CHAPTER 4

Linear optimization problem with fuzzy relation equations
4.1 Background and motivation

The behavior of a fuzzy system is characterized by a system of if-then rules which can be considered as a partial fuzzy function. While associating fuzzy function with logical implication rule, there appear two problems, (i) how this function can be represented, and (ii) how it can be used in calculations. Since a fuzzy function is a fuzzy relation, therefore, it is a common practice to represent a system of fuzzy if-then rules as a fuzzy relation so that the required calculations can be performed using the compositional rule of inference [174]. Moreover, it is required that for each input which coincides with one of the antecedents of existing if-then rules, the computed output should coincide with the corresponding consequence. Thus, there appear the problem of solvability and characterization of solution space of a system of fuzzy relation equations defined so-fourth. It is important in practice to solve a fuzzy relation equation. For example, in fuzzy reasoning, when inference rules and consequences are known, the problem of determining antecedents is usually reduced to solving a fuzzy relation equation. According to Stamou & Tzafestas [147], fuzzy relation equations are the most appropriate mathematical model for the analysis of fuzzy inference system.Perfilieva & Novák [122] identified a relationship between fuzzy inference system and fuzzy relation equations.

The natural approach to capture the properties of system of fuzzy relation equations consists of establishing fuzzy relation equations based on compositional operators chosen from the fuzzy logic algebra of operators, establishing solvability conditions for consistency of systems of fuzzy relation equations, characterization of solution of fuzzy relation equations, determining complete solution etc. In most of these researches,
interest is placed on fuzzy relation equation and its solutions. After the original definition of fuzzy relation advanced by Sanchez in 1976 [133], many researchers are engaged in searching the solution and application of it. Compositional operators such as max-*, max-® (Archimedean/non-Archimedean t-norm), inf-→ were chosen from suitable algebraic structures of operators to explore fuzzy relation equations by several researchers. The fuzzy relation equation is composed of fuzzy input, output and their fuzzy relation. When discussing the solution of the equations, almost all research is from two major points of view: one being an “inverse problem” which performs unknown fuzzy input, the other being an “identification problem” which performs unknown fuzzy relation.

In this chapter, we consider a linear optimization problem with such fuzzy relation equations as constraints. The non-empty solution set of the system of fuzzy relation equations is a lattice, in general, and hence a non-convex set. The existing conventional methods for linear programming problems cannot be applied to decide a fuzzy linear optimization problem. Therefore, an efficient and modified procedure for such problems appears to be necessary. A linear optimization problem with max-® fuzzy relation equations, ® being an Archimedean t-norm, is considered and the optimal solution of the problem is achieved by using two methods. Firstly, a matrix reduction method is applied to minimize a linear programming problem subject to fuzzy relation equations with max-Hamacher composition. Secondly, dynamic programming is explored to minimize a linear optimization model with max-product fuzzy relation equations as constraints.

4.2 The problem

Let $A=\begin{bmatrix} a_{ij} \end{bmatrix}$, $0 \leq a_{ij} \leq 1$, be a $m \times n$ dimensional fuzzy matrix and $b=\begin{bmatrix} b_1, b_2, \ldots, b_n \end{bmatrix}$, $0 \leq b_j \leq 1$, be a $n$-dimensional vector, then the following system of fuzzy relation equations is defined by $A$ and $b$:

$$x^d A = b$$

(4.1)
where $\circ'$ denotes the max-$\otimes$ composition of $x$ and $A$. $\otimes$ denotes an Archimedean t-norm from algebra over residuated lattice $L = ([0,1], \land, \lor, \otimes, \rightarrow, 0, 1)$. It is intended to find a solution vector $x = [x_1, x_2, \ldots, x_m]$, with $0 \leq x_i \leq 1$, such that

$$\max_{i=1}^m (x_i \otimes a_{ij}) = b_j, \forall j = 1, 2, \ldots, n$$  \hspace{1cm} (4.2)

Let $I = \{1, 2, \ldots, m\}$ and $J = \{1, 2, \ldots, n\}$ be the index sets. We are interested in solving the following linear optimization problem with max-$\otimes$ fuzzy relation equations as constraints:

Min $Z = \sum_{i \in I} c_i x_i$  \hspace{1cm} (4.3)

s.t. $\max_{i \in I} (x_i \otimes a_{ij}) = b_j, \forall j \in J$  \hspace{1cm} (4.4)

$0 \leq x_i \leq 1, \forall i \in I$

where, $c = [c_1, c_2, \ldots, c_m] \in \mathbb{R}^m$ is a $m$-dimensional vector, $c_i$ represents the weight (or cost) associated with variable $x_i$, $\forall i \in I$.

Study of fuzzy linear optimization problems is an important ongoing topic of research. Since the feasible domain of fuzzy relation equations is, in general, non-convex, thus for linear optimization models with fuzzy relation equations as constraints, traditional/conventional methods fail. Linear optimization problems with fuzzy relation equations as constraints forms interesting topic of research due to scarcity of methods and fast algorithms for the resolution of fuzzy relation equations. Problem (4.3)-(4.4) subject to fuzzy relation equations with different adjoint couple of operators has been studied by different researchers. Fang and Li [27], in their maiden paper, considered an optimization model with a linear objective function subject to a system of max-min fuzzy relation equations. Feasible domain was first characterized and then the problem was converted to an equivalent problem involving 0-1 integer programming with a branch-and-bound
solution technique. Wu et al. [158] improved Fang and Li’s method by proposing an initial upper bound for the branch-and-bound procedure and rearranging the data in parts so that the branches in the solution tree generated by branch-and-bound procedure are likely to be fathomed by the incumbent upper bound. Loetamonphong and Fang [77] considered the problem (4.3)-(4.4) with max-product composition by deriving special characteristics of its feasible domain and the optimal solutions. Some procedures were proposed for reducing the original problem. The problem was transformed into a 0-1 integer program solvable by branch-and-bound method.

Pandey [105] studied the optimization of fuzzy relation equations with continuous t-norms and with linear objective function. Pandey and Srivastava [103] gave efficient procedure for optimization of linear objective function subject to fuzzy relation equations as constraints and solved the associated 0-1 integer programming problem by branch-and-bound method with forward jumping-tracking technique.

