CHAPTER-V

PRODUCTS OF ITERATED COMPOSITION MULTIPLICATION AND DIFFERENTIATION OPERATORS BETWEEN BERGMAN TYPE SPACES

In this chapter, we discuss the products of iterated composition, multiplication and differentiation operators between Bergman type spaces. In section 5.1, we characterize the boundedness of \( T_{\psi,0,\phi} D^n = T^n_{\psi,\phi} \) acting on weighted Bergman-Nevanlinna spaces. We also provide a sufficient condition for \( T^n_{\psi,\phi} \) to be compact on weighted Bergman-Nevanlinna spaces. In section 5.2, we note that the boundedness and compactness of iterated composition operators induced by the lens and the lunar maps between Hardy and weighted Bergman spaces of the unit disk depends upon the number of iterations and the angle of contact of these maps with the unit circle.

5.1. Composition followed by differentiation between weighted Bergman-Nevanlinna spaces

In this section, we characterize boundedness of \( T^n_{\psi,\phi} \) acting on weighted Bergman-Nevanlinna spaces. We also give a sufficient condition for \( T^n_{\psi,\phi} \) to be compact on weighted Bergman-Nevanlinna spaces.

Weighted Bergman Nevanlinna space \( A^0_\alpha \)

The weighted Bergman Nevanlinna space \( A^0_\alpha \) consists of \( f \in H(D) \) such that

\[
\| f \|_{A^0_\alpha} = \int_D \log^+ |f(z)| dm_\alpha(z) < \infty,
\]

where

\[
\log^+ x = \begin{cases} 
\log x & \text{if } x \geq 1 \\
0 & \text{if } x < 1.
\end{cases}
\]
In fact, \( \|f\|_{A^0_\alpha} \) fails to be a norm, but \( (f, g) \to \|f - g\|_{A^0_\alpha} \) defines a translation invariant metric on \( A^0_\alpha \) and this turns \( A^0_\alpha \) into a complete metric space. The space \( A^0_\alpha \) appears in the limit as \( p \to 0 \) of the weighted Bergman space \( A^p_\alpha \) in the sense of

\[
\lim_{p \to 0} \frac{t^p - 1}{p} = \log^+ t, \quad 0 < t < \infty.
\]

The Bergman-Nevanlinna space \( A^0_\alpha \) contains all the Bergman spaces \( A^p_\alpha \) for all \( p > 0 \). Obviously, the inequality

\[
\log^+(x) \leq \log(1 + x) \leq 1 + \log^+(x); \quad x \geq 0
\]

implies that \( f \in A^0_\alpha \) if and only if

\[
\|f\|_{A^0_\alpha} \asymp \int_D \log(1 + |f(z)|) \, dm_\alpha(z) < \infty.
\]

See [HKZ 00] for more about weighted Bergman spaces and weighted Bergman-Nevanlinna spaces. By the subharmonicity of \( \log(1 + |f(z)|) \), we have

\[
\log(1 + |f(z)|) \leq C_\alpha \frac{\|f\|_{A^0_\alpha}}{(1 - |z|^2)^{\alpha+2}}, \quad z \in \mathbb{D}
\]  

(5.1.1)

for all \( f \in A^0_\alpha \). In particular, equation (5.1.1) tells us that if \( f_n \to f \) in \( A^0_\alpha \), then \( f_n \to f \) locally uniformly. Here locally uniform convergence means the uniform convergence on every compact subset of \( \mathbb{D} \).

We first consider the following lemma of [HKZ 00]. Lemma 5.1.1. Let \( \alpha \in (-1, \infty) \) and \( \beta > 0 \), then there exists a constant \( C = C(\alpha, \beta) \) such that

\[
(1 - |z|^2)^\beta \int_D \frac{dm_\alpha(w)}{|1 - \overline{z}w|^{2+\alpha+\beta}} \asymp 1, \quad z \in \mathbb{D}.
\]

The next lemma is proved in [CKS 08].

