ in section 3.1. We obtain adjoints of $T_{\psi,\psi}^n = T_{\psi,0,\psi} D^n$ on some functional Hilbert spaces of holomorphic functions (like the Hardy space, the Bergman space and the Dirichlet space) in section 3.2. In fact, we generalize the known formulas for the adjoints of composition operators such as Cowen’s adjoint formula, the Gallardo and Montes adjoint formula to generalized composition operators.

In chapter fourth, we analyze the rate of increase of mean values of functions in generalized Hardy spaces. In section 4.1, for each $f$ in the weighted Hardy-Orlicz space $H_{\phi,q}$, $\frac{d||f_r||_{\phi,q}}{dr}$ grows at most like $o(1/1-r)$ as $r \to 1$. In section 4.2, we prove a similar results for functions in generalized Hardy space $H_{\omega,\alpha}$.

In the last and fifth chapter of this thesis, we consider the products of iterated composition, multiplication and differentiation operators between Bergman type spaces. Section 5.1, deals with the boundedness of $T_{\psi,0,\psi} D^n = T_{\psi,\psi}^n$ acting on weighted Bergman-
Nevanlinna spaces. We also provide a sufficient condition for $T^n_{\psi,\varphi}$ to be compact on weighted Bergman-Nevanlinna spaces. In section 5.2, we note that the boundedness and compactness of iterated composition operators induced by the lens and the lunar maps between Hardy and weighted Bergman spaces of the unit disk depends upon the number of iterations and the angle of contact of these maps with the unit circle.