Ghodousian and Khorram [31] focused on problems in which the solutions region is the fuzzy relation equation with max-prod composition and the objective function is linear. After determining the feasible set, an algorithm was presented to optimize the linear objective function on such regions. Ghodousian and Khorram [32] introduced a fuzzy operator constructed by the convex combination of two known operators, max-min and max-average compositions. The operator contains some properties of the two known compositions when it generates the feasible region for linear optimization problems. They investigated linear optimization problems whose feasible region is the fuzzy sets defined with this operator. Firstly, the structure of fuzzy regions was considered and then a method to solve the linear optimization problems with fuzzy equation constraints regarding this operator was presented. Ghodousian and Khorram [33] studied linear objective function optimization with respect to the fuzzy relational inequalities defined by max-min composition in which fuzzy inequality replaces ordinary inequality in the constraints. Fuzzy inequalities enable to attain the optimal points that are better solutions than those resulting from the resolution of the similar problems with ordinary inequality constraints. An algorithm was presented to generate such optimal solutions.
Qu and Wang [125] considered minimization of linear objective functions under the constraints expressed by a system of max-product fuzzy relation equations. Some properties of minimal solutions of a system and some rules for reducing the original problem were presented. An algorithm was derived which enables to find all the optimal solutions for solving the optimization problem. Li [71] studied the problem of minimizing a linear objective function subject to a fuzzy system constraint. By utilizing the fuzzy system's compact system, $k$-form chained solutions and spaces of $k$-form chained solutions were defined. It was proved that any minimal solution of the fuzzy system is a $k$-form chained solution. By using an operation method expressed in tables, basic solutions from the spaces of $k$-form chained solutions were sieved out. Optimal solutions of the studied problem were obtained from the basic solutions.

Guu and Wu [44] solved the problem of minimizing a linear objective function subject to a max-$T$ fuzzy relation equation constraint, were $T$ a continuous t-norm. Two extensions were given based on the results in Fang and Li [27] and a procedure for solving optimization problems with the max-Archimedean t-norm fuzzy relation equations as constraints was discussed.

Molai [91] studied the optimization model of a linear objective function subject to a system of fuzzy relation inequalities (FRI) with the max-Einstein composition operator. Properties of feasible domain, an efficient algorithm to solve the model based on the structure of FRI path, the concept of partial solution, and the branch-and-bound approach to find an optimal solution of the model without explicitly generating all the minimal solutions were discussed. Some sufficient conditions were given and it was shown that under such conditions, some of the optimal components of the model can be directly determined. Procedures were presented to reduce the search domain of an optimal solution of the original problem based on the conditions. Then the reduced domain was decomposed (if possible) into several sub-domains with smaller dimensions which makes the finding of the components of the optimal solution in each sub-domain very easy. In order to obtain an optimal solution of the original problem, they proposed another more efficient algorithm which combines the first algorithm, there procedures, and the
decomposition method. Furthermore, sufficient conditions were suggested that under them, the problem has a unique optimal solution.

Wu and Guu [159] investigated the problem of minimizing a linear objective function subject to fuzzy relation equations and derived a necessary condition for optimal solution. Based on this necessary condition, they proposed rules to simplify the work of computing an optimal solution. Guo and Xia [43] studied an optimization model with one linear objective function and finitely many constraints of fuzzy relation inequalities and proposed a new approach for solving this problem based on a necessary condition of optimality. Compared with the known methods, the proposed algorithm shrinks the searching region and hence obtains an optimal solution fast.

Mashayekhi and Khorram [86] extended Guo and Xia's [43] necessary condition in order to study the finitely many constraints of fuzzy relation inequalities and optimized a linear objective function on this region which is defined by the fuzzy max-min operator. The new condition was provided for removing the unnecessary paths resulting from Guo and Xia's [43] paths. Peeva et al. [120] presented an optimization problem with a linear objective function subject to fuzzy linear system of equations or fuzzy linear system of inequalities, with max-product composition as constraints. They developed methods for constraint resolution that provide algorithm for computing the maximum solution and all minimal solutions, when the fuzzy linear system is consistent, using MATLAB and Java.

4.3 Characterization of feasible domain

The feasible domain of the problem (4.3)-(4.4) is the solution set of (4.4) denoted by \( X(A,b) = \{ x \in [0,1]^n \mid x^T A = b \} \). In \( X(A,b) \), for any \( x^1, x^2 \in X(A,b) \), \( x^1 \leq x^2 \) if and only if \( x^1_i \leq x^2_i \), \( \forall i \in I \). Thus, \( \leq \) is a partial ordering relation on \( X \). Moreover, we call \( x^* \in X(A,b) \) as the maximum solution if \( x \leq x^* \), \( \forall x \in X(A,b) \). Similarly, \( x^* \in X(A,b) \) is a minimal solution if \( x \leq x^* \) implies \( x = x^* \), \( \forall x \in X(A,b) \). According to [66], when
$X(A,b) \neq \emptyset$, it can be completely determined by one maximum solution and a finite number of minimal solutions.

The maximum solution can be computed explicitly by the residual implicator (pseudo complement). The maximum solution can be obtained by assigning

$$\hat{x} = A \rightarrow b = \left[ \min_{j \in J} (a_j \rightarrow b_j) \right]_{i \in I}$$

(4.5)

where

$$a_j \rightarrow b_j = \sup \{ x_i \in [0,1] | x_i \otimes a_j \leq b_j \}$$

(4.6)

If $X(A,b)$ is the set of all minimal solutions, then

$$X(A,b) = \bigcup_{x \in X} \{ x \in X | x \leq \hat{x} \}$$

(4.7)

**Definition 4.3.1.** For any solution $x \in X(A,b)$, $x_i$ is called a binding variable if $x_i \otimes a_j = b_j$ holds for some $j \in J$ and a constraint $j \in J$ is said to be a binding constraint if $x_i \otimes a_j = b_j$ holds for some $i \in I$, $\otimes$ being an Archimedean t-norm. Let $X(A,b) \neq \emptyset$. Then define $I_j = \{ i \in I | x_i \otimes a_j = b_j \}$, $\forall j \in J$ and $J_i = \{ j \in J | x_i \otimes a_j = b_j \}$, $\forall i \in I$.

**Lemma 4.3.2.** If in the $j$th equation $a_j < b_j$, $\forall i \in I$, then the solution set $X(A,b) = \emptyset$.

**Proof.** If in the $j$th equation $a_j < b_j$ holds for all $i \in I$, then for $x_i \neq a_j$, $(x_i \otimes a_j) \leq (1 \otimes a_j) = a_j < b_j$ and for $x_i = a_j$, $(x_i \otimes a_j) = (a_j \otimes a_j) < a_j < b_j$. Thus for both cases, $(x_i \otimes a_j) < b_j$, $\forall i \in I$. Hence, $\max_{i \in I} (x_i \otimes a_j) < b_j$ and there exists no solution for the $j$th equation. Thus $X(A,b) = \emptyset$. 