Lemma 5.1.2. Let \( \alpha \in (-1, \infty) \), \( n \in \mathbb{N} \) and \( 0 < r < 1 \), then there exists a constant \( C = C(\alpha, r) \) such that the following inequality holds:

\[
\log(1 + |f^n(z)|) \leq C \int_{D(z,r)} \frac{\log(1 + |f(w)|)}{(1 - |w|)^{\alpha+2+n}} \, dm_\alpha(w).
\]
Lemma 5.1.3. Let $\alpha \in (-1, \infty)$, $n \in \mathbb{N}$ and $0 < r < 1$ be fixed. If $\mu$ is $(\alpha + 2 + n)$-Carleson measure on $\mathbb{D}$, then there exists a constant $C = C(\alpha, r)$ such that the following inequality holds:

$$\int_{\mathbb{D}} \log(1 + |f^n(w)||) \, d\mu(w) \leq C \int_{\mathbb{D}} \log(1 + |f(w)||) \, d\mu(w).$$

**Proof.** Let $0 < r < 1$ be fixed. Pick a sequence $\{a_n\}$ in $\mathbb{D}$ satisfying the conditions of Lemma 1.2.8. Using Lemma 5.1.2, for $f \in \mathcal{A}_\alpha^0$, we have

$$\int_{\mathbb{D}} \log(1 + |f^n(w)||) \, d\mu(w) \leq \sum_{n=1}^{\infty} \int_{D(a_n, r)} \log(1 + |f^n(w)||) \, d\mu(w)$$

$$\leq \sum_{n=1}^{\infty} \mu(D(a_n, r)) \sup_{w \in D(a_n, r)} \log(1 + |f^n(w)||) \, d\mu(w)$$

$$\leq \sum_{n=1}^{\infty} \frac{\mu(D(a_n, r))}{(1 - |a_n|)^{\alpha + 2 + n}} \int_{D(a_n, 2r)} \log(1 + |f(w)||) \, d\mu(w).$$

Now $\mu$ is $(\alpha + 2 + n)$-Carleson measure on $\mathbb{D}$, so we have

$$\int_{\mathbb{D}} \log(1 + |f^n(w)||) \, d\mu(w) \leq C \sum_{n=1}^{\infty} \int_{D(a_n, 2r)} \log(1 + |f(w)||) \, d\mu(w)$$

$$= CN \int_{\mathbb{D}} \log(1 + |f(w)||) \, d\mu(w).$$

Theorem 5.1.4. Let $\varphi$ be a holomorphic self-map of $\mathbb{D}$. Then the following are equivalent:

1. $T_{\psi, \varphi}^n : \mathcal{A}_\alpha^0 \to \mathcal{A}_\alpha^0$ is bounded.

2. The pull-back measure $m_\alpha \circ \varphi^{-1}$ is a $(\alpha + 2 + n)$-Carleson measure on $\mathbb{D}$.

**Proof.** Suppose that $T_{\psi, \varphi}^n : \mathcal{A}_\alpha^0 \to \mathcal{A}_\alpha^0$ is bounded. Consider the function

$$f_z(w) = \frac{(1 - |z|^2)^{\alpha + 2 + 2n}}{(1 - \overline{z}w)^{\alpha + 2 + n}}, \ z \in \mathbb{D}.$$
Since $A^1_\alpha \subset A^0_\alpha$, by Lemma 5.1.1, we have
\[
|f_z|_{A^0_\alpha} \leq |f_z|_{A^1_\alpha} \asymp (1 - |z|)^{\alpha + 2 + n}
\]
for all $z \in \mathbb{D}$. Also
\[
f'_z(w) = (\alpha + 2 + n)\overline{z} \frac{(1 - |z|^2)^{\alpha + 2 + 2n}}{(1 - \overline{z}w)^{\alpha + 3 + n}}
\]
and
\[
f^{(n)}_z(w) = (\alpha + 2 + n)(\alpha + 3 + n) + \cdots + (\alpha + 2n - 1)(\overline{z})^n \frac{(1 - |z|^2)^{\alpha + 2 + 2n}}{(1 - \overline{z}w)^{\alpha + 2 + 2n}}.
\]
Therefore,
\[
|f^{(n)}_z(w)| = (\alpha + 2 + n)(\alpha + 3 + n) + \cdots + (\alpha + 2n - 1)|\overline{z}|^n \frac{(1 - |z|^2)^{\alpha + 2 + 2n}}{|1 - \overline{z}w|^{\alpha + 2 + 2n}},
\]
and so we have $|f^{(n)}_z(w)| \leq C$ for some constant $C = C(\alpha, n)$.