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Lemma 4.3.3. If \( x \in X(A,b) \), then for each \( j \in J \) there exists \( i_j \in I \) such that 
\[
(x_i \otimes a_{ij}) = b_j \quad \text{and} \quad (x_i \otimes a_{ij}) \leq b_j, \quad \forall i \in I.
\]

Proof. For \( x \in X(A,b) \), \( \max_{i \in I} (x_i \otimes a_{ij}) = b_j, \quad \forall j \in J \). This implies \( (x_i \otimes a_{ij}) \leq b_j, \quad \forall j \in J \).

Therefore, in order to satisfy the equality constraint, there must exist at least one \( i_j \in I \) such that \( (x_i \otimes a_{ij}) = b_j \).

Lemma 4.3.4. If \( X(A,b) \neq \emptyset \), then \( I_j \neq \emptyset, \forall j \in J \).

Proof. From Lemma 4.3.3, we know that there exists at least one \( i_j \in I \) which satisfies constraint \( j \). Therefore, \( I_j \) must contain at least one element.

Lemma 4.3.5. If \( X(A,b) \neq \emptyset \), then \( \tilde{x} \) is the maximum solution.

Proof. Let \( \tilde{x} \in X(A,b) \) and \( \tilde{x}_i > \tilde{x}_i \). Let \( \tilde{x}_i \) belong to \( j \)th equation. Then 
\[
(\tilde{x}_i \otimes a_{ij}) > (\tilde{x}_i \otimes a_{ij}) = b_j.
\]
So, \( \tilde{x}_i \) violates \( j \)th equation, a contradiction.

Lemma 4.3.6. Let \( \tilde{x} \in X(A,b) \) be the maximum solution and \( x \in X(A,b) \). If \( x_i \) is binding in the \( j \)th equation, then \( \tilde{x}_i \) is also binding in the \( j \)th equation. Moreover, if \( \tilde{x}_i \) is not a binding variable, \( x_i \) is also nonbinding for any solution \( x \).

Proof. For \( x \in X(A,b) \), \( \max_{i \in I} (x_i \otimes a_{ij}) = b_j, \quad \forall j \in J \). This implies \( (x_i \otimes a_{ij}) \leq b_j, \quad \forall j \in J \).

Thus, \( (\tilde{x}_i \otimes a_{ij}) \leq b_j, \quad \forall j \in J \). Now if \( x_i \) is binding variable in the \( j \)th equation, \( j \in J \) then \( x_i \otimes a_{ij} = b_j \) holds for \( j \in J \). Since \( x_i \leq \tilde{x}_i \), therefore by the monotonically non-decreasing property of the t-norms, \( b_j = (x_i \otimes a_{ij}) \leq (\tilde{x}_i \otimes a_{ij}) \leq b_j \). This implies \( \tilde{x}_i \otimes a_{ij} = b_j \) and \( \tilde{x}_i \) is also binding in the \( j \)th equation. On the other hand, if \( \tilde{x}_i \) is not
binding in any equation, then \( \bar{x}_i \otimes a_{ij} < b_j, \forall j \in J \). Since \((x_i \otimes a_{ij}) \leq (\bar{x}_i \otimes a_{ij}) < b_j \) holds for any solution \( x \), we have \((x_i \otimes a_{ij}) < b_j, \forall j \in J \). Therefore, \( x_i \) is not a binding variable.

**Lemma 4.3.7.** Let \( x \in X(A, b) \). If for some \( j \in J, b_j = 0 \), then \( x_i \) is binding variable for all \( i \in I \), in the \( j \)th equation.

**Proof.** For \( x \in X(A, b) \), \( \max_{i \in I} (x_i \otimes a_{ij}) = b_j, \forall j \in J \). Thus, \((x_i \otimes a_{ij}) \leq b_j, \forall j \in J \). If for some \( j \in J, b_j = 0 \), then since \((x_i \otimes a_{ij}) \geq 0 \), therefore \((x_i \otimes a_{ij}) = 0 \) for \( j \in J \) and for each \( i \in I \). Thus \( x_i \) is a binding variable for all \( i \in I \), in the \( j \)th equation.

**Theorem 4.3.8.** (Mostert and Shields [94]) A t-norm is continuous and Archimedean if and only if a strictly decreasing and continuous function \( f : [0,1] \rightarrow [0,\infty] \) with \( f(1) = 0 \) exists such that \( x \otimes a = f^{(-1)}(f(a) + f(x)) \), where \( f^{(-1)} \) is the pseudoinverse of \( f \) defined by

\[
f^{(-1)}(y) = \begin{cases} f^{-1}(y) & \text{if } y \leq f(0) \\ 0 & \text{otherwise} \end{cases}
\]

Moreover, representation of \( x \otimes a \) is unique up to a positive multiplicative constant.

**Theorem 4.3.9.** Let \( \tilde{x} \in X(A, b) \) be the maximum solution. For any \( x \in X(A, b) \), if \( x_i \) is a binding variable, then \( x_i = \tilde{x}_i \).

**Proof.** For any \( x \in X(A, b) \), \( \max_{i \in I} (x_i \otimes a_{ij}) = b_j, \forall j \in J \). Since \( x_i \) is a binding variable, \( x_i \otimes a_{ij} = b_j \) for some \( j \in J \). By Lemma 4.3.6, \( \tilde{x}_i \) is also binding in the \( j \)th equation. Hence, \( \tilde{x}_i \otimes a_{ij} = b_j \). Assume \( x_i < \tilde{x}_i \). Since \( b_j > 0 \), Theorem 4.3.8 implies that
\(f(x_i) + f(a_{ij}) \leq f(0)\) and \(x_i \otimes a_{ij} = f^{-1}(f(x_i) + f(a_{ij}))\). As \(f\) is strictly decreasing, and \(x_i < \tilde{x}_i\), we have \(f(\tilde{x}_i) + f(a_{ij}) \leq f(0)\) and \(\tilde{x}_i \otimes a_{ij} = f^{-1}(f(\tilde{x}_i) + f(a_{ij}))\). Hence, we conclude that \(0 < b_j = (x_i \otimes a_{ij}) < (\tilde{x}_i \otimes a_{ij}) = b_j\), which is impossible. Therefore, we have \(x_i = \tilde{x}_i\).

**Theorem 4.3.10.** Let \(\tilde{x} \in X(A, b)\) be the maximum solution. If \(\tilde{x} \in X(A, b)\) is a minimal solution, then either \(\tilde{x}_i = 0\) or \(\tilde{x}_i = \tilde{x}_i\) for each \(i \in I\).

**Proof.** For \(\tilde{x} \in X(A, b)\), \(\tilde{x}_i\) is either binding or non-binding. Let \(\tilde{x}_i\) be non-binding and \(\tilde{x}_i > 0\). Then a solution \(\tilde{x}\) can be constructed by letting \(\tilde{x}_i = 0\) and \(\tilde{x}_k = \tilde{x}_k\), for \(k \in I\), \(k \neq i\). Therefore, \(\tilde{x} \leq \tilde{x}\) implying that \(\tilde{x}\) is not a minimal solution. Hence, \(\tilde{x}_i = 0\) if \(\tilde{x}_i\) is not a binding variable. And if \(\tilde{x}_i\) is a binding variable, by Theorem 4.3.9, \(\tilde{x}_i = \tilde{x}_i\).

**Theorem 4.3.11.** Let \(\tilde{x} \in X(A, b)\) be the maximum solution and \(x \leq \tilde{x}\), then \(x \in X(A, b)\) if and only if \(I_x = \{i \in I \mid x_i = \tilde{x}_i\} \neq \emptyset\) and \(\bigcup_{i \in I_x} J_i = J\).

**Proof.** For \(x \in X(A, b)\), \(\max_{i \in I_x} (x_i \otimes a_{ij}) = b_j\), \(\forall j \in J\). Thus there exists \(i \in I\) such that \(b_j = (x_i \otimes a_{ij}) \leq (\tilde{x}_i \otimes a_{ij}) \leq b_j\). This implies \(b_j = x_i \otimes a_{ij} = \tilde{x}_i \otimes a_{ij}\), i.e. \(x_i = \tilde{x}_i\). Hence \(I_x \neq \emptyset\) and it is easy to see \(\bigcup_{i \in I_x} J_i = J\). Conversely, if \(I_x \neq \emptyset\) and \(\bigcup_{i \in I_x} J_i = J\), then for every \(j \in J\) there exists \(i \in I_x\) such that \(b_j = \tilde{x}_i \otimes a_{ij} = x_i \otimes a_{ij}\), thus \(x \in X(A, b)\).

**Lemma 4.3.12.** Let \(\tilde{x} \in X(A, b)\) be the maximum solution and \(x^*\) be an optimal solution of problem (4.3)-(4.4). If \(c_i < 0\), for some \(k \in I\), then \(x^*_k = \tilde{x}_k\). If \(c_i \leq 0\), \(\forall i \in I\), then \(x^* = \tilde{x}\) is an optimal solution of the problem (4.3)-(4.4).
Proof. For optimal solution \( x^* \in X(A,b) \), we have \( 0 \leq x_i^* \leq \bar{x}_i \). If for some \( k \in I, c_k < 0 \), then \( \sum_{i \in I} c_i x_i^* = \sum_{i \in I, i \neq k} c_i x_i^* + c_k x_k^* > \sum_{i \in I, i \neq k} c_i x_i^* + c_k \bar{x}_k^* \). This contradicts that \( x^* \) is an optimal solution. Hence \( x_k^* = \bar{x}_k \). If \( c_i \leq 0, \forall i \in I \), we have \( \sum_{i \in I} c_i x_i \geq \sum_{i \in I} c_i \bar{x}_i \). Therefore, \( \bar{x} \) is an optimal solution.

Lemma 4.3.13. If \( c_i \geq 0, \forall i \in I \), then one of the minimal solutions of the problem (4.3)-(4.4) is an optimal solution.

Proof. If \( c_i \geq 0, \forall i \in I \), then for any \( x \in X(A,b) \), \( \sum_{i \in I} c_i \bar{x}_i \leq \sum_{i \in I} c_i x_i \leq \sum_{i \in I} c_i \bar{x}_i \). Therefore, a minimal solution gives the smallest value of the objective function. Since \( X(A,b) \) consists of finite number of minimal solutions, hence one of the minimal solution of problem (4.3)-(4.4) is an optimal solution.

4.4 Linear optimization problem with max-Hamacher fuzzy relation equations

Consider the linear optimization problem (4.3)-(4.4) with max-\( \odot \) fuzzy relation equations as constraints where \( \odot \) is considered to be a Hamacher t-norm. Then problem (4.3)-(4.4) can be restated as follows:

\[
\text{Min } Z = \sum_{i \in I} c_i x_i \tag{4.8}
\]

s.t. \( \max_{i \in I} (x_i \odot a_{ij}) = b_j, \forall j \in J \) \tag{4.9}

\( 0 \leq x_i \leq 1, \forall i \in I \)

where
The maximum solution can be obtained by assigning

$$x_i, a_{ij} = \frac{x_i a_{ij}}{x_i + a_{ij} - x_i a_{ij}}$$  \hspace{1cm} (4.10)$$

where

$$a_{ij} \rightarrow b_j = \frac{b_j a_{ij}}{a_{ij} - b_j + a_{ij} b_j}$$  \hspace{1cm} (4.12)$$

Some rules are proposed in the next section to solve the problem (4.8)-(4.9) using matrix reduction method.

**4.5 Matrix reduction method: Rules for reducing the problem**

In this section we employ a value based matrix method to reduce the original problem and find the optimal solution of the problem. A value matrix $M = [m_{ij}]$ is defined as

$$m_{ij} = \begin{cases} c_i \bar{x}_i & \text{if } j \in J_i \\ \infty & \text{otherwise} \end{cases}$$

Define $\bar{J}_i = \{i \in I \mid m_{ij} = c_i \bar{x}_i\}$, $\forall j \in J$ and $\bar{J}_i = \{j \in J \mid m_{ij} = c_i \bar{x}_i\}$, $\forall i \in I$. Some rules are proposed to optimize the problem by employing value matrix.

**Rule 1.** If for some $i \in I$, $m_{ij} \leq 0$, then for any optimal solution $x^*$ assign $x_i^* = \bar{x}_i$.

**Proof.** If for some $i \in I$, $m_{ij} \leq 0$, this implies that $c_i \bar{x}_i < 0$. Thus, from Lemma 4.3.12, $x_i^* = \bar{x}_i$. 
**Rule 2.** If for some $j \in J$, $\overline{I}_j = \{i\}$, a singleton set, then $i$th component of any optimal solution is assigned $\tilde{x}_i$.