Thus
\[
\log(1 + |f^{(n)}_z(w)|) \asymp |f^{(n)}_z(w)| \quad \text{for all } z, w \in \mathbb{D}.
\]

In addition, by Lemma 1.2.7, we have
\[
1 - |z|^2 \asymp |1 - \overline{z}w|
\]
for $w \in D(z, r)$. Thus
\[
|f^{(n)}_z(w)| \asymp |z|^n
\]
for $w \in D(z, r)$. Since $T^n_{\psi, \varphi} : A^0_\alpha \to A^0_\alpha$ is bounded, there exists $C > 0$ such that
\[
||T^n_{\psi, \varphi} f_z||_{A^0_\alpha} \leq C|f_z|_{A^0_\alpha} \asymp (1 - |z|^2)^{\alpha + 2 + n}.
\]

That is,
\[
(1 - |z|^2)^{\alpha + 2 + n} \asymp ||T^n_{\psi, \varphi} f_z||_{A^0_\alpha} \asymp \int_{\mathbb{D}} \log(1 + |f^{(n)}_z(\varphi(z))|) d\mu_\alpha(w)
\]
\[
= \int_{\mathbb{D}} |f^{(n)}_z(w)| d(m_\alpha \circ \varphi^{-1})(w)
\]
\[
\geq \int_{D(z, r)} |f^{(n)}_z(w)| d(m_\alpha \circ \varphi^{-1})(w)
\]
\[
\asymp |z|^n (m_\alpha \circ \varphi^{-1}) D(z, r), \quad \text{for all } z \in \mathbb{D}.
\]
Consequently,
\[
\sup_{z \in D} \frac{|z|^n (m_\alpha \circ \varphi^{-1})(D(z, r))}{(1 - |z|^2)^{a+2+n}} < \infty.
\]

Thus,
\[
\sup_{r_0 < |z| < 1} \frac{(m_\alpha \circ \varphi^{-1})(D(z, r))}{(1 - |z|^2)^{a+2+n}} < \infty. \tag{5.1.2}
\]

On the other hand
\[
\sup_{0 \leq |z| \leq r_0} \frac{(m_\alpha \circ \varphi^{-1})(D(z, r))}{(1 - |z|^2)^{a+2+n}} \leq \frac{(m_\alpha \circ \varphi^{-1})(D)}{(1 - r_0^2)^{a+2+n}}
= \frac{1}{(1 - r_0^2)^{a+2+n}} \int_{\varphi^{-1}(D)} m_\alpha(w)
\leq \frac{1}{(1 - r_0^2)^{a+2+n}} \int_D m_\alpha(w)
= \frac{1}{(1 - r_0^2)^{a+2+n}}. \tag{5.1.3}
\]

From equation (5.1.2) and (5.1.3), we have
\[
\sup_{z \in D} \frac{(m_\alpha \circ \varphi^{-1})(D(z, r))}{(1 - |z|^2)^{a+2+n}} < \infty.
\]

Hence \(m_\alpha \circ \varphi^{-1}\) is an \((\alpha + 2 + n)\) measure on \(D\).

Conversely, suppose that \(m_\alpha \circ \varphi^{-1}\) is an \((\alpha + 2 + n)\) measure on \(D\). Then by Lemma 5.1.3, we have for \(f \in A_\alpha^0\),
\[
\|T_{\psi, \varphi}^n f\|_{A_\alpha^0} = \int_D \log(1 + |f^{(n)}(\varphi(w))|) d m_\alpha(w)
\leq \int_D \log(1 + |f^{(n)}(w)|) d (m_\alpha \circ \varphi^{-1})(w)
\leq C \int_D \log(1 + |f(w)|) d m_\alpha(w)
\asymp \|f\|_{A_\alpha^0}.
\]

\[\blacksquare\]

**Lemma 5.1.5.** Let \(\varphi\) be a holomorphic self-map of \(D\). Then \(T_{\psi, \varphi}^n : A_\alpha^0 \to A_\alpha^0\) is compact if and only if for every sequence \(\{f_n\}\) which is bounded in \(A_\alpha^0\) and converges
to zero uniformly on compact subsets of \( \mathbb{D} \) as \( n \to \infty \), we have \( \| T^n_{\varphi, \varphi} f_n \|_{A_0^\alpha} \to 0 \).

**Proof.** Proof follows on the same lines as the proof of proposition 3.11 in [CoM 95]. We omit the details. \( \blacksquare \)

**Theorem 5.1.6. (Sufficient condition)** Let \( \varphi \) be a holomorphic self-map of \( \mathbb{D} \). Then \( T^n_{\psi, \varphi} : A_0^\alpha \to A_0^\alpha \) is compact if the pull-back measure \( m_\alpha \circ \varphi^{-1} \) is a vanishing \((\alpha + 2 + n)\)-Carleson measure on \( \mathbb{D} \).