**Proof.** If for some $j \in J$, $\overline{I}_j = \{i\}$, a singleton set, this implies that $j$th equation can only be satisfied by variable $x_i$, i.e. $x_i$ is the only binding variable for the $j$th equation. By Theorem 4.3.9, $x_i = \tilde{x}_i$.

**Rule 3.** If $\overline{I}_s \subsetneq \overline{I}_t$ for some $s, t \in J$ in the value matrix, then $t$th column of the value matrix $M$ can be deleted.

**Proof.** If $\overline{I}_s \subsetneq \overline{I}_t$ for some $s, t \in J$ in the value matrix, then all the variables that are binding in the $s$th equation are also binding in the $t$th equation. Thus column corresponding to the $t$th equation can be deleted from consideration.

**Rule 4.** If $\phi \neq \overline{I}_p \subsetneq \overline{I}_q$ for some $p, q \in I$ and $0 < c_q \tilde{x}_q < c_p \tilde{x}_p$, then there exists an optimal solution $x^*$ with $x^*_p = 0$.

**Proof.** Let $x^*$ be any optimal solution. Since $\overline{I}_p \subsetneq \overline{I}_q$, this implies that $J_p \subsetneq J_q$. Also, since $0 < c_q \tilde{x}_q < c_p \tilde{x}_p$, thus $c_p > 0$. This implies that $x^*_p = 0$ or $x^*_p = \tilde{x}_p$. If $x^*_p = 0$, then proof is complete. If $x^*_p = \tilde{x}_p > 0$, then if $x^*_q = 0$, then we can construct a solution vector $x$, equal to $x^*$, except for $x_p = 0$ and $x_q = \tilde{x}_q$. Since $J_p \subseteq J_q$, the constraints satisfied by $x^*_p = \tilde{x}_p$ are also satisfied $x_q = \tilde{x}_q$. Thus, $x$ is a solution of the problem. Also, $Z(x) - Z(x^*) = \sum_{i \in I} c_i x_i^* - \sum_{i \in I} c_i x_i = c_p\tilde{x}_p^* - c_q\tilde{x}_q = c_p\tilde{x}_p - c_q\tilde{x}_q > 0$. This contradicts the assumption of $x^*$ being an optimal solution. Therefore, if $0 < c_q \tilde{x}_q < c_p \tilde{x}_p$, then for optimal solution $x^*$, $x^*_p = 0$. 

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If \( x^*_p = \bar{x}_p \), then if \( x^*_q = \bar{x}_q > 0 \), then we can construct a solution vector \( x \), equal to \( x^* \), except for \( x_p = 0 \). Since \( J_p \subseteq J_q \), the constraints satisfied by \( x^*_p = \bar{x}_p \) are also satisfied by \( x_p = \bar{x}_q \). Thus, \( x \) is a solution of the problem. Also, \( Z(x^*) - Z(x) = \sum_{i \in I} c_i x^*_i - \sum_{i \in I} c_i x_i = c_p \bar{x}_p > 0 \). This contradicts the assumption of \( x^* \) being an optimal solution. Therefore, if \( 0 < c_p \bar{x}_q < c_p \bar{x}_p \), then for optimal solution \( x^* \), \( x^*_p = 0 \).

**Rule 5.** Let \( \hat{I} \subseteq I \) and \( \bigcup_{i \in I} J_i = \hat{J} \). If \( p \in I \), \( p \notin \hat{I} \), \( \hat{J}_p \subseteq \hat{J} \) and \( \sum_{i \in I} c_i \bar{x}_i < c_p \bar{x}_p \), then there exists an optimal solution \( x^* \) with \( x^*_p = 0 \).

**Proof.** Similar to that of Rule 4.

**Rule 6.** If for some \( p \in I \), \( c_p > 0 \) and \( \tilde{J}_p = \phi \), then \( p \)th component of any optimal solution can be assigned value 0.

**Proof.** If for some \( p \in I \), \( \tilde{J}_p = \phi \), then \( x_p \) is not a binding variable for any constraint. Since \( c_p > 0 \), thus \( p \)th component of solution vector can be assigned value 0.

**Algorithm 1: For obtaining optimal solution of the problem (4.8)-(4.9)**

Step 1: Compute the maximum solution \( \bar{x} \) by (4.11). If \( \bar{x} \odot A = b \), continue. Otherwise stop, problem is inconsistent.

Step 2: Compute the index sets \( J_i, \forall i \in I \). Obtain the value matrix \( M \).

Step 3: Apply Rule 1-Rule 6 to reduce the problem as far as possible. If values of all components of the optimal solution are determined, obtain the optimal value of the problem. Else go to Step 4.
Step 4: Apply branch-and-bound method on the remaining value matrix and find the undetermined components of the optimal solution as discussed in Fang and Li [27]. Obtain the optimal value of the problem.

Example 4.5.1. Consider the following linear programming problem subject to max-Hamacher t-norm fuzzy relation equations with \( x \otimes a = \frac{x \cdot a}{x + a - x \cdot a} \), which is one type of the max-Archimedean t-norm compositions:

\[
\begin{aligned}
\text{Min } Z &= 3x_1 - 2x_2 + 1.5x_3 + 0.5x_4 + 2x_5 + 1.1x_6 + 2x_7 + 1.5x_8 \\
A &= \begin{bmatrix}
0.60 & 0.20 & 0.30 & 0.84 & 0.60 & 0.12 \\
0.20 & 0.70 & 0.10 & 0.70 & 0.20 & 0 \\
0.60 & 0.60 & 1 & 0.30 & 0.20 & 0.30 \\
1 & 1 & 0.40 & 0 & 0.50 & 0.10 \\
0.60 & 0.90 & 0.30 & 0.10 & 0.60 & 0 \\
0.80 & 0 & 0.20 & 0.70 & 0.80 & 0 \\
0.80 & 0.80 & 0.60 & 0.70 & 0.20 & 0.20 \\
1 & 0.60 & 0.40 & 0.60 & 0.50 & 0.10
\end{bmatrix}
\end{aligned}
\]

\( b = [0.30 \ 0.40 \ 0.20 \ 0.40 \ 0.30 \ 0.10] \)

Step 1: Compute the maximum solution \( \bar{x} \)
\( \bar{x} = [0.3750 \ 0.4828 \ 0.1304 \ 0.2857 \ 0.3750 \ 0.3243 \ 0.1667 \ 0.2857] \)

Since \( \bar{x} \circ A = b \), problem is consistent.

Step 2: Compute the index sets \( J_i, \forall i \in I \).
\( J_1 = \{1,3,5,6\}, J_2 = \{2,4\}, J_3 = \{6\}, J_4 = \{3\}, J_5 = \{1,3,5\}, J_6 = \{1,5\}, J_7 = \{6\}, J_8 = \{3\} \).

Obtain the value matrix \( M \).
Step 3: Rule 1 is applied to the value matrix $M$ to reduce it as far as possible. In the matrix $M$, $m_{22} = m_{24} = -0.9655 < 0$. Let us denote $x^*$ as the optimal solution. Thus from Rule 1, $x_2^* = \tilde{x}_2 = 0.4828$. Column 2\textsuperscript{nd}, 4\textsuperscript{th} and row 2\textsuperscript{nd} are deleted. After reduction, reduced matrix $M$ is obtained as

\[
\begin{array}{cccccc}
\text{Equation} & 1 & 2 & 3 & 4 & 5 \\
\hline
x_1 & 1.1250 & 1.1250 & 1.1250 & 1.1250 \\
x_2 & \infty & -0.9655 & \infty & -0.9655 & \infty & \infty \\
x_3 & \infty & \infty & \infty & \infty & \infty & 0.1957 \\
x_4 & \infty & \infty & \infty & \infty & \infty & 0.1429 \\
x_5 & 0.7500 & \infty & 0.7500 & \infty & 0.7500 & \infty \\
x_6 & 0.3568 & \infty & \infty & \infty & \infty & 0.3568 \\
x_7 & \infty & \infty & \infty & \infty & \infty & \infty \\
x_8 & \infty & \infty & \infty & \infty & \infty & 0.4286 \\
\end{array}
\]

Now Rule 3 can be applied to the reduced matrix $M$. Note that $I_1 = \{1,5,6\} = I_5 = \{1,5,6\}$. Thus, according to Rule 3, 5\textsuperscript{th} column (or 1\textsuperscript{st} column) can be deleted. After modification reduced matrix $M$ is obtained as

\[
\begin{array}{ccc}
\text{Equation} & 3 & 5 & 6 \\
\hline
x_1 & 1.1250 & 1.1250 & 1.1250 \\
x_3 & \infty & \infty & \infty \\
x_4 & \infty & 0.1429 & \infty \\
x_5 & 0.7500 & 0.7500 & 0.7500 \\
x_6 & 0.3568 & \infty & 0.3568 \\
x_7 & \infty & \infty & \infty \\
x_8 & \infty & 0.4286 & \infty \\
\end{array}
\]
In matrix $M$, $\vec{J}_2 = \vec{J}_3$, $0 < c_3 \bar{x}_3 < c_7 \bar{x}_7$, and $\vec{J}_8 = \vec{J}_4$, $0 < c_4 \bar{x}_4 < c_8 \bar{x}_8$, thus from Rule 4, delete rows corresponding to the 7th and 8th variable and assign $x_7^* = x_8^* = 0$. After further reduction, $M$ is obtained as

\begin{equation}
\begin{array}{ccc}
\text{Equation} & 1 & 3 & 6 \\
\hline
x_1 & 1.1250 & 1.1250 & 1.1250 \\
x_3 & \infty & \infty & 0.1957 \\
x_4 & \infty & 0.1429 & \infty \\
M = x_5 & 0.7500 & 0.7500 & \infty \\
x_6 & 0.3568 & \infty & \infty \\
x_7 & \infty & \infty & 0.3333 \\
x_8 & \infty & 0.4286 & \infty \\
\end{array}
\end{equation}

Now in the updated matrix $M$, we have $\vec{J}_1 = \{1,3,6\} \subseteq \bigcup_{i=3,4,5} \vec{J}_i = \{1,3,6\}$ and also $c_3 \bar{x}_3 + c_4 \bar{x}_4 + c_5 \bar{x}_5 = 1.0886$, $c_1 \bar{x}_1 = 1.1250$, i.e. $0 < c_3 \bar{x}_3 + c_4 \bar{x}_4 + c_5 \bar{x}_5 < c_1 \bar{x}_1$. Thus from Rule 5, delete row corresponding to 1st variable and assign $x_1^* = 0$. Obtain modified matrix $M$.

\begin{equation}
\begin{array}{ccc}
\text{Equation} & 1 & 3 & 6 \\
\hline
x_3 & \infty & \infty & 0.1957 \\
M = x_4 & \infty & 0.1429 & \infty \\
x_5 & 0.7500 & 0.7500 & \infty \\
x_6 & 0.3568 & \infty & \infty \\
\end{array}
\end{equation}
In the current matrix $M$, $\bar{I}_6 = \{3\}$, a singleton set. From Rule 2, $x^*_3 = \hat{x}_3 = 0.1304$. Delete row corresponding to $3^{rd}$ variable and column corresponding to $6^{th}$ equation and modify matrix $M$.

Equation 1  3
\[
M = \begin{bmatrix}
x_4 & \infty & 0.1429 \\
x_5 & 0.7500 & 0.7500 \\
x_6 & 0.3568 & \infty
\end{bmatrix}
\]

For matrix $M$, $\bigcup_{i=4,6} \bar{J}_i = \{1,3\} = \bar{J}_3 = \{1,3\}$ and $c_4 \hat{x}_4 + c_6 \hat{x}_6 = 0.4997$, $c_5 \hat{x}_5 = 0.7500$, i.e. $0 < c_4 \hat{x}_4 + c_6 \hat{x}_6 < c_5 \hat{x}_5$. From Rule 5, delete row corresponding to $5^{th}$ variable and assign $x^*_5 = 0$. Modified matrix $M$ is

Equation 1  3
\[
M = \begin{bmatrix}
x_4 & \infty & 0.1429 \\
x_6 & 0.3568 & \infty
\end{bmatrix}
\]

In the above matrix, from Rule 2, assign $x^*_4 = \hat{x}_4 = 0.2857$ and $x^*_6 = \hat{x}_6 = 0.3243$. All the components of the optimal solution are determined. The optimal solution is obtained as $x^* = [0 \ 0.4828 \ 0.1304 \ 0.2857 \ 0 \ 0.3243 \ 0 \ 0]$ with optimal value of the objective function as $Z(x^*) = -0.2701$.