**Proof.** Suppose that \( m_\alpha \circ \varphi^{-1} \) is a vanishing \((\alpha + 2 + n)\)-Carleson measure on \( \mathbb{D} \). Then

\[
\frac{(m_\alpha \circ \varphi^{-1})(D(a, r))}{(1 - |a|^2)^{\alpha + 2 + n}} \to 0 \quad \text{as} \quad |a| \to 1.
\]

Suppose that \( \{f_m\} \) is a bounded sequence in \( A_0^\alpha \) that converges to zero uniformly on compact subsets of \( \mathbb{D} \). Let \( \{a_n\} \) be a sequence as in Lemma 1.2.8 such that \( |a_1| < |a_2| < |a_3| \cdots \). Then for each \( \epsilon > 0 \), we have

\[
(m_\alpha \circ \varphi^{-1})(D(a_n, r)) < \epsilon (1 - |a_n|^2)^{\alpha + 2 + n}
\]

for all \( a_n \in \mathbb{D} \) such that \( |a_n| > r \). Thus

\[
\| T^n_{\psi, \varphi} f_m \|_{A_0^\alpha} \lesssim \int_{\mathbb{D}} \log (1 + |f_m^n(\varphi(z))|) d m_\alpha(z)
\]

\[
= \int_{\mathbb{D}} \log (1 + |f_m^n(z)|) d (m_\alpha \circ \varphi^{-1})(z)
\]

\[
= \int_{|z| \leq r_0} \log (1 + |f_m^n(z)|) d (m_\alpha \circ \varphi^{-1})(z)
\]

\[
+ \int_{|z| > r_0} \log (1 + |f_m^n(z)|) d (m_\alpha \circ \varphi^{-1})(z).
\] (5.1.4)

Since \( \{f_m\} \) is a bounded sequence in \( A_0^\alpha \) that converges to zero uniformly on compact subsets of \( \mathbb{D} \),

\[
\lim_{m \to \infty} \int_{|z| \leq r_0} \log (1 + |f_m^n(z)|) d (m_\alpha \circ \varphi^{-1})(z) = 0,
\]

whereas

\[
\int_{|z| > r_0} \log (1 + |f_m^n(z)|) d (m_\alpha \circ \varphi^{-1})(z)
\]
\[
\leq \sum_{n=k+1}^{\infty} \int_{D(a_n,r)} \log(1 + |f_{m}^{(n)}(z)|) d(m_{\alpha} \circ \varphi^{-1})(z)
\]
\[
\leq \sum_{n=k+1}^{\infty} (m_{\alpha} \circ \varphi^{-1})(D(a_n,r)) \sup_{z \in D(a_n,r)} \log(1 + |f_{m}^{(n)}(z)|)
\]
\[
\leq \sum_{n=k+1}^{\infty} \frac{(m_{\alpha} \circ \varphi^{-1})(D(a_n,r))}{(1 - |a_n|^2)^{2+\alpha}} \int_{D(a_n,2r)} \log(1 + |f_{m}(z)|) d\alpha(z)
\]
\[
\leq \epsilon C N \int_{D} \log(1 + |f_{m}(z)|) d\alpha(z)
\]
\[
= \epsilon C N ||f_{m}||_{A_{0}^{\alpha}}.
\]
Since \( \epsilon > 0 \) is arbitrary, we have \( ||T_{\varphi, \psi}^{n} f_{m}||_{A_{0}^{\alpha}} \to 0 \) as \( m \to \infty \). Hence \( T_{\varphi, \psi}^{n} : A_{0}^{\alpha} \to A_{0}^{\alpha} \) is compact.

5.2. Iterated composition operators induced by the lens and the lunar maps between Hardy and weighted Bergman spaces

In this section, we analyze that the boundedness and compactness of iterated composition operators induced by the lens and the lunar maps between Hardy and weighted Bergman spaces of the unit disk depends upon the number of iterations and the angle of contact of these maps with the unit circle.

Let \( \varphi \) be a holomorphic self-map of \( \mathbb{D} \). Denote by \( \varphi^{[n]} \) the \( n \)-th iterate of \( \varphi \), that is
\[
\varphi^{[n]} = \varphi \circ \varphi \circ \cdots \circ \varphi \quad (n \text{ times}).
\]

The connection between iterations of \( \varphi \) and composition operators comes from the equation
\[
C_{\varphi^{[n]}} = C_{\varphi}^{[n]}.
\]
For \( 0 < \gamma < 1 \), we consider two self-maps \( \varphi_{\gamma} \) and \( \psi_{\gamma} \) of \( \mathbb{D} \) defined respectively, as
\[
\varphi_{\gamma}(z) = 1 - (1 - z)^{\gamma} \quad \text{and} \quad \psi_{\gamma}(z) = \frac{(\lambda(z))^{\gamma} - 1}{(\lambda(z))^{\gamma} + 1},
\]
where \( \lambda(z) = (1 + z)/(1 - z) \). The maps \( \varphi_{\gamma} \) has an angle of contact of \( \gamma \pi \) at 1 on \( \partial \mathbb{D} \) and is known as the lunar map, whereas the map \( \psi_{\gamma} \) has an angle of contact of
\( \gamma \pi \) at 1 and \(-1\) on \( \partial \mathbb{D} \) and is known as the \textit{lens map}. Let \( \varphi = \varphi_\gamma \) or \( \psi_\gamma \). Then using elementary calculations, we have

\[
(1 - |z|)^\gamma \leq 1 - |\varphi(z)| \leq |1 - z|^\gamma.
\]

(5.2.1)

for all \( z \in \mathbb{D} \) such that \( |z| \to 1 \). Moreover,

\[
\varphi_{\gamma}^n = 1 - (1 - z)^\gamma^n
\]

(5.2.2)

and

\[
\psi_{\gamma}^n = \frac{(\lambda(z))^\gamma^n - 1}{(\lambda(z))^\gamma^n + 1}.
\]

(5.2.3)

**Boundedness and Compactness of \( \mathcal{C}_{\varphi}^{[n]} \)**

The next result is an easy consequence of main result in [Sm 96].

**Theorem 5.2.1.** Let \( 1 \leq p \leq q \), \( \alpha, \beta \geq -1 \), \( n \) be a non-negative integer and \( \varphi \) be a univalent self-map of \( \mathbb{D} \). Then

(i) \( \mathcal{C}_{\varphi}^{[n]} : \mathcal{A}_\alpha^p \to \mathcal{A}_\beta^q \) is bounded if and only if

\[
\sup_{z \in \mathbb{D}} \frac{(1 - |z|)^{2+\beta}}{(1 - |\varphi_{\gamma}^n(z)|)^{2+\alpha}} < \infty.
\]

(5.2.4)

(ii) \( \mathcal{C}_{\varphi}^{[n]} : \mathcal{A}_\alpha^p \to \mathcal{A}_\beta^q \) is compact if and only if

\[
\lim_{|z| \to 1} \frac{(1 - |z|)^{2+\beta}}{(1 - |\varphi_{\gamma}^n(z)|)^{2+\alpha}} = 0.
\]

(5.2.5)

**Theorem 5.2.2.** Let \( \alpha \geq -1 \), \( 0 < \gamma < 1 \), \( n \) be a non-negative integer and \( \varphi = \varphi_\gamma \) or \( \psi_\gamma \). Then

(i) \( \mathcal{C}_{\varphi}^{[n]} : \mathcal{A}_\alpha^p \to \mathcal{A}_\beta^q \) is bounded if and only if

\[
\gamma^n \leq \frac{(\beta + 2)p}{(\alpha + 2)q}.
\]

(5.2.6)
(ii) \( C_\varphi^{[n]} : A_\alpha^p \to A_\beta^q \) is compact if and only if
\[
\gamma^n < \frac{(\beta + 2)p}{(\alpha + 2)q}.
\] (5.2.7)