4.6 Linear optimization problem with max-product fuzzy relation equations

Consider the linear optimization problem (4.3)-(4.4) with max-$\oplus$ fuzzy relation equations as constraints where $\oplus$ is considered to be a product t-norm. Then problem (4.3)-(4.4) can be restated as follows:
\[
\text{Min } Z = \sum_{i \in I} c_i x_i \tag{4.13}
\]

\[
\text{s.t. } \max_{i \in I} \left( x_i \otimes a_{ij} \right) = b_j, \ \forall \ j \in J \tag{4.14}
\]

\[0 \leq x_i \leq 1, \ \forall \ i \in I \]

where

\[x_i \otimes a_{ij} = x_i \cdot a_{ij} \tag{4.15}\]

The maximum solution can be obtained by assigning

\[
\tilde{x} = \left[ \min_{j \in J} (a_{ij} \rightarrow b_j) \right]_{i \in I} \tag{4.16}
\]

where

\[a_{ij} \rightarrow b_j = \begin{cases} 
\frac{b_j}{a_{ij}} & \text{if } a_{ij} > b_j \\
1 & \text{if } a_{ij} < b_j
\end{cases} \tag{4.17}\]

In next section, a dynamic characterization of the constraints designing the feasible domain that make the fuzzy linear programming problem decidable over it by providing optimal decision is presented. We propose an interrelated multistage dynamic programming procedure to solve the problem (4.13)-(4.14).

### 4.7 Concept of dynamic programming

The dynamic programming technique is a mathematical technique that decomposes a multistage decision problem into a sequence of interrelated single stage decision problems. It can deal with discrete variables, non-convex, non-continuous and non-differentiable functions. It makes use of the concept of sub-optimization and the principle of optimality in solving the problems. In traditional theory, dynamic programming
models are generally formulated in terms of state variables, decision variables, return function and transformation function.

We present a concept of dynamic programming problem to solve fuzzy linear programming problem. Before considering the representation of a multistage decision process, we first consider a single stage fuzzy decision problem characterized by input parameters $S$ (or data) and certain decision variables $x$ and certain output parameters $S'$ representing the outcome obtained as a result of making the decision. The input parameters are known as input state variables, and the output parameters are called output state variables. Finally there is return or objective function $Z$ which measures the effectiveness of the decisions that are made and the output that results from these decisions. Figure 4.1 represents a single stage decision process.

The output is related to the input through a stage transformation function by $T = t(S, x)$ and the return function represented as $R = r(S, x)$.

Figure 4.1: A single stage fuzzy decision problem
Any multistage fuzzy decision process can now be represented as shown in Figure 4.2.

![Diagram](image)

**Figure 4.2: A multistage fuzzy decision problem**

For the \( i \)th stage, the input state vector is denoted by \( S_i \) and the output state vector as \( S_{i+1} \). Since the system is a serial one, the output from stage \( i \) must be equal to the input to stage \( i+1 \). Hence the state transformation and return functions are

\[
S_{i+1} = t_i(S_i, x_i) \quad \text{and} \quad R_i = r_i(S_i, x_i) \tag{4.18}
\]

where \( x_i \) denotes the decision variable at stage \( i \). The objective of a multistage decision problem is to find \( x_1, x_2, \ldots, x_m \) so as to optimize some function of the individual stage returns, i.e.

\[
Z(x_1, x_2, \ldots, x_m) = \sum_{i \in I} R_i = \sum_{i \in I} r_i(S_i, x_i) \tag{4.19}
\]

satisfying the equations (4.18), \( \forall i = 1, 2, \ldots, m \).
Suppose that the desired objective is to minimize the $m$-stage objective function $Z$ which gives the sum of the individual stage returns. Consider the first sub-problem by starting at the first stage, $i = 1$. If the optimum is denoted as $Z_1^*$, we have

$$Z_1^* = \min_{x_1} [R_i(S_1, x_1)] \quad (4.20)$$

Next, consider the second sub-problem by grouping the first two stages together. If $Z_2^*$ denotes the optimum value of the second sub-problem, then we have

$$Z_2^* = \min_{x_1, x_2} [R_2(S_2, x_2) + R_i(S_1, x_1)] = \min_{x_2} [R_2(S_2, x_2) + Z_1^*] \quad (4.21)$$

This idea can be repeated up to the $m$th sub-problem and thus, we have

$$Z_m^* = \min_{x_1, \ldots, x_m} [R_m(S_m, x_m) + (R_{m-1}, x_{m-1}) + \ldots + R_2(S_2, x_2) + R_1(S_1, x_1)] \quad (4.22)$$

which can be written as

$$Z_m^* = \min_{x_m} (R_m + Z_{m-1}^*) \quad (4.23)$$

**Algorithm 2: Dynamic programming for optimizing problem (4.13)-(4.14)**

Step 1: Given matrix $A, b$ and cost vector $c$, find the maximum solution $\bar{x}$ by using (4.16). Check the feasibility of maximum solution. If $\bar{x} \cdot A \neq b$, problem is infeasible, stop.

Step 2: Compute sets $J_i = \{j \in J : \bar{x}_i \cdot a_{ij} = b_j\}, \forall i \in I$.

Step 3: Let $S = \{J_i\}_{i \in I}$ be the set of collection of sets $J_i, \forall i \in I$. 

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Step 4: Find $c_i \tilde{x}_i, \forall i \in I$. Arrange $c_i \tilde{x}_i, \forall i \in I$ in ascending order and correspondingly rearrange $x_i, \forall i \in I$. Let $\tilde{I}$ be the new index set. Let $x^*$ be the optimal solution.

Step 5: For $i^* = 1, 2, \ldots, m$,

Set $S_{i^*} = S$

If ($S_{i^*} \neq \emptyset$)

{ 
  If ($J_{i^*} = \emptyset$)
    
    If $c_{i^*} < 0$, set $x_{i^*}^* = \tilde{x}_{i^*}$ else set $x_{i^*}^* = 0$.
  
  } 

If ($J_{i^*} \neq \emptyset$)

{ 
  If $J_{i^*} \cup \tilde{J} \subseteq J_k$ and $c_i \tilde{x}_i < c_i \tilde{x}_{i^*} + \sum_{p \in P} c_p \tilde{x}_p$, where $P \subseteq \tilde{I}$, $i \notin P$,
    
    $\tilde{j} = \bigcup_{p \in P} J_p, k \in \tilde{I}, k \neq i^*, k \notin P$, Set $x_{i^*}^* = 0, x_{i^*}^* = 0, \forall p \in P$
  
  Else
    
    Set $x_{i^*}^* = \tilde{x}_{i^*}, J_i = J_i - J_{i^*} \forall i \in I, S = \{J_i - J_{i^*}\}_{i \in I}$
  
  }

Return $R_{i^*} = c_{i^*} x_{i^*}^*$

Step 6: Obtain the optimal solution $x^*$ and optimal value of the objective function $Z^*$.