**Proof.** (i) Assume that \( \varphi = \varphi_\gamma \) or \( \psi_\gamma \), \( 0 < \gamma < 1 \), and equation (5.2.6) holds. Then using equations (5.2.1) – (5.2.3), there exists some \( r_0, 0 < r_0 < 1 \) such that
\[
(1 - |z|^2)^\gamma^n \leq 1 - |\varphi^{[n]}(z)|^2 \leq |1 - z|^\gamma^n
\] (5.2.8)
for \( z \in \mathbb{D} \setminus r_0 \mathbb{D} \). First suppose that equation (5.2.6) holds. Then by equation (5.2.8), we have
\[
\sup_{z \in \mathbb{D}} \frac{(1 - |z|)^{2+\beta}}{(1 - |\varphi^{[n]}(z)|)^{(2+\alpha)p}} \leq \sup_{z \in \mathbb{D}} \frac{(1 - |z|)^{2+\beta}}{(1 - |z|)^{\gamma n(2+\alpha)p}} < \infty
\]
and so by Theorem 5.2.1, \( C_\varphi^{[n]} : A_\alpha^p \to A_\beta^q \) is bounded. Conversely, suppose that \( C_\varphi^{[n]} : A_\alpha^p \to A_\beta^q \) is bounded. Then
\[
\sup_{|z| > r_0} \frac{(1 - |z|)^{2+\beta}}{(1 - |\varphi^{[n]}(z)|)^{(2+\alpha)p}} \leq \sup_{z \in \mathbb{D}} \frac{(1 - |z|)^{2+\beta}}{(1 - |\varphi^{[n]}(z)|)^{(2+\alpha)p}} < \infty.
\] (5.2.9)
Using second inequality of (5.2.8) in (5.2.9) and taking \( z = x \in (r_0, 1) \), we have
\[
\sup_{x \in (r_0, 1)} (1 - x)^{(2+\beta) - \gamma n(2+\alpha)p} < \infty,
\]
which is possible only if equation (5.2.6) holds.

(ii) Suppose that equation (5.2.7) holds. Then by equation (5.2.8), we have
\[
\lim_{|z| \to 1} \frac{(1 - |z|)^{2+\beta}}{(1 - |\varphi^{[n]}(z)|)^{(2+\alpha)p}} \leq \lim_{|z| \to 1} \frac{(1 - |z|)^{2+\beta}}{(1 - |z|)^{\gamma n(2+\alpha)p}} = 0
\]
and so by Theorem 5.2.1 (ii), \( C_\varphi^{[n]} : A_\alpha^p \to A_\beta^q \) is compact. Conversely, suppose that \( C_\varphi^{[n]} : A_\alpha^p \to A_\beta^q \) is compact. Then for every sequence \( \{z_m\} \) in \( \mathbb{D} \), we have
\[
\frac{(1 - |z_m|)^{2+\beta}}{(1 - |z_m|)^{\gamma n(2+\alpha)p}} \leq \frac{(1 - |z_m|)^{2+\beta}}{(1 - |\varphi^{[n]}(z_m)|)^{(2+\alpha)p}} \to 0
\] (5.2.10)
as \( |z_m| \to 1 \). In particular, if we consider the sequence \( z_m = \frac{m}{m+1} \). Then \( \{z_m\} \in \mathbb{D} \) and \( |z_m| \to 1 \) as \( m \to \infty \). Thus by equation (5.2.10), we have
\[
\lim_{m \to \infty} \left(1 - \frac{m}{m+1}\right)^{(2+\beta) - \gamma n(2+\alpha)p} = 0,
\]
which is possible only if equation (5.2.7) holds.

As an application of Theorem 5.2.2, we see that angle of contact of lens and lunar maps and \( n \) play significant role in boundedness and compactness of \( C^{[n]}_{\varphi} \) between Hardy and Bergman spaces. For \( 0 < \gamma < 1 \), let \( \varphi = \varphi_\gamma \) or \( \psi_\gamma \) and \( n \geq 1 \). Let \( \theta \) be the angle of contact. Then \( \theta = \gamma \pi \), and so

\[
C^{[n]}_{\varphi} : A^p_\alpha \to A^q_\beta \text{ is bounded } \iff \theta \leq \left( \frac{(\beta + 2)p}{(\alpha + 2)q} \right)^{1/n} \pi,
\]

\[
C^{[n]}_{\psi} : A^p_\alpha \to A^q_\beta \text{ is compact } \iff \theta < \left( \frac{(\beta + 2)p}{(\alpha + 2)q} \right)^{1/n} \pi.
\]

Taking \( n = 1, 2, 3, 4 \), \( \alpha = 0 \) and \( \beta = -1 \) and \( p = q = 2 \) respectively, we get

<table>
<thead>
<tr>
<th>( n )</th>
<th>( C^{[n]}_{\varphi} : A^2 \to H^2 \text{ is bounded} )</th>
<th>( C^{[n]}_{\psi} : A^2 \to H^2 \text{ is compact} )</th>
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<td>( \theta &lt; \frac{\pi}{2} )</td>
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<tr>
<td>2</td>
<td>( \theta \leq \sqrt{1/2}\pi )</td>
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<td>4</td>
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