**Example 4.7.1.** Consider the following linear programming problem subject to max-product t-norm fuzzy relation equations with $x \oplus a = x \cdot a$, which is one type of the max-Archimedean t-norm compositions:
Min $Z = 5x_1 + 2.4x_2 + 1.8x_3 + 1.5x_4 + 1.2x_5 + x_6$

$$
A = \begin{bmatrix}
0.8 & 0.4 & 0.64 & 0.6 & 0.5 & 0.3 & 0.1 \\
0.3 & 0.8 & 1.0 & 0.2 & 0.875 & 0.75 & 0.6 \\
1.0 & 0.9 & 0.8 & 0.4 & 0.7 & 0.6 & 0.1 \\
0.7 & 0.4 & 0.7 & 0.5 & 0.7 & 0.96 & 0.84 \\
1.0 & 0.9 & 0.7 & 0.6 & 0.5 & 0.6 & 0.5 \\
0.8 & 0.2 & 0.5 & 0.9 & 0.8 & 0.4 & 0.6
\end{bmatrix}
$$

$b = [0.8 \ 0.72 \ 0.64 \ 0.63 \ 0.56 \ 0.48 \ 0.42]$

Step 1: Maximum solution is given by $\tilde{x} = [1 \ 0.64 \ 0.8 \ 0.5 \ 0.8 \ 0.7]$.
Since $\tilde{x} \cdot A = b$, thus the maximum solution is feasible.

Step 2: $J_1 = \{1,3\}$, $J_2 = \{3,5,6\}$, $J_3 = \{1,2,3,5,6\}$, $J_4 = \{6,7\}$, $J_5 = \{1,2,6\}$, $J_6 = \{4,5,7\}$.

Step 3: Obtain set $S = \{J_i\}_{i=1}^n$.
\[ S = \{J_1 = \{1,3\}, J_2 = \{3,5,6\}, J_3 = \{1,2,3,5,6\}, J_4 = \{6,7\}, J_5 = \{1,2,6\}, J_6 = \{4,5,7\}\}. \]

Step 4: $(c_i \tilde{x}_i)_{i=1}^n = [5 \ 1.536 \ 1.44 \ 0.75 \ 0.96 \ 0.7]$.
Arranging set $(c_i \tilde{x}_i)_{i=1}^n$ in ascending order, we get
\[ (c_i \tilde{x}_i)_{i=1}^n = [0.7 \ 0.75 \ 0.96 \ 1.44 \ 1.536 \ 5]. \]
Index set $\overline{T} = \{6,4,5,3,2,1\}$.

Multistage decision fuzzy problem is shown in Figure 4.3.
Step 5-Stage 1: \( J_6 = \{4,5,7\}, S_6 = S \),
Assign \( x_6^* = \bar{x}_6 = 0.7 \),
\( S = \{J_i - J_6 \}_{i=1} = \{ J_1 = \{1,3\}, J_2 = \{3,6\}, J_3 = \{1,2,3,6\}, J_4 = \{6\}, J_5 = \{1,2,6\}, J_6 = \phi \} \),
\( R_6 = c_6x_6^* = 0.7, Z^* = R_6 = 0.7 \).

Stage 2: \( J_4 = \{6\}, S_4 = S \),
Note that \( J_4 + J_5 \subseteq J_3 \) and \( c_3\bar{x}_3 < c_4\bar{x}_4 + c_5\bar{x}_5 \), therefore, set \( x_4^* = 0, x_5^* = 0 \),
\( R_4 = c_4x_4^* = 0, R_5 = c_5x_5^* = 0, Z^* = 0.7 + R_4 + R_5 = 0.7 \).

Stage 3: Since \( x_3^* \) has been determined in Stage 2, move to the next stage.

Stage 4: \( J_3 = \{1,2,3,6\}, S_3 = S \),
Assign \( x_3^* = \bar{x}_3 = 0.8 \),
\( S = \{J_i - J_3 \}_{i=1} = \{ J_1 = \phi, J_2 = \phi, J_3 = \phi, J_4 = \phi, J_5 = \phi, J_6 = \phi \} = \phi \),
\( R_3 = c_3x_3^* = 1.44, Z^* = 0.7 + R_3 = 2.14 \).

Stage 5: Since \( J_2 = \phi \) and \( c_2 > 0 \), thus set \( x_2^* = 0 \).
\( R_2 = c_2x_2^* = 0, Z^* = 2.14 + R_2 = 2.14 \).
Stage 6: Since \( J_i = \emptyset \) and \( c_i > 0 \), thus set \( x_i^* = 0 \).

\[ R_i = c_i x_i^* = 0, \quad Z^* = 2.14 + R_i = 2.14. \]

Step 6: Thus optimal solution is \( x^* = [0 \ 0 \ 0.8 \ 0 \ 0 \ 0.7] \) and optimal value of the objective function is obtained as \( Z^* = 2.14 \).

4.8 Conclusion

This chapter deals with minimization of linear optimization problem subject to fuzzy relation equations with a max-\( \oplus \) composition, where \( \oplus \) is an Archimedean t-norm. Some properties and conditions for the existence of solution are studied. Matrix reduction method and dynamic programming are used to obtain the optimal solution of the linear optimization problem with max-Hamacher and max-product fuzzy relation equations as constraints respectively. For the optimization problem with max-Hamacher fuzzy relation equations, maximum solution is computed and the concept of value matrix is applied to find the optimal solution of the problem. Some rules are employed to reduce the size of the value matrix. Numerical example shows that the optimal solution can be obtained efficiently by the proposed procedure.

Dynamic programming procedure is designed to optimize a linear programming problem with max-product fuzzy relation equations as constraints. At present, the proposed algorithm is efficient for problems with unique optimal solutions. There is no need to decompose the problem into two sub-problems depending upon the positive and negative cost coefficients. Fuzzy dynamic programming uses the concept of sub-optimization and principle of optimality in solving the problem. Moreover, the presented matrix reduction method and dynamic programming procedure is applicable for finding the optimal solution and optimal value of the objective function subject to fuzzy relation equations with max-\( \oplus \) composition, where \( \oplus \) stands for any Archimedean t-norm.